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The Chebotarov theorem for Galois coverings of Axiom A flows

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Abstract. We consider G (Galois) coverings of Axiom *A* flows (restricted to basic sets) and prove an analogue of Chebotarev's theorem. The theorem provides an asymptotic formula for the number of closed orbits whose Frobenius class is a given conjugacy class in G. An application answers a question raised by *J.* Plante. The basic method is then extended to compact group extensions and applied to frame bundle flows defined on manifolds of variable negative curvature.

0. *Introduction*

In recent years Hejhal [**8**], Huber [**9**] and Sarnak [**19**] have derived very precise results in the asymptotic analysis of closed orbits for geodesic flows on surfaces of constant negative curvature. The techniques they used were originally developed by Selberg in [**20**]. Similar results for geodesic flows on manifolds of (variable) negative curvature were announced by Margulis in [**11**] which were achieved, it seems, by a rather different approach. Further progress was made by Gangolli [**7**] who extended Selberg's analysis to symmetric spaces of rank one. Bowen approached these problems for the more general case of Axiom A flows via his own work on symbolic dynamics. Bowen obtained asymptotic formulae of the Chebychev rather than the Hadamard-de la Valee-Poussin kind [**2**]. He also extended a combinatorial result of Manning, for diffeomorphisms, which is central to a more delicate analysis.

This paper, like [**12**], [**13**], [**14**], [**16**] belongs to the Bowen tradition. It should also be made clear that Ruelle's book [**18**] has played an essential part in our recent work on the asymptotic analysis of closed orbits of Axion A flows.

Unfortunately, there is a forfeit to be paid for the generality of our results (in [**13**], [**14**] and in this paper). We are unable to obtain the orders of approximation exhibited in [**8**], [**19**] etc. It is our hope that further study of the relevant zeta functions will yield a deeper understanding of this problem.

Here we are concerned with proving an analogue of Chebotarov's theorem (theorem 3) for Galois coverings of Axiom A flows. This theorem has already been proved by Sarnak [**19**] and Sunada [**21**] for the special case of geodesic flows on compact locally symmetric manifolds of negative curvature.

It should be remarked that the first author has given a different interpretation of Dirichlet's theorem (which is usually regarded as a specialisation of Chebotarov's theorem) in terms of the spatial equidistribution of closed orbits. This work appears in [**13**].

In § 7 we provide an application of our main result to homology and prove a conjecture of J. Plante's to the effect that an Anosov flow with non-wandering set the entire manifold has the property that its closed orbits generate the first homology group.† This application was pointed out to us by Sunada to whom we are grateful.

The final sections are concerned with compact, as opposed to finite, group extensions of Axiom A flows. We show that the earlier Chebotarov distribution theorem has an analogue in this setting and we apply our results to frame bundle flows associated with oriented manifolds of negative sectional curvature. In particular we show that when the frame bundle flow is topologically mixing, holonomy classes associated with closed geodesics are uniformly distributed in $SO(d-1)$ where d is the dimension.

1. *Galois coverings and Axiom A flows*

Let M be a compact Riemannian manifold and let $\phi_t: M \to M$ ($t \in \mathbb{R}$) be a C^1 flow. A compact invariant set Ω containing no fixed points is called *hyperbolic* if the tangent bundle restricted to Ω can be written as the Whitney sum of three $D\phi_t$-invariant continuous sub-bundles

$$T_\Omega M = E + E^s + E^u,$$

where E is the one-dimensional bundle tangent to the flow and where (for constants $C, \lambda > 0$),

(a) $\|D\phi_t(v)\| \le Ce^{-\lambda t}\|v\|$ for $v \in E^s$, $t \ge 0$

(b) $\|D\phi_{-t}(v)\| \le Ce^{-\lambda t}\|v\|$ for $v \in E^u$, $t \ge 0$.

A hyperbolic set Ω is called *basic* if the periodic orbits of $\phi_t|_\Omega$ are dense in Ω, $\phi_t|_\Omega$ is a topologically transitive flow, and there is an open set $U \supset \Omega$ with $\Omega = \bigcap_{t\in\mathbb{R}} \phi_t U$.

We shall always take a basic set Ω to be non-trivial i.e. Ω is not a topological circle.

Let $\tilde{\phi}_t: \tilde{M} \to \tilde{M}$ be an Axiom A flow and assume that G is a finite group of diffeomorphisms of $\tilde{M}$ which act freely. Furthermore, assume that $\tilde{\phi}_t g = g\tilde{\phi}_t$ for all $t \in \mathbb{R}$, $g \in G$. Then we may define a flow ϕ on the quotient manifold $M = \tilde{M}/G$ by $\phi_t(Gx) = G(\tilde{\phi}_t x)$. Let $\pi_G: \tilde{M} \to M$ be the covering map, then we can write $\phi_t \pi_G = \pi_G \tilde{\phi}_t$. One can show that ϕ is also an Axiom A flow.

We shall be concerned with a basic set $\tilde{\Omega} \subset \tilde{M}$ which is G-invariant. It then follows that $\Omega = \tilde{\Omega}/G$ is a basic set for the flow ϕ_t. In future we shall understand $\tilde{\phi}_t$, ϕ_t as the flows restricted to $\tilde{\Omega}$, Ω respectively.

