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FOURIER TRANSFORMS OF UNBOUNDED MEASURES

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1. Introduction. One of the basic objects of study in harmonic analysis is the Fourier transform (or Fourier-Stieltjes transform) μ of a bounded (complex) measure μ on the real line R:

$$(1.1)\qquad \hat{\mu}(t) = \int_{-\infty}^{\infty} e^{-itx} d\mu(x).$$

More generally, if μ is a bounded measure on a locally compact abelian group G, then its Fourier transform is the function

$$(1.2)\qquad \hat{\mu}(\hat{x}) = \int_G \overline{[x, \hat{x}]} d\mu(x) \quad (\hat{x} \in \hat{G})$$

where $\hat{G}$ is the dual group of G and $[x, \hat{x}] = \hat{x}(x)$. One answer to the question "Which functions can be represented as Fourier transforms of bounded measures?" was given by the following criterion due to Schoenberg [**11**] for the real line and Eberlein [**5**] in general: f is a Fourier transform of a bounded measure if and only if there is a constant M such that

$$(1.3)\qquad \left| \int_G f\phi \right| \leqq M \sup_{x \in G} |\hat{\phi}(\hat{x})|$$

for all $\phi \in L^1(G)$, where $\hat{\phi}(\hat{x}) = \int_G [x, \hat{x}]\phi(x)dx$.

The integrals (1.1) and (1.2) do not exist if μ is unbounded, and so the question arises as to the existence of a meaningful notion of Fourier transform in the case of unbounded measures. One could, of course, interpret (1.1) or (1.2) as holding in a summability sense, and this has sometimes been done. (See [**4**], [**12**], and [**7**, **8**].) But Argabright and Gil de Lamadrid [**1**] have recently proposed a very general definition of a Fourier transform. They defined a measure μ to be *transformable* if there exists a measure $\hat{\mu}$ on $\hat{G}$ such that, for every $\phi \in C_c(G)$ (the continuous functions with compact support), $\hat{\phi} \in L^2(\hat{\mu})$ and

$$(1.4)\qquad \int_G \phi * \tilde{\phi}(x) d\mu(x) = \int_{\hat{G}} |\phi(-\hat{x})|^2 d\hat{\mu}(\hat{x}),$$

where $\tilde{\phi}(x) = \overline{\phi(-x)}$ and $*$ denotes convolution. If μ is transformable, then the measure $\hat{\mu}$ occurring on the right side of (1.4) is called the *Fourier transform* of μ.

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This definition of Fourier transform is easily seen to be a generalization of both the Fourier-Stieltjes transform (1.2) and the classical Fourier transform of L^p functions, $1 \leqq p \leqq 2$. It also encompasses the representation of unbounded positive definite functions as Fourier transforms of positive unbounded measures [**4**], [**12**].

Argabright and Gil de Lamadrid showed that any Fourier transform μ must be *translation-bounded* in the sense that

$$(1.5) \qquad \sup_{\hat{x}\in\hat{G}}|\hat{\mu}|(\hat{x} + C) < \infty$$

for every compact set C in $\hat{G}$. They also established extended versions of the Poisson Summation Formula and the Inversion Theorem for Fourier transforms.

The present paper has two main purposes. The first is to describe a class of measures which are transformable in the sense of (1.4). If $\sum_\alpha[|\mu|(K_\alpha)]^r < \infty$, where $1 \leqq r \leqq 2$ and the K_α's are certain subsets of the group G related to its structure as described in § 3, then μ will be shown to be transformable. The second purpose is to generalize the Schoenberg-Eberlein criterion (1.3) to unbounded measures. If $2 \leqq q \leqq \infty$ and there is a constant M such that

$$(1.6) \qquad \left|\int_G f\phi\right| \leqq M\left[\sum_\alpha \sup_{x\in K_\alpha} |\hat{\phi}(\hat{x})|^q\right]^{1/q}$$

for every $\phi \in C_c(G)$, then it will be shown that f is a Fourier transform.

Some of the main theorems of this paper are generalizations of results of Finbarr Holland [**7**, **8**] to the context of groups. Therefore we devote §2 to a short exposition of his work on amalgams of L^p and l^q. In § 3 we show how to extend some of his definitions and results to groups. Then in § 4 we apply these to prove the results stated in the above paragraph.

