The Asymptotic Expansions of the Spherical Harmonics

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§ 1. *Associated Legendre Functions as Integrals involving Bessel Functions.* Let

$$J \equiv \int_C e^{\lambda z} I_{-m} (\lambda \sqrt{z^2 - 1})\, \lambda^{-n-1}\, d\lambda,$$

where C denotes a contour which begins at $-\infty$ on the real axis, passes positively round the origin, and returns to $-\infty$, amp $\lambda = -\pi$ initially, and $R(z) > 0$, z being finite and $\neq 1$. [If $R(z) > 0$ and z is finite, then $R(z \pm \sqrt{z^2 - 1}) > 0$.] Then if $I_{-m} (\lambda \sqrt{z^2 - 1})$ be expanded in ascending powers of λ, and if the resulting expression be integrated term by term, it is found that

$$J = 2\pi i \frac{2^m z^{m+n} (z^2 - 1)^{-\frac{1}{2}m}}{\Pi(m+n)\,\Pi(-m)} F\left(\frac{-m-n}{2}, \frac{1-m-n}{2}, 1-m, 1-\frac{1}{z^2}\right)$$

$$= 2\pi i \frac{P_n^m(z)}{\Pi(m+n)}. *$$

From the equation

$$K_m(z) = \frac{\pi}{2\sin m\pi} \{I_{-m}(z) - I_m(z)\}$$

it follows that

$$\int_C e^{\lambda z} K_m (\lambda \sqrt{z^2 - 1})\, \lambda^{-n-1}\, d\lambda$$

$$= 2\pi i \frac{\pi}{2\sin m\pi} \left\{ \frac{P_n^m(z)}{\Pi(m+n)} - \frac{P_n^{-m}(z)}{\Pi(-m+n)} \right\}$$

$$= 2\pi i\; e^{-m\pi i} \frac{1}{\Pi(m+n)} Q_n^m(z)$$

$$= 2\pi i\; e^{m\pi i} \frac{1}{\Pi(n-m)} Q_n^{-m}(z).$$

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* *Cf.* Barnes, *Quart. Journ. of Math.*, Vol. 39, p. 120. The notation employed for the Associated Legendre Functions is that of Hobson, *Phil. Trans.*, Vol. 187, A.

Therefore, since $K_{-m} = K_m$,

$$Q_n^m(z) = \frac{1}{2\pi i} e^{m\pi i}\,\Pi(n+m)\int_C e^{\lambda z} K_m(\lambda\sqrt{z^2-1})\,\lambda^{-n-1}\,d\lambda\,, \quad \text{.........(i)}$$

where z is finite and $\neq 1$, and $R(z) > 0$.

If z passes once round the point $z = 1$ in the negative direction, we denote the resulting value of $Q_n^m(z)$ by $Q_n^m(z, +1-)$; then

$$Q_n^m(z, +1-) = \frac{1}{2\pi i} e^{m\pi i}\,\Pi(n+m)\int_C e^{\lambda z} K_m(\lambda e^{-i\pi}\sqrt{z^2-1})\,\lambda^{-n-1} d\lambda \quad \text{(ii)}$$

Again, if in the integral

$$J \equiv \int_C e^{\lambda z} I_{n+\frac{1}{2}}(\lambda)\,\lambda^{m-\frac{1}{2}}\,d\lambda\,,$$

where $R(z) > 1$, $I_{n+\frac{1}{2}}(\lambda)$ be expanded in ascending powers of λ, and if the resulting expression be integrated term by term, it is found that

$$J = \frac{2\pi i\, z^{-m-n-1}}{2^{n+\frac{1}{2}}\,\Pi(n+\frac{1}{2})\,\Pi(-m-n-1)} \times F\left(\frac{m+n+1}{2},\ \frac{m+n+2}{2},\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right)$$

$$= -2\pi i\sqrt{\left(\frac{2}{\pi}\right)}\frac{\sin(m+n)\pi}{\pi}\,e^{-m\pi i}(z^2-1)^{-\frac{1}{2}m}\,Q_n^m(z).$$