We call $\tilde{\phi}_t$ a regular or *Galois covering* of ϕ_t with *covering transformations G.* In particular, we refer to $\tilde{\phi}$ as a *G-covering* of ϕ_t.

2. *Zeta functions and L-functions*

Let $\tilde{\phi}$ be a G-covering of ϕ and let h be the topological entropy of ϕ. The zeta function $\zeta(s)$ for ϕ is defined by

$$\zeta(s) = \prod_\tau (1 - N(\tau)^{-s})^{-1},$$

† Since writing this paper we have noticed that Fried obtains the same result at least for the case where there is a global cross-section consisting of a closed submanifold. (c.f. *Comment Math. Helvetic* **57** (1982), 237–259.

where this Euler product is over closed ϕ orbits τ of least period $\lambda(\tau)$ and $N(\tau)=e^{\lambda(\tau)h}$. This product converges for $\mathcal{R}(s)>1$ [**14**].

For any closed ϕ-orbit τ let $\tilde{\tau}_1,\ldots,\tilde{\tau}_n$ be the closed $\tilde{\phi}$-orbits satisfying $\pi_G\tilde{\tau}_i=\tau$. Since G is Galois, $n\,||G|$ and if $n\cdot l=|G|$ then $\lambda(\tilde{\tau}_i)=l\lambda(\tau)$, $1\le i\le n$. Choose $x\in\tilde{\tau}_i$ then $\pi_G x=\pi_G\tilde{\phi}_{\lambda(\tau)}x$ and so there exists a unique $\gamma(\tilde{\tau}_i)\in G$ such that $\gamma(\tilde{\tau}_i)x=\tilde{\phi}_{\lambda(\tau)}x$. The *Frobenius element* $\gamma(\tilde{\tau}_i)$ is independent of the choice of $x\in\tilde{\tau}_i$. If $g\tilde{\tau}_i=\tilde{\tau}_j$ then $\gamma(\tilde{\tau}_j)=g\gamma(\tilde{\tau}_i)g^{-1}$ so that the Frobenius class, the conjugacy class of $\gamma(\tilde{\tau}_i)$, is well defined.

Let R_χ be an irreducible representation with irreducible character $\chi=\text{trace } R_\chi$. R_χ is unique (up to equivalence). Let d_χ be the degree of the representation R_χ. The regular representation of G can be written

$$R=\sum_\chi \oplus d_\chi R_\chi \qquad \Big(|G|=\sum_\chi d_\chi^2\Big).$$

If R_χ is a representation of G with character χ we define (up to equivalence) $R_\chi(\tau)=R_\chi(\gamma(\tilde{\tau}_i))$. For characters χ of G we then define (following Artin) L-functions by the Euler product

$$L(s,\chi)=\prod_\tau \det\left(I-\frac{R_\chi(\tau)}{N(\tau)^s}\right)^{-1}$$

which converges for $\mathcal{R}(s)>1$. If χ_0 is the (principal) trivial character then $L(s,\chi_0)=\zeta(s)$.

For characters χ_1 and χ_2 we note that

$$\log L(s,\chi_1+\chi_2)=\sum_\tau\sum_{n=1}^\infty \frac{1}{n}\frac{\chi_1(\gamma(\tilde{\tau})^n)+\chi_2(\gamma(\tilde{\tau})^n)}{N(\tau)^{ns}}=\log L(s,\chi_1)+\log L(s,\chi_2),$$

where $\tilde{\tau}$ is an arbitrary $\tilde{\phi}$ closed orbit such that $\pi_G\tilde{\tau}=\tau$. Therefore $L(s,\chi_1+\chi_2)=L(s,\chi_1)L(s,\chi_2)$.

If $H\subset G$ is a subgroup and χ is a character of H we can decompose G into H cosets $G=\bigcup_{i=1}^m H\alpha_i$ and define the *induced character* χ^* of G by

$$\chi^*(g)=\sum_{\alpha_i g\alpha_i^{-1}\in H}\chi(\alpha_i g\alpha_i^{-1}).$$

The following is a well known result of Frobenius (see also Brauer [**5**]):

PROPOSITION 1. *Each non-trivial character χ of G is a rational combination of characters χ_i^* induced from non-trivial characters χ_i of cyclic subgroups H_i.*

Thus there exist integers $n, n_1,\ldots,n_k$ with $n\chi=\sum_{i=1}^k n_i\chi_i^*$ and in particular

$$L(s,\chi)^n=\prod_{i=1}^k L(s,\chi_i^*)^{n_i}. \tag{2.1}$$

When we study the meromorphic domain of L-functions it will prove easier to deal with cyclic covering groups. Therefore, bearing in mind (2.1), we need the following.

PROPOSITION 2. *Let χ be a character of the subgroup $H\subset G$ and let $L(s,\chi)$ be the L-function with respect to $\pi_H\colon \tilde{M}\to\tilde{M}/H$. Then $L(s,\chi^*)=L(s,\chi)$.*

Proof. Let τ be a closed ϕ-orbit and choose $z\in\tau$. Assume that $\tilde{\tau}_1,\ldots,\tilde{\tau}_n$ are the closed $\tilde{\phi}$ orbits satisfying $\pi_G\tilde{\tau}_i=\tau$ and let $\gamma(\tilde{\tau}_1)=g$. In particular, this means that if $x \in \tilde{\tau}_1$ and $\pi_G(x) = z$ then $\tilde{\tau}_1 \cap \pi_G^{-1}(z) = \{x, gx, \ldots, g^{l-1}x\}$ and $\tilde{\tau}_i \cap \pi_G^{-1}(z) = \{g_i x, g_i gx, \ldots, g_i g^{l-1}x\}$ for some $g_i \in G$. (Here $|g| = l$ and $l \cdot n = |G|$).