2. Amalgams of L^p and l^q on the real line. If f is a measurable function on R and $1 \leqq p, q \leqq \infty$, define

$$||f||_{p,q} = \left[\sum_{-\infty}^{\infty}\left[\int_n^{n+1} |f(x)|^p dx\right]^{q/p}\right]^{1/q}$$

$$||f||_{\infty,q} = \left[\sum_{-\infty}^{\infty} \sup_{n\leqq x\leqq n+1} |f(x)|^q\right]^{1/q}$$

and let

$$(L^p, l^q) = \{f; \|f\|_{p,q} < \infty\}, \quad (C_0, l^q) = C_0 \cap (L^\infty, l^q).$$

These spaces were introduced and studied systematically by Holland [**7**], although certain special cases had been used earlier by Wiener [**13**] ($p = 2$, $q = \infty$), [**14**], ($p = \infty$, $q = 1$ and $p = 1$, $q = \infty$) and Cooper [**4**] ($p = 2$, $q = 1$), and certain related spaces had been used by Pitt [**9**] and Benedek and Panzone [**2**].

We list here some of Holland's results.

THEOREM A. *(L^p, l^q) is a Banach space and for $1 \leqq p, q < \infty$ its dual space is isometrically isomorphic to $(L^{p'}, l^{q'})$, where $p^{-1} + (p')^{-1} = 1$.*

THEOREM B. *If T is a continuous linear functional on (C_0, l^q), where $1 \leqq q \leqq \infty$, then there exists a measure $\mu \in M_{q'}$ such that*

$$T(\phi) = \int_{-\infty}^{\infty} \phi d\mu \quad (\phi \in (C_0, l^q))$$

where $M_r = \{$complex measures μ; $\sum_{-\infty}^{\infty}[|\mu|([n, n+1])]^r < \infty\}$.

THEOREM C. *Let $1 \leqq p, q \leqq 2$. If $f \in (L^p, l^q)$, then $\int_{-N}^{N} e^{-itx} f(x)dx$ converges to an element $\hat{f} \in (L^{q'}, l^{p'})$ as $N \to \infty$. There is a constant $M_{p,q}$ such that*

$$\| \hat{f} \|_{q',p'} \leqq M_{p,q} \| f \|_{p,q} \ (f \in (L^p, l^q)).$$

THEOREM D. *If $1 \leqq q \leqq 2$ and $\mu \in M_q$, then*

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{-itx} d\mu(x)$$

exists in the sense of Cesàro summability.

3. Amalgams on groups. Let G be a locally compact abelian group. The structure theorem for such groups [**6**, Theorem 24.30] allows us to write $G = R^a \times G_1$, where a is a nonnegative integer and G_1 is a group which contains a compact open subgroup H. (If G_1 is compact, we take $H = G_1$. If G_1 is discrete and infinite, we take $H = \{0\}$. Otherwise H is arbitrary but fixed. If the relationship between H and G needs to be made explicit, then we write $H = H(G)$.) We normalize the Haar measure m on G_1 so that $m(H) = 1$.

The dual group of G can then be written as $\hat{G} = R^a \times \hat{G}_1$. If A is the annihilator of H, $A = \{\hat{s} \in \hat{G}_1; [s, \hat{s}] = 1$ for all $s \in H\}$, then A is a compact open subgroup of $\hat{G}_1$. (Since H is open, G_1/H is discrete, and so A, being its dual group, is compact. Since H is compact, its dual $\hat{G}_1/A$ is discrete, and thus A is open.) We can therefore make the choice $A = H(\hat{G})$. This is consistent with the conventions in the above paragraph and with the inversion theorems for Fourier transforms.

Define $K = [0, 1]^a \times H$ and $L = [0, 1)^a \times H$. We can then write G as a disjoint union

$$G = \cup_{\alpha \in J} L_\alpha$$

where $L_\alpha = g_\alpha + L$ and each g_α is of the form $(n_1, \ldots, n_a, t)$ with $n_i \in Z$, $t \in G_1$, (the collection of t's being a transversal of H in G_1). It will sometimes be convenient to use the (nondisjoint) decomposition $G = \cup_{\alpha \in J} K_\alpha$, where $K_\alpha = g_\alpha + K$.

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Definition. If $f \in L^p(C)$ for any compact subset C of G and $1 \leqq p, q \leqq \infty$, define

$$||f||_{p,q} = \left[\sum_{\alpha \in J} \left[\int_{K_\alpha} |f|^p\right]^{q/p}\right]^{1/q}$$

$$||f||_{\infty,q} = \left[\sum_{\alpha \in J} \sup_{x \in K_\alpha} |f(x)|^q\right]^{1/q}$$

and let

$$(L^p, l^q) = \{f; \ \ ||f||_{p,q} < \infty\}, \ (C_0, l^q) = C_0 \cap (L^\infty, l^q)$$

where C_0 denotes the continuous functions on G which vanish at infinity.

This definition clearly reduces to Holland's when $G = R$. Other reasons for choosing to define (L^p, l^q) via the particular decomposition $G = \bigcup_\alpha K_\alpha$ will be seen in the proofs of Theorems 3.1, 3.3, and 4.3.

We begin our study of the amalgams (L^p, l^q) by listing some relations between them.