Therefore

$$\int_C e^{\lambda z} K_{n+\frac{1}{2}}(\lambda)\,\lambda^{-m-\frac{1}{2}} d\lambda = 2\pi i\sqrt{\left(\frac{\pi}{2}\right)}\frac{e^{m\pi i}(z^2-1)^{\frac{1}{2}m}}{\pi\cos n\pi}$$

$$\times\left\{\sin(-m+n)\pi\, Q_n^{-m}(z) - \sin(m+n)\pi\, Q_{-n-1}^{-m}(z)\right\}$$

$$= 2\pi i\sqrt{\left(\frac{\pi}{2}\right)}(z^2-1)^{\frac{1}{2}m}\,P_n^{-m}(z).$$

Thus

$$P_n^{-m}(z) = \frac{1}{2\pi i}\sqrt{\left(\frac{2}{\pi}\right)}(z^2-1)^{-\frac{1}{2}m}\int_C e^{\lambda z} K_{n+\frac{1}{2}}(\lambda)\,\lambda^{-m-\frac{1}{2}}\,d\lambda, \quad \text{......(iii)}$$

where $R(z) > 1$.

§ 2. *The Asymptotic Expansions for n large.* Now deform C into the contour K (Fig. 1), which has initial and final directions

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making angles $-\pi+\phi$ and $\pi-\phi$ with the positive real axis, where $0<\phi<\frac{1}{2}\pi$. Then if in (i) we expand $K_m(\lambda\sqrt{z^2-1})$ by means of its asymptotic expansion

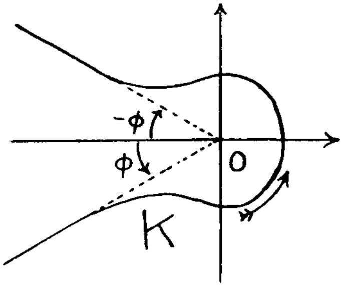

Fig. 1

$$\sqrt{\left(\frac{\pi}{2\lambda\sqrt{z^2-1}}\right)}\, e^{-\lambda\sqrt{z^2-1}} \left\{1+\frac{4m^2-1^2}{1\,!\,(8\lambda\sqrt{z^2-1})}+\frac{(4m^2-1^2)(4m^2-3^2)}{2\,!\,(8\lambda\sqrt{z^2-1})^2}\right.$$
$$\left. +\ldots+\frac{(4m^2-1^2)(4m^2-3^2)\ldots\{4m^2-(2s-3)^2\}}{(s-1)\,!\,(8\lambda\sqrt{z^2-1})^{s-1}}+R_s\right\},$$

where

$$R_s=\frac{1}{s\,!\,\Gamma(m+\frac{1}{2}-s)(2\lambda\sqrt{z^2-1})^s}$$
$$\times\int_0^\infty e^{-\zeta}\zeta^{m-\frac{1}{2}+s}\,d\zeta\int_0^1 s(1-t)^{s-1}\left(1+\frac{\zeta t}{2\lambda\sqrt{z^2-1}}\right)^{m-\frac{1}{2}-s}dt,$$

and integrate term by term, we find* that $Q_n^m(z)$ is equal to the first s terms in

$$e^{m\pi i}\sqrt{\left(\frac{\pi}{2\sqrt{z^2-1}}\right)}(z-\sqrt{z^2-1})^{n+\frac{1}{2}}\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}$$
$$\times F\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ n+\frac{3}{2},\ -\frac{z-\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right)\quad \ldots\ldots\ldots\text{(iv)}$$

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* The existence of a relation between the asymptotic expansions of the Bessel Functions and those of the Spherical Harmonics was suggested by Dr John Dougall, *Proc. of the Edin. Math. Soc.*, Vol. 18, p. 52.