The closed orbits $\tilde{\tau}_1, \ldots, \tilde{\tau}_n$ lie over closed orbits $\tau_1, \ldots, \tau_n$ in $\tilde{\Omega}/H$ where $\pi_H(\tilde{\tau}_i) = \tau_i$. The images of the $g_i g^j x$ under $\pi_H : \tilde{\Omega} \to \tilde{X}/H$ are of the form $Hg_i g^j \in \tau_i$. To eliminate repeated images suppose that, after relabelling if necessary, $Hg_1Kx, \ldots, Hg_mKx$ are distinct where we define $K = \{e, g, g^2, \ldots, g^{l-1}\}$. Next, choose the least positive integer l_i for which $Hg_i g^{l_i} x = Hg_i x$ then

$$Hg_iKx = \bigcup_{j=0}^{l_i - 1} Hg_i g^j x.$$

Evidently $G = \bigcup_{i=1}^{m} \bigcup_{j=0}^{l_i-1} Hg_i g^j$ is a decomposition of G into H cosets. Since l_i is the least positive integer for which $g_i g^{l_i} g_i^{-1} \in H$ we see that $\pi_H(\tilde{\tau}_i) = \tau_i$ with $\lambda(\tilde{\tau}_i) = l_i \lambda(\tau_i)$. Therefore the Frobenius elements of H are given by $\gamma_H(\tilde{\tau}_i) = g_i g^{l_i} g_i^{-1}$. If χ is a character of H then

$$\begin{aligned} \chi^*(g^p) &= \sum_{\substack{g_i g^j g^p g^{-j} g_i^{-1} \in H \\ 1 \le j \le l_i}} \chi(g_i g^j g^p g^{-j} g_i^{-1}) \\ &= \sum_{l_i | p} l_i \chi(g_i g^p g_i^{-1}). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{\chi^*(\gamma(\tilde{\tau})^p)}{pN(\tau)^{ps}} &= \sum_{p=1}^{\infty} \frac{1}{pN(\tau)^{ps}} \sum_{l_i | p} l_i \chi(g_i \gamma(\tilde{\tau})^p g_i^{-1}) \\ &= \sum_{i=1}^{m} l_i \sum_{k=1}^{\infty} \frac{\chi(g_i \gamma(\tilde{\tau})^{kl_i} g_i^{-1})}{l_i k N(\tau)^{kl_i s}} = \sum_{i=1}^{m} \sum_{k=1}^{\infty} \frac{\chi(\gamma_H(\tilde{\tau}_i)^p)}{kN(\tau_i)^{ks}}, \end{aligned}$$

where $\tau_1, \ldots, \tau_m$ are closed orbits in $\tilde{M}/H$ which lie over τ and $\tilde{\tau}_i$ is an arbitrary closed orbit in $\tilde{M}$ which lies over τ_i. Summing over all closed ϕ-orbits we see that

$$\sum_{\tau} \sum_{p=1}^{\infty} \frac{\chi^*(\gamma(\tilde{\tau})^p)}{pN(\tau)^{ps}} = \sum_{\omega} \sum_{p=1}^{\infty} \frac{\chi(\gamma_H(\tilde{\omega})^p)}{pN(\omega)^{ps}},$$

where the ω are closed orbits in $\tilde{M}/H$ and $\tilde{\omega}$ is an arbitrary closed orbit in $\tilde{M}$ with $\pi_H(\tilde{\omega}) = \omega$. Hence $L(s, \chi^*) = L(s, \chi)$ and this completes the proof. □

Applying the above proposition to (2.1) gives

PROPOSITION 3. *If χ is an (irreducible) non-trivial character of G then $L(s, \chi)^n$ is a product of integral powers of L-functions defined with respect to non-trivial characters of cyclic subgroups of G.*

We end this section by relating zeta functions for $\tilde{\phi}$ and ϕ. Since π_G is $|G|$-to-1 it follows that $\tilde{\phi}$ and ϕ have the same topological entropy h. As before, we define

$$\tilde{\zeta}(s) = \prod_{\tilde{\tau}} (1 - N(\tilde{\tau})^{-s})^{-1},$$

where $\tilde{\tau}$ are closed $\tilde{\phi}$-orbits of least period $\lambda(\tilde{\tau})$ and $N(\tilde{\tau}) = e^{\lambda(\tilde{\tau})h}$. Then $\tilde{\zeta}(s)$ and $\zeta(s) = L(s, \chi_0)$ are related by the following.

PROPOSITION 4. $\tilde{\zeta}(s) = \prod_{\chi \text{ irreducible}} L(s, \chi)^{d_\chi}$.