(3.1) $\quad (L^p, l^{q_1}) \subset (L^p, l^{q_2}) \quad$ if $q_1 \leqq q_2$

(3.2) $\quad (L^{p_2}, l^q) \subset (L^{p_1}, l^q) \quad$ if $p_1 \leqq p_2$

(3.3) $\quad (L^p, l^p) = L^p$

(3.4) $\quad (L^p, l^q) \subset L^p \cap L^q \quad$ if $q \leqq p$.

The last relation follows from the first two relations which are consequences of the following inequalities.

(3.5) $\quad ||f||_{p,q_2} \leqq ||f||_{p,q_1} \quad$ if $q_1 \leqq q_2$

(3.6) $\quad ||f||_{p_1,q} \leqq ||f||_{p_2,q} \quad$ if $p_1 \leqq p_2$.

The inequality (3.5) follows from Jensen's inequality while (3.6) is easily proved using Holder's inequality, remembering that the Haar measure of K_α is 1.

THEOREM 3.1. *Let C be a compact subset of G, and let $1 \leqq p, q \leqq \infty$. Then there is a function $g \in C_c(G)$ such that $g \equiv 1$ on C and $\hat{g} \in (L^p, l^q)$.*

Proof. Write $C \subset C_1 \times C_2$ where C_1 is compact in R^a and C_2 is compact in G_1. Then C_2 is covered by a finite number of cosets of H:

$$C_2 \subset \bigcup_{i=1}^k s_i + H = F, \text{ say.}$$

Let $g_2 = \chi_F$, the characteristic function of F. Then $g_2 \in C_c(G_1)$. Since $g_2(s) = \sum_{i=1}^k \chi_H(s - s_i)$, we have

$$\hat{g}_2(\hat{s}) = \sum_{i=1}^k \overline{[s_i, \hat{s}]}\hat{\chi}_H(\hat{s}) = \chi_A(\hat{s}) \sum_{i=1}^k \overline{[s_i, \hat{s}]}.$$

Since the support of $\hat{g}_2$ is A, we have $\hat{g}_2 \in (L^p, l^q)(\hat{G}_1)$. Let g_1 be a function which is equal to 1 on C_1, has compact support, and is an a-fold product of functions which are m times differentiable functions of a real variable (where m is to be chosen). Then $\hat{g}_1$ is an a-fold product of continuous functions which are $0(x^{-m})$ as $x \to \pm \infty$. Such functions are in $(L^p, l^q)(R)$ for sufficiently large m. (Specifically, $m > (p + q)p^{-2}$ if $p, q < \infty$ or $m > q^{-1}$ if $p = \infty$.) Therefore $\hat{g}_1 \in (L^p, l^q)(R^a)$. If g is defined on G by $g(s, t) = g_1(s)g_2(t)$, then g and $\hat{g}$ possess the desired properties.

THEOREM 3.2. *(L^p, l^q) is a Banach space and for $1 \leqq p, q < \infty$ its dual space is isometrically isomorphic to $(L^{p'}, l^{q'})$.*

The proof of this theorem is virtually identical with the proof of Theorem A given in [**7**].

THEOREM 3.3. *Translation is a bounded operator on (L^∞, l^1). Specifically, if $f_t(x) = f(x - t)$ and $f \in (L^\infty, l^1)$, then*

$$\| f_t\|_{\infty,1} \leqq 2^a\| f \|_{\infty,1}.$$

Proof.

$$\| f_t\|_{\infty,1} = \textstyle\sum_{\alpha \in J} \sup_{x \in K_\alpha} | f(x - t)| = \sum_\alpha \sup_{x \in t+K_\alpha} | f(x)|$$

where $K_\alpha = g_\alpha + K$ and $K = [0, 1]^a \times H$.

If $t = g_\alpha$ for some α, then clearly $\| f_t\|_{\infty,1} = \| f \|_{\infty,1}$. Otherwise $t + K_\alpha$ intersects the interiors of at most 2^a K_β's and so we can write

$$\| f_t\|_{\infty,1} = \textstyle\sum_{\alpha \in J} \sup_{x \in t+K_\alpha} | f(x)| \leqq 2^a \sum_{\beta \in J} \sup_{x \in K_\alpha} | f(x)| = 2^a\| f \|_{\infty,1}.$$

THEOREM 3.4. *Suppose $f \in L^r(G)$ and* supp $(f) \subset C$, *where C is compact and $1 \leqq r \leqq 2$. Then*

$$\| \hat{f} \|_{\infty,r'} \leqq M\| \hat{f} \|_{r'}$$

where M is a constant depending only on C and r.