plus a remainder

$$\rho_s = \frac{1}{2\pi i}\, e^{m\pi i} \sqrt{\left(\frac{\pi}{2\sqrt{z^2-1}}\right)}\, \Pi\,(n+m)\, \frac{1}{s\,!\,\Gamma\,(m+\frac{1}{2}-s)}$$

$$\times \frac{1}{(2\sqrt{z^2-1})^s} \int_K e^{\lambda\,(z-\sqrt{z^2-1})}\, \lambda^{-n-\frac{3}{2}-s}\, d\lambda$$

$$\times \int_0^\infty e^{-\zeta}\, \zeta^{m-\frac{1}{2}+s}\, d\zeta \int_0^1 s\,(1-t)^{s-1} \left(1+\frac{\zeta t}{2\lambda\sqrt{z^2-1}}\right)^{m-\frac{1}{2}-s} dt, \quad \ldots\text{(v)}$$

where we assume that $-\frac{1}{2}\pi < \text{amp}\, \sqrt{z^2-1} < \frac{1}{2}\pi$ and make $\text{amp}\,\zeta = \text{amp}\, \sqrt{z^2-1}$ so that

$$\text{amp}\left(\frac{\zeta}{\sqrt{z^2-1}}\right) = 0\,;$$

thus the singularity of the λ-integrand in (v) lies on the negative real axis.

Note.—If m is half an odd integer, series (iv) terminates.

In (v) change the order of integration; thus we obtain

$$\rho_s = \frac{1}{2\pi i}\, e^{m\pi i} \sqrt{\left(\frac{1}{2\sqrt{z^2-1}}\right)}\, \Pi\,(n+m)\, \frac{1}{s\,!\,\Gamma\,(m+\frac{1}{2}-s)\,(2\sqrt{z^2-1})^s}$$

$$\times \int_0^\infty e^{-\zeta}\, \zeta^{m-\frac{1}{2}+s}\, d\zeta \int_0^1 s\,(1-t)^{s-1}\, dt I, \quad \ldots\ldots\ldots\ldots\text{(vi)}$$

where

$$I \equiv \int_K e^{\lambda\,(z-\sqrt{z^2-1})}\, \lambda^{-n-\frac{3}{2}-s} \left(1+\frac{\zeta t}{2\lambda\sqrt{z^2-1}}\right)^{m-\frac{1}{2}-s} d\lambda. \quad \ldots\ldots\text{(vii)}$$

Now for any finite value of ζ we can deform the contour K so that

$$\left|\frac{\zeta t}{2\lambda\sqrt{z^2-1}}\right| < 1\,;$$

if then the last term of the integrand in (vii) be expanded in descending powers of λ, and if the resulting expression be integrated term by term, we have

$$I = (z-\sqrt{z^2-1})^{n+\frac{1}{2}+s}\; 2\pi i\, \frac{1}{\Pi\,(n+\frac{1}{2}+s)}$$

$$\times F\,(-m+\tfrac{1}{2}+s\,;\; n+\tfrac{3}{2}+s\,;\; -\zeta t v),$$

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where $v = \dfrac{z - \sqrt{z^2-1}}{2\sqrt{z^2-1}}$ and

$$F(\alpha\,;\ \gamma\,;\ z) \equiv 1 + \frac{\alpha}{\gamma\,.\,1!}z + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)\,.\,2!}z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)\,.\,3!}z^3 + \ldots$$

If s be taken so large that $R(-m+\tfrac{1}{2}+s) > -1$ and $R(n)$ so large that $R(n+m) > -1$, then

$$I = (z - \sqrt{z^2-1})^{n+\frac{1}{2}+s}\, 2\pi i \frac{1}{\Pi(n+\frac{1}{2}+s)} \times \int_0^1 \frac{e^{-\lambda\zeta t v}\,\lambda^{-m-\frac{1}{2}+s}(1-\lambda)^{n+m}\,d\lambda}{B(n+m+1,\ -m+\frac{1}{2}+s)}$$

as can easily be seen by expanding the exponential term in powers of λ and integrating term by term. For the sake of clearness we may assume for the time being that z is real and >1 and that amp $\sqrt{z^2-1} = 0$. Then, substituting this value of I in (vi), and changing the order of integration, we find that