Proof. Observe that for any $\mu \in \mathbb{C}$ and $g \in G$ (of order l):

$$\begin{aligned}\frac{|G|}{l}\log(1-\mu^l) &= \sum_{n=1}^{\infty}\frac{\mu^{nl}}{nl}|G| \\ &= \sum_{n=1}^{\infty}\frac{\mu^n}{n}\sum_{\chi}\bar{\chi}(e)\chi(g^n) = \sum_{\chi}\sum_{n=1}^{\infty}\frac{\mu^n}{n}d_\chi\chi(g^n) \\ &= \sum_{\chi}\sum_{n=1}^{\infty}\frac{d_\chi}{n}\mu^n \operatorname{Trace} R_\chi(g^n).\end{aligned}$$

Thus $(1-\mu^l)^{|G|/l} = \prod_\chi \det(I-\mu R_\chi(g))^{d_\chi}$. Take $\mu = N(\tau)^{-s}$ and $g = \gamma(\tilde{\tau}_i)$ where $\pi_G\tilde{\tau}_i = \tau$ then we have

$$(1-N(\tilde{\tau}_i)^{-s})^{-|G|/l} = \prod_\chi \det\left(I - \frac{R_\chi(\tau)}{N(\tau)^s}\right)^{-d_\chi} = \prod_{\pi_G(\tilde{\tau})=\tau}(1-N(\tilde{\tau})^{-s})^{-1}.$$

Taking products over all closed ϕ-orbits proves the result. □

3. *Symbolic dynamics for Axiom A flows*

Let A be a $k \times k$ irreducible 0–1 matrix and let

$$X_A = \left\{x \in \prod_{n=-\infty}^{+\infty}\{1, 2, \ldots, k\}: A(x_n, x_{n+1}) = 1 \quad \text{for all } n \in \mathbb{Z}\right\}$$

have the compact zero dimensional product topology. Let $\sigma: X_A \to X_A$ be the homeomorphism such that $(\sigma x)_n = x_{n+1}$ then σ is called a *shift of finite type.*

We can define a metric on X_A by $d(x, y) = (\frac{1}{2})^n$ where n is the least positive integer for which $x_i = y_i$, $|i| \le n$.

A function $f: X_A \to \mathbb{R}$ is called *Hölder continuous* if there exists $C, \alpha > 0$ such that $|f(x) - f(y)| \le Cd(x, y)^\alpha$. If f is strictly positive we define

$$X_f = \{(x, y) | 0 \le y \le f(x)\},$$

where $(x, f(x))$ and $(\sigma x, 0)$ are identified.

The *suspended flow* $\sigma_{f,t}: X_f \to X_f$ is defined by $\sigma_{f,t}(x, y) = (x, y + t)$, with appropriate identifications.

A closed set $S \subset \Omega$ is called a *global cross-section* to ϕ if each ϕ-orbit intersects S infinitely often (both in the future and in the past) and there exists $\delta > 0$ such that

$$\inf\{t > 0 | \phi_t x \in S\} > \delta \qquad \text{for all } x \in S.$$

For any $\varepsilon > 0$, Bowen constructed disjoint (local) cross-sections $S_1, \ldots, S_k \subseteq \Omega$ such that $S = \bigcup_{i=1}^k S_i$ is a global cross-section with diam $S_i < \varepsilon$, $i = 1, \ldots, k$. (Bowen refers to S as ε-small). He also constructed a shift of finite type (X_A, σ) and a continuous surjection $p: X_A \to S$ such that $p\{x \in X_A | x_0 = i\} = S_i$. Furthermore, if $x \in X_A$ such that $x_0 = i$, $x_1 = j$ then $\phi_{f(x)}p(x) = p\sigma(x) \in S_j$, where

$$f(x) = \inf\{t > 0 | \phi_t p(x) \in S_j\}.$$

Bowen's construction also allows us to assume (for an arbitrary $\varepsilon > 0$) that $f < \varepsilon$. With these constructions Bowen [**3**] showed:

Proposition 5. *For an Axiom A flow $\phi_t: \Omega \to \Omega$ (restricted to a basic set) and $\varepsilon > 0$ there exists the following: an ε-small cross-section S; a shift of finite type $\sigma: X_A \to X_A$,* *a Hölder continuous function $f>0$ and a continuous surjective map $p\colon X_f\to\Omega$ with $p\sigma_{f,t}=\phi_t p$ and $p(X_A)=S$. Furthermore p is at most N-to-1 (for some N) and p is a measure isomorphism (with respect to the measures of maximal entropy for σ_f and ϕ).*

Unfortunately, $p\colon X_f\to\Omega$ does not give a one-one correspondence between closed σ^f-orbits and closed ϕ-orbits. Nevertheless, by extending a technique due to Manning [**10**], for the Axiom A diffeomorphism case, Bowen showed how to account for this discrepancy. It suffices for us to note the following: Let E be the set of all closed ϕ-orbits τ which are *not* the p images of *exactly one* closed σ_f orbit (of the same period) then

$$\sum_{\tau\in E}\sum_{n=1}^{\infty}\frac{1}{nN(\tau)^{sn}}\quad\text{and}\quad\sum_{\tau'\in p^{-1}E}\sum_{n=1}^{\infty}\frac{1}{nN(\tau')^{sn}}$$

both converge for $\mathcal{R}(s)>1-\varepsilon$, for some $\varepsilon>0$.