Proof. By Theorem 3.1 there exists a real function $g \in C_c(G)$ with $g \equiv 1$ on C and $\hat{g} \in (L^\infty, l^1)$. Then

$$\hat{f}(\hat{x}) = \int_C f(x)[x, \hat{x}]dx = \int_G f(x)g(x)[x, \hat{x}]dx = \int_{\hat{G}} \hat{f}(\hat{t})\hat{g}(\hat{t} - \hat{x})d\hat{t}$$

by the Parseval formula. So Holder's inequality yields

$$|f(x)|^{r'} \leqq \int_{\hat{G}} |\hat{f}(\hat{t})|^{r'} |\hat{g}(\hat{t} - \hat{x})|dt \cdot \left[\int_{\hat{G}} |\hat{g}(\hat{t} - \hat{x})|d\hat{t}\right]^{r'/r}$$
$$= \int_{\hat{G}} |\hat{f}(\hat{t})|^{r'} |\hat{g}(\hat{x} - \hat{t})|d\hat{t}\left[\int_{\hat{G}} |\hat{g}(\hat{t})|d\hat{t}\right]^{r'/r}$$
$$= \int_{\hat{G}} |\hat{f}(\hat{t})|^{r'} |\hat{g}_{\hat{t}}(\hat{x})|d\hat{t} \cdot ||\hat{g}||_1{}^{r'/r}.$$

Thus

$$\|\hat{f}\|_{\infty,r'}^{r'} = \sum_{\alpha} \sup_{x \in K\alpha} |\hat{f}(\hat{x})|^{r'}$$

$$\leqq \sum_{\alpha} \sup_{x \in K_\alpha} \int_{\hat{G}} |\hat{f}(\hat{t})|^{r'} |\hat{g}_{\hat{t}}(\hat{x})| d\hat{t} \cdot \|\hat{g}\|_1^{r'/r}$$

$$= \int_{\hat{G}} |\hat{f}(\hat{t})|^{r'} \|\hat{g}_{\hat{t}}\|_{\infty,1} d\hat{t} \cdot \|\hat{g}\|_1^{r'/r} \leqq 2^a \|\hat{g}\|_{\infty,1} \|\hat{f}\|_{r'}^{r'} \|\hat{g}\|_1^{r'/r}$$

by Theorem 3.3. Therefore the inequality holds with

$$M = 2^{a/r'} \|\hat{g}\|_{\infty,1}^{1/r'} \|\hat{g}\|_1^{1/r}.$$

THEOREM 3.5. *If $f \in (L^p, l^1)$ where $1 \leqq p \leqq 2$, then $\hat{f} \in (L^\infty, l^{p'})$. There is a constant A_p such that*

$$\|\hat{f}\|_{\infty,p'} \leqq A_p \|f\|_{p,1} \quad (f \in (L^p, l^1)).$$

Proof. Write $f = \sum_{\alpha \in J} f_\alpha$, where $\text{supp}(f_\alpha) \subset K_\alpha$. This series converges in L^1 and so $\hat{f} = \sum \hat{f}_\alpha$ is uniformly convergent. Applying Theorem 3.4 to f_α with $C = K_\alpha$, we obtain

$$\|\hat{f}_\alpha\|_{\infty,p'} \leqq A_p \|\hat{f}_\alpha\|_{p'}.$$

(Note that A_p is independent of α because each K_α is a translate of $K = [0, 1]^a \times H$. The corresponding functions $g = g_\alpha$ in the proof of Theorem 3.4 can be chosen to be translates of each other, and so $|\hat{g}_\alpha|$ is independent of α.)

The Hausdorff-Young inequality [**6**, (31.21)] then gives

$$\|\hat{f}_\alpha\|_{\infty,p'} \leqq A_p \|f_\alpha\|_p.$$

Thus

$$\sum_\alpha \|\hat{f}_\alpha\|_{\infty,p'} \leqq A_p \|f\|_{p,1}.$$

Since $(L^\infty, l^{p'})$ is a Banach space, this shows that

$$\hat{f} \in (L^\infty, l^{p'}) \text{ and } \|\hat{f}\|_{\infty,p'} \leqq A_p \|f\|_{p,1}.$$

4. Unbounded measures. The word "measure" will mean a set function μ which is locally a complex measure, i.e., for each compact subset C of G, $\mu_C(E) = \mu(E \cap C)$ is a complex measure (in the usual sense of the word [**10**, Ch. 6]) on the Borel subsets of G. This is consistent with the point of view taken by Argabright and Gil de Lamadrid in discussing transformable measures [**1**] which is the continuous functional point of view of Bourbaki [**3**]. (The functional $\mu(f) = \int_G f d\mu \equiv \int_C f d\mu_C$, where $C = \text{supp}(f)$, is a continuous linear functional on $C_c(G)$ topologized as the inductive limit of the spaces $C(G, A) = \{f \in C_c(G); \text{supp}(f) \subset A\}$, A compact in G, i.e., for each compact A there is a constant M_A such that $|\mu(f)| \leqq M_A \|f\|_\infty$ for every $f \in C(G, A)$. Indeed $M_A = |\mu|(A)$.)