$$\rho_s = T_{s+1} \frac{\int_0^1 s(1-t)^{s-1}\,dt \int_0^1 \lambda^{-m-\frac{1}{2}+s}(1-\lambda)^{n+m}\,J d\lambda}{\Gamma(m+\frac{1}{2}+s)\,B(n+m+1,\ -m+\frac{1}{2}+s)},$$

where

$$T_{s+1} \equiv e^{m\pi i}\sqrt{\left(\frac{\pi}{2\sqrt{z^2-1}}\right)}(z-\sqrt{z^2-1})^{n+\frac{1}{2}} \times \frac{\Pi(n+m)\,\Gamma(m+\frac{1}{2}+s)}{\Pi(n+\frac{1}{2}+s)\,s!\,\Gamma(m+\frac{1}{2}-s)}\,.\,v^s$$

is the $(s+1)^{\text{th}}$ term in the expansion (iv) of $Q_n^m(z)$ and

$$J \equiv \int_0^\infty e^{-\zeta(1+\lambda t v)}\,\zeta^{m\ \ \frac{1}{2}+s}\,d\zeta = \frac{\Gamma(m+\frac{1}{2}+s)}{(1+\lambda t v)^{m+\frac{1}{2}+s}}.$$

Thus

$$\rho_s = T_{s+1}\frac{\int_0^1 s(1-t)^{s-1}\,dt \int_0^1 \frac{\lambda^{-m-\frac{1}{2}+s}(1-\lambda)^{n+m}}{(1+\lambda t v)^{m+\frac{1}{2}+s}}d\lambda}{B(n+m+1,\ -m+\frac{1}{2}+s)}\,. \quad \ldots\text{(viii)}$$

From this value of ρ_s we see that the expansion (iv) and (viii) for $Q_n^m(z)$ is valid in every part of the z-plane to which z can

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approach by a path at no point of which $1+\lambda tv$ has the value zero. But this expression can only be zero (since $0 \leqq \lambda t \leqq 1$) when v is real and $\leqq -1$; *i.e.* when $\dfrac{z}{\sqrt{z^2-1}}$ is real and $\leqq -1$.

Let $u \equiv \dfrac{z}{\sqrt{z^2-1}}$; then $z = \dfrac{\pm u}{\sqrt{u^2-1}}$. Thus if u is real and >1 or <-1, z is real, while if u is real and $-1<u<1$, z is purely imaginary. Conversely, if z is real and $|z|>1$, u is real, while if z is real and $|z|<1$, u is purely imaginary. It follows that, when z is complex, u must be complex, and vice versa. Therefore the expression for $Q_n^m(z)$ is valid at any point to which z can approach without passing the real axis to the right of $z=1$ or to the left of $z=-1$ with values of z and $\sqrt{z^2-1}$ which make $z/\sqrt{z^2-1}$ negative. In particular, if a cross-cut is taken along the real axis from -1 to $-\infty$, the expression is valid for $-2\pi<\text{amp}\,(z-1)<2\pi$, $z \neq 1$. The corresponding expression for $Q_n^m(z, +1-)$ will be valid for $0<\text{amp}\,(z-1)<4\pi$.

Now let M be the maximum value of $|(1+\lambda tv)^{-m-\frac{1}{2}-s}|$ for $0 \leqq \lambda t \leqq 1$, then if $n=\alpha+i\beta$, $m=\sigma+i\tau$,

$$\left|\frac{\rho_s}{T_{s+1}}\right| \leqq M \frac{\int_0^1 s(1-t)^{s-1}\,dt \int_0^1 \lambda^{-\sigma-\frac{1}{2}+s}(1-\lambda)^{\alpha+\sigma}\,d\lambda}{|B(n+m+1,\ -m+\frac{1}{2}+s)|}$$

$$= M \frac{B(\alpha+\sigma+1,\ -\sigma+\frac{1}{2}+s)}{|B(n+m+1,\ -m+\frac{1}{2}+s)|}.$$