Let $\tilde\phi$ be a Galois G-covering of ϕ and let (X_f,σ_f) be a suspended flow corresponding to $\phi\colon\Omega\to\Omega$, as in proposition 5. Let B_i be a neighbourhood of S_i such that $B_i\cap B_j=\varnothing$ when $i\neq j$. Providing ε is sufficiently small we may assume that $\pi_G^{-1}B_i$, $i=1,\dots,k$ is a trivial bundle over B_i. We define $\tilde S=\pi_G^{-1}S$ then $\tilde S$ is a global cross-section for $\tilde\phi$. We can identify $\tilde S$ with $S\times G$ and define $\tilde p\colon X_A\times G\to\tilde S$ by $\tilde p(x,g)=(px,g)$.

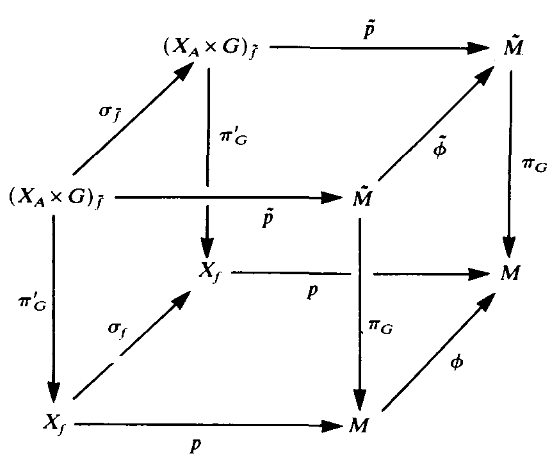
Define $\alpha\colon X_A\to G$ so that $\phi_{f(x)}(x,g)=(\sigma x,\alpha(x)g)$ then α depends only on x_0 and x_1 i.e. $\alpha(x)=\alpha(x_0,x_1)$.

We can define a 'shift' $\tilde\sigma\colon X_A\times G\to X_A\times G$ by

$$\tilde\sigma(x,g)=(\sigma x,\alpha(x)g).$$

Then $\tilde p$ extends to a continuous surjective map $\tilde p\colon(X_A\times G)_{\tilde f}\to\tilde\Omega$ such that $\tilde p\tilde\sigma_{\tilde f,t}=\tilde\phi_t p$ (where we define $\tilde f(x,g)=f(x)$). Furthermore, $\tilde p$ is at most N-to-1 and is a measure isomorphism (with respect to the measures of maximal entropy). We can define $\pi'_G\colon(X_A\times G)_{\tilde f}\to X_f$ by $\pi'_G((x,g),t)=(x,t)$ then $\pi'_G\tilde\sigma_{\tilde f,t}=\sigma_{f,t}\pi'_G$.

Since the maps $p,\tilde p,\pi_G,\pi'_G$ are all bounded-to-one this means that the flows σ_f, $\tilde\sigma_{\tilde f}$, ϕ and $\tilde\phi$ all have the same topological entropy [**1**]. The above equations are illustrated in the commutative diagram:



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4. *The weak-mixing case*

An Axiom A flow $\phi_t:\Omega\to\Omega$ is called (topologically) *weak-mixing* if there is no non-trivial solution to $F(\phi_t x)=e^{iat}F(x)$, where a is non-zero and $F\in C(\Omega)$. (In fact, it is equivalent to consider Borel measurable solutions.) In particular, if any of the flows ϕ, $\tilde{\phi}$, σ_f, and $\tilde{\sigma}_{\tilde{f}}$ are weak-mixing then all of them are. When $\phi|_\Omega$ is not weak-mixing there exists a least positive $a>0$ called the eigenfrequency.

THEOREM 1. *When ϕ, $\tilde{\phi}$ are weak-mixing then ζ, $\tilde{\zeta}$ have non-zero analytic extensions to a neighbourhood of $\mathcal{R}(s)\geq 1$ except for simple poles at $s=1$.*

The proof may be found in [**13**], [**14**], or [**16**].

THEOREM 2. *When ϕ, $\tilde{\phi}$ are weak-mixing and χ is a non-trivial irreducible character then $L(s,\chi)$ has a non-zero analytic extension to a neighbourhood of $\mathcal{R}(s)\geq 1$.*

Proof. By proposition 3 we need only consider the case where G is a cyclic group. Let $L_f(s,\chi)$ be the L-function for the covering σ_f of ϕ. Then

$$\frac{L_f(s,\chi)}{L(s,\chi)}=\exp\left(\sum_{\tau'\in p^{-1}E}\sum_{n=1}^{\infty}\frac{\chi(\gamma(\tau')^n)}{nN(\tau')^{ns}}-\sum_{\tilde{\tau}\in E}\sum_{n=1}^{\infty}\frac{\chi(\gamma(\tau)^n)}{nN(\tau)^{ns}}\right).$$

Since the summations converge absolutely for $\mathcal{R}(s)>1-\varepsilon$ this is a non-zero, analytic function on $\mathcal{R}(s)>1-\varepsilon$. It suffices therefore to prove the theorem for $L_f(s,\chi)$.

Furthermore, it is easy to see that

$$\begin{aligned}L_f(s,\chi)&=\prod_{\tau}\det\left(I-\frac{R_\chi(\gamma(\tilde{\tau}))}{N(\tau)^s}\right)^{-1}\\&=\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\mathrm{Fix}_n}\chi(\alpha_n(x))\exp\left[-shf^n(x)\right]\right),\end{aligned}$$

where $\alpha_n(x)=\alpha(x)\alpha(\sigma x)\cdots\alpha(\sigma^{n-1}x)$; $f^n(x)=f(x)+\cdots+f(\sigma^{n-1}x)$ and $\mathrm{Fix}_n=\{x|\sigma^n x=x\}$.