Before exhibiting a class of unbounded measures that are transformable, we state, for ease of reference, a Parseval formula for transforms of bounded measures.

LEMMA 4.1. (Extended Parseval Formula) *Suppose that* μ *is a bounded complex measure on* G *and let* $\check{\phi}$ *denote the inverse Fourier transform of* $\phi \in L^1(\hat{G})$: $\check{\phi}(x) = \int_{\hat{G}}[x, \hat{x}]\phi(\hat{x})d\hat{x}$, $x \in G$. *Then*

$$(4.1) \qquad \int_G \overline{\check{\phi}(x)}d\mu(x) = \int_{\hat{G}} \overline{\phi(\hat{x})}\hat{\mu}(\hat{x})d\hat{x}$$

holds whenever $\phi \in L^1(\hat{G})$ *and*

$$(4.2) \qquad \int_G \overline{\phi(x)}d\mu(x) = \int_{\hat{G}} \overline{\hat{\phi}(\hat{x})}\hat{\mu}(\hat{x})d\hat{x}$$

holds whenever $\phi \in L^1(G)$, $\hat{\phi} \in L^1(\hat{G})$, *and* ϕ *is continuous.*

Proof. (4.1) is a straightforward consequence of Fubini's Theorem and (4.2) follows from (4.1) and the Inversion Theorem.

Definition. Let $M_r = M_r(G)$ be the set of all measures μ on G such that

$$\|\mu\|_{M_r} = [\textstyle\sum_{\alpha \in J}[|\mu|(K_\alpha)]^r]^{1/r} < \infty.$$

THEOREM 4.2. *Let* $\mu \in M_r$, $1 \leqq r \leqq 2$. *Then*
(i) μ *is transformable (in the sense of* (1.4)),
(ii) $\hat{\mu}$ *is a function,* $\hat{\mu} \in (L^{r'}, l^\infty)$, *and there is a constant* A_r *such that*

$$\|\hat{\mu}\|_{r',\infty} \leqq A_r\|\mu\|_{M_r} \ (\mu \in M_r).$$

Proof. Let $\mathscr{V} = \{V_\alpha : \alpha \in I\}$ be the set of all finite unions of the sets K_β, $\beta \in J$. For each $\alpha \in I$, define a finite measure μ_α on G by

$$\mu_\alpha(E) = \mu(E \cap V_\alpha) \ (E \text{ a Borel set in } G)$$

and let

$$(4.3) \qquad T_\alpha(\phi) = \int_{\hat{G}} \overline{\phi(\hat{x})}\hat{\mu}_\alpha(\hat{x})d\hat{x} \quad (\phi \in (L^r, l^1)(\hat{G})).$$

The integral in (4.3) exists since $\hat{\mu}_\alpha \in L^\infty(\hat{G})$ and $(L^r, l^1) \subset L^1(\hat{G})$. Theorem 3.2 gives $\|T_\alpha\| = \|\hat{\mu}_\alpha\|_{r',\infty}$. Using (4.1) we also have

$$(4.4) \qquad T_\alpha(\phi) = \int_G \overline{\check{\phi}(x)}d\mu_\alpha(x) \quad (\phi \in (L^r, l^1)(\hat{G}))$$

and so

$$|T_\alpha(\phi) - T_\beta(\phi)| \leqq \sum_{n\in J} \int_{K_n} |\check{\phi}(x)| d|\mu_\alpha - \mu_\beta|(x)$$

$$\leqq \sum \sup_{x\in K_n} |\check{\phi}(x)| \cdot |\mu_\alpha - \mu_\beta|(K_n)$$

$$\leqq \left[\sum \sup_{x\in K_n} |\check{\phi}(x)|^{r'}\right]^{1/r'} \cdot \left[\sum (|\mu_\alpha - \mu_\beta|(K_n))^r\right]^{1/r}$$

$$= ||\check{\phi}||_{\infty,r'}\left[\sum (|\mu_\alpha - \mu_\beta|(K_n))^r\right]^{1/r} \leqq A_r||\phi||_{r,1}\left[\sum (|\mu_\alpha - \mu_\beta|(K_n))^r\right]^{1/r}$$

using Theorem 3.5. Thus

$$\|T_\alpha - T_\beta\| \leqq [\textstyle\sum_n (|\mu_\alpha - \mu_\beta|(K_n))^r]^{1/r}.$$

If $V_\alpha \supset V_\beta$, then $(\mu_\alpha - \mu_\beta)(K_n) = \mu(K_n)$ if $K_n \subset V_\alpha \backslash V_\beta$ and is 0 otherwise. Therefore

$$\|T_\alpha - T_\beta\| \leqq A_r[\textstyle\sum_{K_n \subset V_\alpha \backslash V_\beta} (|\mu|(K_n))^r]^{1/r}$$

which can be made arbitrarily small since $\mu \in M_r$. Hence $\|\hat{\mu}_\alpha - \hat{\mu}_\beta\|_{r',\infty} \to 0$ along $\mathscr{V}$.