But, if $|\text{amp}\,\kappa|<\pi$ and $|\text{amp}\,(\kappa+\delta)|<\pi$,

$$\Gamma(\kappa+\delta)/\Gamma(\kappa) \rightarrow \kappa^{\delta}$$

as $\kappa \rightarrow \infty$. Hence, if $\chi = \text{amp}\,n$,

$$\left|\frac{\Gamma(n+\frac{3}{2}+s)}{\Gamma(n+m+1)}\right| \times \frac{\Gamma(\alpha+\sigma+1)}{\Gamma(\alpha+\frac{3}{2}+s)} \rightarrow \left(\frac{|n|}{\alpha}\right)^{\frac{1}{2}+s-\sigma} e^{\chi\tau}$$

as $n \rightarrow \infty$, provided that $|\text{amp}\,n| = |\chi| < \frac{1}{2}\pi$. But $|n|/\alpha = \sec\chi$ is then finite; therefore $\rho_s = T_{s+1} \times$ *a quantity which remains finite as* $n \rightarrow \infty$, provided that $|\text{amp}\,n| < \frac{1}{2}\pi$.

It follows that, in those parts of the z-plane in which the expression for $Q_n^m(z)$ is valid, but where ρ_s does not $\rightarrow 0$ as $s \rightarrow \infty$, the series in (iv) is asymptotic for large values of n, since the $(s+1)^{\text{th}}$ term in the series can be made arbitrarily small by increasing n, provided $|\text{amp}\, n| < \frac{1}{2}\pi$.

If m, n, and z are real, z positive, $z/\sqrt{z^2-1}$ positive, and $0 < v < 1$, $\{i.e.$ if $z > 3/(2\sqrt{2})\}$, then $1 + \lambda t v \geqq 1$, so that, from (viii),

$$\frac{\rho_s}{T_{s+1}} < \frac{\int_0^1 s(1-t)^{s-1}\, dt \int_0^1 \lambda^{-m-\frac{1}{2}+s}(1-\lambda)^{n+m}\, d\lambda}{B(n+m+1,\; -m+\frac{1}{2}+s)} = 1.$$

Since $v < 1$, T_{s+1} and therefore ρ_s can be made arbitrarily small by increasing s, so that the series (iv) converges to the value $Q_n^m(z)$. By the theory of analytical continuation it follows that this is always true when the expression for $Q_n^m(z)$ is valid, provided that $|v| < 1$. For other parts of the region of validity the series, while not convergent, is asymptotic in n.

In the corresponding region of validity the asymptotic series for $Q_n^m(z, +1-)$ is

$$e^{(m+\frac{1}{2})\pi i} \sqrt{\left(\frac{\pi}{2\sqrt{z^2-1}}\right)} (z + \sqrt{z^2-1})^{n+\frac{1}{2}} \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}$$

$$\times F\left(\tfrac{1}{2}-m,\; \tfrac{1}{2}+m,\; n+\tfrac{3}{2},\; \frac{z+\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right). \quad \ldots\ldots\ldots\text{(ix)}$$

Finally, from the formmla

$$e^{\frac{1}{2}m\pi i} P_n^m(z) = \frac{i}{\pi} e^{-\frac{3}{2}m\pi i} Q_n^m(z) - \frac{i}{\pi} e^{-\frac{1}{2}m\pi i} Q_n^m(z, +1-),$$

it follows that the expansion

$$P_n^m(z) = e^{-\frac{1}{2}m\pi i} \frac{e^{i\pi/4}}{\sqrt{(2\pi\sqrt{z^2-1})}} \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}$$

$$\times \left[\begin{array}{l} e^{-\frac{1}{2}m\pi i + i\pi/4} (z-\sqrt{z^2-1})^{n+\frac{1}{2}} F\left(\frac{1}{2}-m,\; \frac{1}{2}+m,\; n+\frac{3}{2},\; -\dfrac{z-\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right) \\ + e^{\frac{1}{2}m\pi i - i\pi/4} (z+\sqrt{z^2-1})^{n+\frac{1}{2}} F\left(\frac{1}{2}-m,\; \frac{1}{2}+m,\; n+\frac{3}{2},\; \dfrac{z+\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right) \end{array}\right] \quad \text{(x)}$$

holds asymptotically at those parts of the z-plane in which the expressions for $Q_n^m(z)$ and $Q_n^m(z, +1-)$ are both valid, and where both series are not convergent. In particular, if $R(z) > 0$, the series (x) holds asymptotically if $0 < \text{amp}(z-1) < 2\pi$.