Since G is abelian each irreducible character χ satisfies $\chi(\alpha_n(x))=\prod_{i=0}^{n-1}\chi(\alpha(\sigma^i x))$.

A general theorem (theorem 1of [**13**] and theorem 3 of [**16**]) implies that $L_f(s,\chi)$ has an analytic extension to a neighbourhood of $1+it$ unless

$$htf+\frac{2\pi\theta(x)}{|G|}=2\pi K(x)+w(\sigma x)-w(x), \tag{4.1}$$

where $w\in C(X_A)$, K is integer-valued and

$$\chi(\alpha(x))=\exp\left(2\pi i\theta(x)/|G|\right).$$

(Here $|G|$ is the order of the cyclic group G and we choose θ to take values in $\{0,1,\ldots,|G|-1\}$.)

If (4.1) has a solution then $|G|th$ is an eigenfrequency for σ_f so that $t=0$ and (4.1) becomes

$$\chi(\alpha(x))=H(\sigma x)/H(x)$$

where $H(x)=e^{iw(x)}$. Hence $\tilde{\sigma}$ is not ergodic, contradicting the fact that $\tilde{\sigma}_{\tilde{f}}$ is weak-mixing. This contradiction shows that for all $t\in\mathbb{R}$, $L(s,\chi)$ is analytic in a neighbourhood of $1+it$. This completes the proof. □

5. *The case when ϕ, $\tilde{\phi}$ are not weak-mixing*

Let $\tilde{\phi}$ be a G (Galois) covering of ϕ and assume $\tilde{\phi}$ and ϕ are not weak-mixing with least eigenfrequencies a and Na respectively (where $N \| G|$). Let $\tilde{\sigma}_{\tilde{f}}$ and σ_f be the corresponding suspended flows. Here f can be taken to be the constant function $2\pi/Na$ [2]. Then for $\chi \in \hat{G}$,

$$L(s,\chi)=\exp\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\mathrm{Fix}_n} e^{-sn2\pi/Na}\chi(\alpha_n(x)).$$

We may assume, without loss of generality, that $\alpha(x)=\alpha(x_0,x_1)$. Then

$$\begin{aligned}&\sum_{\mathrm{Fix}_n} e^{-s2\pi n/Na}\chi(\alpha_n(x))\\ &\quad=\sum_{\mathrm{Fix}_n}\mathrm{Trace}\,(e^{-s2\pi n/Na}R_\chi(\theta(x_0,x_1))\dots R_\chi(\theta(x_{n-1},x_0))),\end{aligned} \tag{5.1}$$

where R_χ is a representation corresponding to χ. Let M denote the matrix

$$\begin{pmatrix} R_\chi(\theta(1,1)) & \cdots & R_\chi(\theta(1,n))\\ \vdots & & \vdots\\ R_\chi(\theta(n,1)) & \cdots & R_\chi(\theta(n,n))\end{pmatrix},$$

then (5.1) is equal to Trace $e^{-s2\pi n/Na}M^n$.

Therefore we have the closed form

$$L(s,\chi)=\exp\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Trace}\,(e^{-s\pi n/Na}M^n)=\frac{1}{\det(I-e^{-s2\pi/Na}M)}.$$

In particular $L(s,\chi)$ is meromorphic in $\mathbb{C}$ and non-zero. Since

$$L(s,\chi)=\exp\sum_{\tau}\sum_{n=1}^{\infty}\frac{\chi(\gamma(\tilde{\tau})^n)}{N(\tau)^{ns}},$$

where $N(\tau)=e^{\lambda(\tau)h}=e^{2\pi kh/Na}$, for some positive integer k, it follows that $L(s,\chi)$ has period iNa/h.

Let $\tilde{F}\tilde{\phi}_t=e^{iat}\tilde{F}$ where $\tilde{F}\in C(\tilde{\Omega})$ and, by transitivity, $|\tilde{F}|=1$, say. Replace $\tilde{F}$ by $\tilde{F}_g$ where $\tilde{F}_g(x)=\tilde{F}(gx)$, $g\in G$, then by the simplicity of the eigenfrequency $\tilde{F}_g=\eta(g)\cdot\tilde{F}$ where $\eta: G\to\mathbb{C}$ defines a one-dimensional representation. Furthermore, taking Nth powers

$$\tilde{F}^N\tilde{\phi}_t=e^{iNat}\tilde{F}^N$$

so that we conclude that $\tilde{F}^N=F\circ\pi_G$ where $F\phi_t=e^{iaNt}F$. Hence $\tilde{F}^N$ is G-invariant and $\eta^N=1$.

PROPOSITION 6. (i) *Each $L(s,\chi)$ is simply periodic with period iNa/h.*

(ii) *η has order N.*

(iii) *$L(s,\eta^i)$ has a nowhere zero analytic extension to $\mathcal{R}(s)>1-\delta$ (some $\delta>0$), except for simple poles at $\{1+ia(nN+j)/h: n\in\mathbb{Z}\}$.*

(iv) *If χ is a character of G and $\chi\neq\eta^j$, $j=0,1,\dots,N-1$ then $L(s,\chi)$ has a nowhere vanishing analytic extension to $\mathcal{R}(s)>1-\delta$.*

Proof. We have already established (i) and (ii).