Let $\hat{\mu} = \lim_\alpha \hat{\mu}_\alpha$ in $(L^{r'}, l^\infty)$. Then the above gives

$$\|\hat{\mu}\|_{r',\infty} \leqq A_r[\textstyle\sum_\alpha (|\mu|(K_\alpha))^r]^{1/r} = A_r\|\mu\|_{M_r}.$$

To prove (i) we must show that

$$\int_G \phi * \tilde{\phi}(x) d\mu(x) = \int_{\hat{G}} |\hat{\phi}(-\hat{x})|^2 \hat{\mu}(\hat{x}) d\hat{x} \quad (\phi \in C_c(G)),$$

or, equivalently,

$$\text{(4.5)} \quad \int_G \phi * \check{\phi}(x) d\mu(x) = \int_{\hat{G}} |\hat{\phi}(\hat{x})|^2 \hat{\mu}(\hat{x}) d\hat{x} \quad (\phi \in C_c(G)).$$

First let us check that the integral on the right side of (4.5) exists. Since $\mu \in (L^{r'}, l^\infty)$ it will exist if $|\hat{\phi}|^2 \in (L^r, l^1)$. Now $\phi \in C_c(G) \subset (L^p, l^q)$ for all $p, q \geqq 1$. We claim that $\hat{\phi} \in (L^{q'}, l^{p'})$. To prove this, write $\phi = \sum_\alpha \phi_\alpha$ as in the proof of Theorem 3.5. Then

$$\|\hat{\phi}_\alpha\|_{q',p'} \leqq \|\hat{\phi}_\alpha\|_{\infty,p'} \leqq A_p\|\phi_\alpha\|_p$$

and so

$$\textstyle\sum_\alpha \|\hat{\phi}_\alpha\|_{q',p'} \leqq A_p\|\phi\|_{p,1} < \infty.$$

Since $(L^{q'}, l^{p'})$ is a Banach space, this shows that $\hat{\phi} = \sum \hat{\phi}_\alpha \in (L^{q'}, l^{p'})$. In particular $\hat{\phi} \in (L^{2r}, l^2)$ and so

$$\sum_\alpha \left[\int_{K_\alpha} (|\hat{\phi}(\hat{x})|^2)^r d\hat{x}\right]^{1/r} = \sum_\alpha \left[\int_{K_\alpha} |\hat{\phi}(\hat{x})|^{2r} d\hat{x}\right]^{2/2r} < \infty,$$

i.e., $|\hat{\phi}|^2 \in (L^r, l^1)$.

To prove (4.5) we apply (4.2) to the bounded measure μ_α. This gives

$$(4.6) \qquad \int_G \overline{\phi * \tilde{\phi}(x)} d\mu_\alpha(x) = \int_{\hat{G}} |\tilde{\phi}(\hat{x})|^2 \hat{\mu}_\alpha(\hat{x}) d\hat{x} \quad (\phi \in C_c(G)).$$

It is clear from the definition of μ_α, and the fact that $\phi * \tilde{\phi}$ has compact support, that the left side of (4.6) converges to the left side of (4.5). The same is true of the right sides because $|\hat{\phi}|^2 \in (L^r, l^1)$ and $\|\hat{\mu}_\alpha - \hat{\mu}\|_{r',\infty} \to 0$. Therefore μ is transformable with Fourier transform $\hat{\mu}$.

Remark. It follows from (i) of Theorem 4.2 and a result of Argabright and Gil de Lamadrid [**1**] that if $\mu \in M_r$ then $\hat{\mu}$ is translation-bounded (see (1.5)). But an easier way of seeing this is to note that $\hat{\mu} \in (L^{r'}, l^\infty) \subset (L^1, l^\infty)$. It is not hard to see that the class of translation-bounded functions is precisely $(L^1, l^\infty) = \{f; \sup_\alpha \int_{K_\alpha} |f| < \infty\}$.