The expansion (x) was obtained by Hobson * for z, m, and n real: asymptotic expansions in terms of descending powers of n when z, m, n, are complex have been obtained by Barnes † and Watson. ‡

When $R(n)$ is negative, an asymptotic expansion can be obtained by means of the formula $P_n^m(z) = P_{-n-1}^m(z)$.

To distinguish between those regions of the z-plane in which the series (iv) is convergent and those in which it is asymptotic we can proceed as follows. At the boundary of the two regions

$$1 \pm \frac{z}{\sqrt{z^2-1}} = 2e^{i\theta},$$

where θ is real: thus

$$z^2 = 1 + e^{-\frac{3}{2}i\theta}/(8i\sin\tfrac{1}{2}\theta),$$

so that

$$8(x^2-y^2) = 5 + 4\sin^2\tfrac{1}{2}\theta$$

$$16xy = -\cot\tfrac{1}{2}\theta\,.\,(1-4\sin^2\tfrac{1}{2}\theta).$$

The elimination of θ between these two equations leads to the equation

$$128(x^2-y^2)(x^2+y^2)^2 - 320x^2y^2 - 336(x^2-y^2)^2 + 288(x^2-y^2) - 81 = 0,$$

from which it follows that the lines $x = \pm y$ are asymptotes.

If $y=0$, then $x = \pm\sqrt{3}/2$, $\pm\sqrt{3}/2$, $\pm 3\sqrt{2}/4$. At each of the points $(\pm\sqrt{3}/2, 0)$ there are two tangents which make angles $\pm 60°$ with the real axis. The curve is shown in Fig. 2.

The quantities $\left|\dfrac{z \pm \sqrt{z^2-1}}{2\sqrt{z^2-1}}\right|$ remain either <1 or >1 unless z crosses the curve. In the region between the two branches of the

* *Phil. Trans.*, 187 A. (1896), pp. 485–489.

† *Quart. Journ. of Maths.*, XXXIX (1908), pp. 143–174.

‡ *Proc. Camb. Phil. Soc.*, XXII (1918), pp. 277–308.

curve they are both <1. In the regions on the right and left $\left|\dfrac{z-\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right|<1$ and $\left|\dfrac{z+\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right|>1$. Within the loops both quantities are >1.

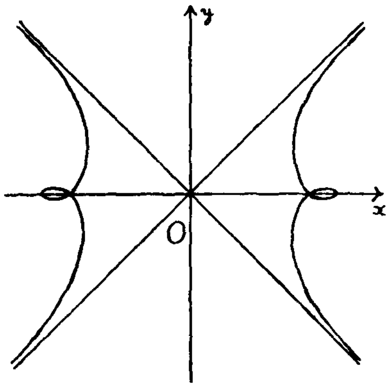

Fig. 2

§ 3. *The Asymptotic Expansions for m large.* In (iii) expand $K_{n+\frac{1}{2}}(\lambda)$ asymptotically and integrate term by term: thus we find that $P_n^{-m}(z)$ is equal to the first s terms in

$$\frac{1}{\Pi(m)}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, m+1,\, \frac{1-z}{2}\right) \quad \text{.........(xi)}$$

plus a remainder ρ_s which can, as in the previous section, be put in the form

$$T_{s+1}\,\frac{\displaystyle\int_0^1 s\,(1-t)^{s-1}\,dt \int_0^1 \lambda^{s-n-1}\,(1-\lambda)^{m+n}\left(1+\lambda t\,\frac{z-1}{2}\right)^{-n-s-1} d\lambda}{B\,(m+n+1,\, s-n)},$$

where T_{s+1} is the $(s+1)^{\text{th}}$ term in (xi). This expression is valid so long as $(z-1)/2$ is not real and <-1; *i.e.* provided that z is not real and <-1.