(iii) Let $H=\text{Ker}\,\eta$ then $G/H=\langle\eta\rangle$ is cyclic. Furthermore $\tilde{\Omega}/H$ is a G/H Galois covering of Ω and so by proposition 4

$$\zeta_{\tilde{\Omega}/H}(s)=\prod_{j=0}^{N-1} L(s,\eta^j),$$

where $\zeta_{\tilde{\Omega}/H}$ has period ia/h and $L(s,\eta^j)$ has period iaN/h. Assume $L(s,\eta)$ has a pole at $1+ia/h$ then this is equivalent to

$$\chi(\gamma(\tau))\,e^{it\lambda(\tau)}=1, \qquad \text{for all } \tau\notin E.$$

Thus $L(s,\eta^j)$ has its first pole at $1+(ia/h)j$.

(iv) Let S be the set of irreducible characters, excluding $\eta,\ldots,\eta^{N-1},\chi_0$, then by proposition 4

$$\tilde{\zeta}(s)=\prod_{j=0}^{N-1} L(s,\eta^j)\prod_{\chi\in S} L(s,\chi)^{d_\chi},$$

and since the poles of $\tilde{\zeta}(s)$ and $\prod_{j=0}^{N-1} L(s,\eta^j)$ are the same we see that $\prod_{\chi\in S} L(s,\chi)^{d_\chi}$ is analytic for $\mathscr{R}(s)>1-\varepsilon$. Since each L-function is non-zero we conclude that $L(s,\chi)$ is analytic in this domain (when $\chi\neq\eta^j, j=0,1,\ldots,N-1$). □

6. *Main theorem*

As usual take $\tilde{\phi}$, ϕ to be Axiom A flows with entropy h and G a (Galois) covering group. Assume $\tilde{\Omega}$ is the basic set for $\tilde{\phi}$ and $\Omega=\tilde{\Omega}/G$ is the basic set for ϕ.

Given $g\in G$ write $C=C(g)$ for its conjugacy class and define a complex function

$$\zeta_C(s)=\prod_{\substack{\tau\\ \gamma(\tilde{\tau})\in C}}(1-N(\tau)^{-s})^{-1},$$

(where $\pi_G(\tilde{\tau})=\tau$). Then

$$\log\zeta_C(s)=\sum_{\substack{\tau\\ \gamma(\tilde{\tau})\in C}}\sum_{n=1}^{\infty}\frac{1}{nN(\tau)^{sn}}.$$

By the orthogonality relation for characters

$$\sum_{\chi\text{ irreducible}}\chi(g^{-1})\sum_{n,\tau}\frac{\chi(\gamma(\tilde{\tau}))}{N(\tau)^{sn}}=\sum_{n,\tau}\sum_{\chi}\bar{\chi}(g)\frac{\chi(\gamma(\tilde{\tau}))}{nN(\tau)^{sn}}=\frac{|G|}{|C|}\sum_{\substack{n,\tau\\ \gamma(\tilde{\tau})\in C}}\frac{1}{nN(\tau)^{sn}}.$$

Thus

$$(|G|/|C|)\log\zeta_C(s)=\textstyle\sum_\chi \chi(g^{-1})\log L(s,\chi),$$

and differentiating:

$$\frac{|G|}{|C|}\frac{\zeta_C'(s)}{\zeta_C(s)}=\sum_\chi \chi(g^{-1})\frac{L'(s,\chi)}{L(s,\chi)}.$$

By theorem 2 and proposition 6

$$\frac{\zeta_C'(s)}{\zeta_C(s)}=\frac{|C|}{|G|}\frac{\zeta'(s)}{\zeta(s)}+\eta(s), \tag{6.1}$$

where if $\tilde{\phi}$, ϕ are weak-mixing $\eta(s)$ is analytic in a neighbourhood of $\mathscr{R}(s)\geq 1$,

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and when $\tilde{\phi}$, ϕ are not weak-mixing $\eta(s)$ is analytic in $\mathcal{R}(s) \geq 1-\delta$, except possibly at the points $1+nia/h$, $n \neq 0$.

Let τ^k be a formal power of a closed orbit τ. Define $\Lambda(\tau^k) = \log N(\tau) = h\lambda(\tau)$ and

$$\Lambda_C(\tau^k) = \begin{cases} \Lambda(\tau^k) & \text{if } \gamma(\tilde{\tau}) \in C. \\ 0 & \text{otherwise.} \end{cases}$$

Equation (6.1) can be written as

$$\sum_{\tau} \sum_{n=1}^{\infty} \frac{\Lambda_C(\tau^n)}{N(\tau)^{ns}} = \frac{|C|}{|G|} \sum_{\tau} \sum_{n=1}^{\infty} \frac{\Lambda(\tau^n)}{N(\tau)^{ns}} + \eta(s).$$

Introducing,

$$F_C(t) = \sum_{N(\tau)^k \leq t} \Lambda_C(\tau^k),$$
$$F(t) = \sum_{N(\tau)^k \leq t} \Lambda(\tau^k),$$

gives

$$\int_1^{\infty} t^{-s}\, dF_C(t) = \frac{|C|}{|G|} \int_1^{\infty} t^{-s}\, dF(t) + \eta(s).$$