THEOREM 4.3. *If Φ is a continuous linear functional on (C_o, l^q), then there exists a measure $\mu \in M_{q'}$ such that*

$$(4.7) \qquad \Phi(f) = \int_G f d\mu \quad (f \in (C_0, l^q)).$$

Proof. Let $C_\alpha = C(K_\alpha)$, the continuous functions on K_α with the usual topology, and let $f_\alpha = f \mid K_\alpha$. Now (C_o, l^q) is isometrically isomorphic to a closed subspace S of $(\Pi_\alpha C_\alpha, l^q)$ via $f \to \{f_\alpha\}$, where $\{f_\alpha\} \in S$ if and only if $f_\alpha = f_\beta$ on $K_\alpha \cap K_\beta$. If Φ_α is a continuous linear functional on C_α, then the Riesz Representation Theorem gives a finite measure μ_α on K_α such that

$$\Phi_\alpha(g) = \int_{K_\alpha} g d\mu_\alpha \quad \text{whenever} \quad g \in C_\alpha.$$

If $\Phi \in (C_0, l^q)^* = (S, l^q)^*$, extend Φ by the Hahn-Banach Theorem to a functional Φ in $(\Pi_\alpha C_\alpha, l^q)^* = (\Pi_\alpha C_\alpha^*, l^q)$. There exist $\Phi_\alpha \in C_\alpha^*$ such that

$$\Phi(f) = \sum_\alpha \Phi_\alpha(f_\alpha) = \sum_\alpha \int_{K_\alpha} f_\alpha d\mu_\alpha \quad (f \in (C_0, l^q)).$$

Define μ by

$$\mu(E) = \textstyle\sum_\alpha \mu_\alpha(E \cap K_\alpha),$$

the domain of μ consisting of those Borel sets in G for which the series converges. Clearly $\mu_C(E) = \mu(E \cap C)$ is a complex measure whenever C is a compact subset of G.

We give a detailed proof of (4.7) for the case $a = 2$. For general a the proof is similar but there are 2^a groups of terms in the corresponding sums. For $a = 2$ we index $\alpha \in J$ as $\alpha = (m, n, \beta)$ where $m, n \in \mathbf{Z}$ and $\beta \in \mathbf{I}$. If $\{t_\beta; \beta \in I\}$ is a transversal of H in G_1, we can write

$$\begin{aligned}
K_\alpha = K_{mn\beta} &= \{(x, y, t); m \leqq x \leqq m + 1, n \leqq y \leqq n + 1, t \in t_\beta + H\} \\
L_\alpha = L_{mn\beta} &= \{(x, y, t); m \leqq x < m + 1, n \leqq y < n + 1, t \in t_\beta + H\} \\
V_{mn\beta} &= \{(m, y, t); n \leqq y < n + 1, t \in t_\beta + H\} \\
H_{mn\beta} &= \{(x, n, t); m \leqq x < m + 1, t \in t_\beta + H\} \\
P_{mn\beta} &= \{(m, n, t); t \in t_\beta + H\}
\end{aligned}$$

so that

$$K_{mn\beta} = L_{mn\beta} \cup V_{m+1,n,\beta} \cup H_{m,n+1,\beta} \cup P_{m+1,n+1,\beta}$$

is a disjoint union.

Now suppose that $E \subset L_{mn\beta}$ is a Borel set. Since $\mu(E) = \sum \mu_\alpha(E \cap K_\alpha)$ and the cosets $t_\beta + H$ are disjoint, we have

$$\mu(E) = \mu_{mn\beta}(E) + \mu_{m-1,n,\beta}(E \cap K_{m-1,n,\beta}) + \mu_{m,n-1,\beta}(E \cap K_{m,n-1,\beta}) + \mu_{m-1,n-1,\beta}(E \cap K_{m-1,n-1,\beta})$$

or

$$\mu(E) = \mu_{mn\beta}(E) + \mu_{m-1,n,\beta}(E \cap V_{mn\beta}) + \mu_{m,n-1,\beta}(E \cap H_{mn\beta}) + \mu_{m-1,n-1,\beta}(E \cap P_{mn\beta}) \tag{4.8}$$

whenever $E \subset L_{mn\beta}$. Therefore

$$\begin{aligned}
\Phi(f) &= \sum_{m,n,\beta} \int_{K_{mn\beta}} f_{mn\beta}\, d\mu_{mn\beta} = \sum \Big[\int_{L_{mn\beta}} f_{mn\beta} d\mu_{mn\beta} + \int_{V_{m+1,n,\beta}} f_{mn\beta} d\mu_{mn\beta} \\
&\quad + \int_{H_{m,n+1,\beta}} f_{mn\beta} d\mu_{mn\beta} + \int_{P_{m+1,n+1,\beta}} f_{mn\beta} d\mu_{mn\beta} \Big] \\
&= \sum \int_{L_{mn\beta}} f_{mn\beta} d\mu_{mn\beta} + \sum \int_{V_{mn\beta}} f_{m-1,n,\beta} d\mu_{m-1,n,\beta} \\
&\quad + \sum \int_{H_{mn\beta}} f_{m,n-1,\beta} d\mu_{m,n-1,\beta} + \sum \int_{P_{mn\beta}} f_{m-1,n-1,\beta} d\mu_{m-1,n-1,\beta} \\
&= \sum \Big[\int_{L_{mn\beta}} f_{mn\beta} d\mu_{mn\beta} + \int_{V_{mn\beta}} f_{mn\beta} d\mu_{m-1,n,\beta} \\
&\quad + \int_{H_{mn\beta}} f_{mn\beta} d\mu_{m,n-1,\beta} + \int_{P_{mn\beta}} f_{mn\beta} d\mu_{m-1,n-1,\beta} \Big]
\end{aligned}$$