As in §2 we can show that, if $|z-1|<2$, the series (xi) is a convergent expansion for $P_n^{-m}(z)$, while for points outside the circle the expansion is asymptotic in m, provided that $|\operatorname{amp} m|<\frac{1}{2}\pi$.

The corresponding expansion for $Q_n^{-m}(z)$ can be deduced from this.

§ 4. *Asymptotic Expansions of the Hypergeometric Function.* In the integral

$$\int_K e^{\lambda(z+\frac{1}{2})}\, W_{k,m}(\lambda)\, \lambda^{\frac{\alpha+\beta-2\gamma-1}{2}}\, d\lambda, \qquad \text{(xii)}$$

where $k=(1-\alpha-\beta)/2$, $m=(\alpha-\beta)/2$, expand the confluent Hypergeometric Function $W_{k,m}(\lambda)$ in terms of its asymptotic expansion,* and integrate term by term. It is found that the integral is equal to the first s terms in the expansion of

$$\frac{2\pi i}{\Gamma(\lambda)}\, z^{\gamma-1}\, F(\alpha, \beta, \gamma, z) \qquad \text{(xiii)}$$

plus a remainder which can be put in the form

$$\rho_s = T_{s+1}\, \frac{\int_0^1 s\,(1-t)^{s-1}\,dt \int_0^1 \dfrac{\lambda^{\beta+s-1}(1-\lambda)^{\gamma-\beta-1}}{(1-\lambda t z)^{\alpha+s}}\,d\lambda}{B(\beta+s, \gamma-\beta)},$$

where T_{s+1} is the $(s+1)^{\text{th}}$ term of (xiii)

It can then be shown that, if $|z|<1$, the integral (xii) is equal to the expression (xiii), while, for any other point in the region bounded by the real axis from $+1$ to $+\infty$, the integral, and therefore the hypergeometric series, is asymptotic in γ, provided that $|\operatorname{amp}\gamma| < \frac{1}{2}\pi$.

Elementary Proof of the Asymptotic Expansion of the Hypergeometric Function. These results can also be obtained from the formula

$$B(\beta, \gamma-\beta)\, F(\alpha, \beta, \gamma, z) = \int_0^1 \zeta^{\beta-1}(1-\zeta)^{\gamma-\beta-1}(1-z\zeta)^{-\alpha}\, d\zeta. \qquad \text{(xiv)}$$

The expression $(1-z\zeta)^{-\alpha}$ is expanded by the Binomial Theorem in the form †

$$1+\frac{\alpha}{1\,!}z\zeta+\frac{\alpha(\alpha+1)}{2\,!}(z\zeta)^2+\ldots+\frac{\alpha(\alpha+1)\ldots(\alpha+s-2)}{(s-1)\,!}(z\zeta)^{s-1}$$
$$+\frac{\alpha(\alpha+1)\ \ (\alpha+s-1)}{s\,!}(z\zeta)^s\int_0^1 s(1-t)^{s-1}(1-tz\zeta)^{-\alpha-s}\,dt$$

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* *Cf.* Whittaker and Watson's Analysis, Chap. XVI.

† *Cf.* Prof. G. A. Gibson, *Proc. Edin. Math. Soc.*, Vol. XXXVIII.

and integrated term by term. The proof that the series so obtained is asymptotic is exactly the same as above. The expansions for the Spherical Harmonics are particular cases of this expansion.

By employing the alternative forms of the hypergeometric function and the expressions for the analytical continuations of the hypergeometric function it is possible to deduce various other asymptotic expansions for the hypergeometric function.