(since $f_\alpha = f_\beta$ on $K_\alpha \cap K_\beta$)

$$= \sum \int_{L_{mn\beta}} f_{mn\beta} d\mu \quad (by\ 4.8)$$

$$= \int f d\mu.$$

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THEOREM 4.4. *Let $1 \leqq r \leqq 2$, and suppose there is a constant M such that*

$$(4.9) \qquad \left|\int_G f\phi\right| \leqq M\|\hat{\phi}\|_{\infty,r'}$$

whenever $\phi \in C_c$. Then f is a Fourier transform, i.e., there exists $\mu \in M_r(\hat{G})$ such that $f = \hat{\mu}$.

Proof. The inequality (4.9) shows that the linear functional

$$T(\hat{\phi}) = \int_G f\phi$$

is continuous on the subspace $\{\hat{\phi} \in (C_0, l^{r'}); \phi \in C_c\}$ of $(C_0, l^{r'})$. Use the Hahn-Banach Theorem to extend T to a continuous linear functional on $(C_0, l^{r'})$. Then Theorem 4.3 yields a measure $\mu \in M_r$ such that

$$(4.10) \qquad \int_G f\phi = \int_G \hat{\phi} d\mu \quad (\phi \in C_c(G), \hat{\phi} \in (C_0, l^{r'})).$$

Combining Theorem 3.5 with inequality (4.9) we get a constant B_r such that

$$\left|\int_G f\phi\right| \leqq B_r\|\phi\|_{r,1} \quad (\phi \in C_c, \hat{\phi} \in (C_0, l^{r'})).$$

This shows that the linear functional $F(\phi) = \int f\phi$ is continuous on a dense subspace of (L^r, l^1) and so $f \in (L^{r'}, l^\infty)$. Consequently (4.10) is valid whenever $\phi \in (L^r, l^1)$.

Given $\psi \in C_c$ we know that $|\check{\psi}|^2 \in (L^r, l^1)$ as in the proof of Theorem 4.2. Applying (4.10) with $\phi(x) = |\check{\psi}(x)|^2$ we obtain

$$\int_{\hat{G}} \overline{\psi * \tilde{\psi}(\hat{x})} d\mu(\hat{x}) = \int_G |\check{\psi}(x)|^2 f(x) dx$$

and this is, by definition, the statement that $f = \check{\mu}$.

REFERENCES

1. L. Argabright and J. Gil de Lamadrid, *Fourier analysis of unbounded measures on locally compact abelian groups*, Memoirs Amer. Math. Soc. *145*, Providence, R.I., 1974.

2. A. Benedek and R. Panzone, *The spaces L^p with mixed norm*, Duke Math. J. *28* (1961), 301–324.

3. N. Bourbaki, *Eléments de mathématique. Intégration*, Chaps. 1–4 (1952), Chap. 6 (Act. Sci. et Ind. 1175, 1281, Hermann, Paris, 1959).

4. J. L. B. Cooper, *Positive definite functions of a real variable*, Proc. London Math. Soc. (3) *10* (1960), 53–66.

5. W. F. Eberlein, *Characterizations of Fourier-Stieltjes transforms*, Duke Math. J. *22* (1955), 465–468.

6. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I (1963), Vol. II (1970) (Springer-Verlag, New York).

7. F. Holland, *Harmonic analysis on amalgams of L^p and l^q*, J. London Math. Soc. (2) *10* (1975), 295–305.

8. ——— *On the representation of functions as Fourier transforms of unbounded measures*, Proc. London Math. Soc. (3) *30* (1975), 347–365.

9. H. R. Pitt, *On Wiener's general harmonic analysis*, Proc. London Math. Soc. (2) *46* (1940), 1–18.

10. W. Rudin, *Real and complex analysis* (McGraw-Hill, New York, 1966).

11. I. J. Schoenberg, *A remark on a preceding note by Bochner*, Bull. Amer. Math. Soc. *40* (1934), 277–278.

12. J. Stewart, *Unbounded positive definite functions*, Can. J. Math. *21* (1969), 1309–1318.

13. N. Wiener, *On the representation of functions by trigonometric integrals*, Math. Z. *24* (1926), 575–616.

14. ——— *Tauberian theorems*, Ann. of Math. *33* (1932), 1–100.

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