§ 5. *Expansion of a function in a series of Legendre Polynomials.* Laurent * gave the following proof of the validity of the expansion

$$f(x) = \sum_0^\infty A_n P_n(x), \qquad \text{...............(xv)}$$

where
$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f(\mu) P_n(\mu)\, d\mu.$$

Let Σ_n denote the first $n+1$ terms of the series; then, by Christoffel's Formula,

$$\Sigma_n = \frac{n+1}{2}\int_{-1}^{1} f(\mu)\frac{P_{n+1}(x)P_n(\mu) - P_n(x)P_{n+1}(\mu)}{x-\mu}\, d\mu.$$

In this expression substitute the asymptotic expansions for the Legendre Polynomials, and put $x=\cos\theta$, $\mu=\cos\phi$. It can then be shown that

$$\Sigma_n = \frac{1}{\pi}\int_0^\pi f(\cos\phi)\, V_n \sin\phi d\phi + \frac{P}{n}, \qquad \text{.........(xvi)}$$

where

$$V_n = \frac{\sin\{(n+1)(\phi-\theta)\}\sin\tfrac{1}{2}(\phi+\theta) - \sin\tfrac{1}{2}(\phi-\theta)\cos\{(n+1)(\phi+\theta)\}}{\sqrt{(\sin\theta\sin\phi)}\,.\,2\sin\tfrac{1}{2}(\phi-\theta)\sin\tfrac{1}{2}(\phi+\theta)}$$

and P remains finite when $n \to \infty$. Hence, by the theory of Dirichlet Integrals, when $n \to \infty$, the series (xv) is equal to

$$\tfrac{1}{2}\{f(\cos\overline{\theta+0}) + f\cos\overline{\theta-0})\} \text{ or } \tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

provided $-1<x<1$.

It has been pointed out that this proof, as it stands, is invalid, because the asymptotic expansions do not hold when $\phi=0$ or π. This difficulty can, however, be removed as follows.

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* *Jour. de Math.* (3) 1. 1875, p. 394.

From (xiv) we have, when n is zero or a positive integer,

$$F(\tfrac{1}{2}, \tfrac{1}{2}, n+\tfrac{3}{2}, \tfrac{1}{2}+iy) = \frac{\int_0^1 \zeta^{-\frac{1}{2}}(1-\zeta)^n \{1-\zeta(\tfrac{1}{2}+iy)\}^{-\frac{1}{2}}\, d\zeta}{B(\tfrac{1}{2}, n+1)}$$

But $\quad | 1-\zeta(\tfrac{1}{2}+iy) | \geqq \tfrac{1}{2}, \ (0 \leqq \zeta \leqq 1);$

therefore $\quad | \{1-\zeta(\tfrac{1}{2}+iy)\}^{-\frac{1}{2}} | \leqq \sqrt{2},$

so that $\quad | F(\tfrac{1}{2}, \tfrac{1}{2}, n+\tfrac{3}{2}, \tfrac{1}{2}+iy) | < \sqrt{2}.$

For real values of z between -1 and $+1$ the arguments of the hypergeometric functions in (x) are of the form $\tfrac{1}{2}+iy$; therefore, since $P_n(\cos\phi)\sqrt{(2\pi\sin\phi)}$ is continuous at $\phi=0$ and $\phi=\pi$

$$| P_n(\cos\phi)\sqrt{(2\pi\sin\phi)} | \leqq \frac{\Pi(n)}{\Pi(n+\tfrac{1}{2})} \times 2\sqrt{2}, \ 0 \leqq \phi \leqq \pi.$$

If now ϵ be chosen so small that $\epsilon < \theta < \pi - \epsilon$, it follows that, with suitable restrictions on $f(x)$ between -1 and $+1$,

$$\frac{n+1}{2} f(\cos\phi) \frac{P_{n+1}(\cos\theta)P_n(\cos\phi) - P_n(\cos\theta)P_{n+1}(\cos\phi)}{\cos\theta - \cos\phi} \sin\phi$$

remains finite for $0 \leqq \phi \leqq \epsilon$ and for $\pi - \epsilon \leqq \phi \leqq \pi$ as $n \to \infty$: thus the integrals between these limits can be made arbitrarily small by decreasing ϵ, and, when $n \to \infty$, the remaining part of (xvi) gives the required result.