Since $\zeta'(s)/\zeta(s)$ has residue -1 at $s=1$ we see that $\int_1^{\infty} t^{-s}\, dF_C(t)$ has residue $|C|/|G|$. When $\tilde{\phi}$, ϕ are weak-mixing we have from the Ikehara–Wiener Tauberian theorem [**22**]:

PROPOSITION 7. *For $\tilde{\phi}$, ϕ weak-mixing*

$$F_C(t) \sim \frac{|C|}{|G|} t,$$
$$F(t) \sim t \qquad \text{as } t \to \infty.$$

When $\tilde{\phi}$, ϕ are not weak-mixing (6.1) gives

PROPOSITION 8. *For $\tilde{\phi}$, ϕ not weak-mixing*

$$F(t) \sim \frac{2\pi h}{Na} \left(\sum_{e^{2\pi hn/Na} \leq t} e^{2\pi hn/Na} \right),$$
$$F_C(t) \sim \frac{|C|}{|G|} \frac{2\pi h}{Na} \left(\sum_{e^{2\pi hn/Na} \leq t} e^{2\pi hn/Na} \right).$$

Details of similar calculations may be found in [**12**], [**14**]. Now define

$$\pi(t) = \text{Card}\, \{\tau \mid N(\tau) \leq t\},$$
$$\pi_C(t) = \text{Card}\, \{\tau \mid N(\tau) \leq t,\ \gamma(\tilde{\tau}) \in C\},$$

where τ is a closed ϕ-orbit and $\pi_G(\tilde{\tau}) = \tau$. Then by manipulations of partial sums (c.f. [**12**, pp. 50–52]) we have that

$$F(t) \sim \pi(t) \log t \quad \text{and} \quad F_C(t) \sim \pi_C(t) \log t.$$

This leads to our main result.

THEOREM 3. *Let $\tilde{\phi}$, ϕ be Axiom A flows with $\tilde{\phi}$ a G-covering of ϕ where G is a Galois group. Let $\tilde{\Omega}$ be a $\tilde{\phi}$ basic set which is G-invariant and let $\Omega=\tilde{\Omega}/G$. Then*

$$\pi_C(t)\sim\frac{|C|}{|G|}\pi(t),$$

where:

(i) *if $\tilde{\phi}$, ϕ are weak-mixing then*

$$\pi(t)\sim t/\log t;$$

(ii) *if $\tilde{\phi}$, ϕ are not weak-mixing then*

$$\pi(t)\sim\frac{1}{\log t}\cdot\frac{2\pi h}{Na}\left(\sum_{e^{2\pi hn/Na}\leq t} e^{2\pi hn/Na}\right),$$

(*where a, Na are the least eigenfrequencies of $\tilde{\phi}$, ϕ*).

Example (Twisted orbits). Let $\phi_t: M\rightarrow M$ be a (topologically) weak-mixing Axiom A flow. Let F^u be the vector bundle over M each of whose fibres is the space of k-dimensional frames corresponding to E^u_x, where $k=\dim E^u_x$. Furthermore, the vector bundle can be decomposed as $F^u=A\oplus B$ according to the orientation of the frames. Although $D\phi_t$ preserves F^u we observe that this need not be true for A and B i.e. $D\phi_t(A_x)=A_{\phi_t x}$ or $B_{\phi_t x}$ and $D\phi_t(B_x)=B_{\phi_t x}$ or $A_{\phi_t x}$. We say that a closed orbit τ is twisted if $D\phi_{\lambda(\tau)}A_x=B_x$, for $x\in\tau$. We can relate this to our above work by constructing a $\mathbb{Z}_2$-extension $\tilde{\phi}_t: M\times\mathbb{Z}_2\rightarrow M\times\mathbb{Z}_2$ of the flow according to the action of $D\phi_t$ in permuting A_x and B_x. A simple necessary and sufficient condition for the flow $\tilde{\phi}$ to be (topologically) weak-mixing is that the unstable bundle E^u should *not* be orientable. By theorem 3 we now have the following: *If E^u is not orientable then closed orbits are equally distributed between twisted and untwisted orbits.* (It is obvious that if E^u is oriented then none of the closed orbits can be twisted.)

7. *An application to homology*

We conclude with an application of theorem 3 to the distribution of closed orbits (considered as 1 cycles) among the cosets of a cofinite subgroup of $H_1(M,\mathbb{Z})$ when the basic set $\Omega=M$. Here we are indebted to Sunada who drew our attention to this application and who, in particular, made the observation that our theorem can be used to answer a question raised by J. Plante in [**15**] (see also [**21**]).

We assume, throughout this section, that $\Omega=M$. In particular ϕ is an Anosov flow. Let $\tilde{\tilde{M}}$ be the universal homology covering of M i.e. $H_1=H_1(M,\mathbb{Z})$ acts freely on $\tilde{\tilde{M}}$ and $M=\tilde{\tilde{M}}/H_1$. Let H_0 be a cofinite subgroup of H_1 and define $\tilde{M}=\tilde{\tilde{M}}/H_0$ so that $M=(\tilde{\tilde{M}}/H_0)/(H_1/H_0)$ i.e. $M=\tilde{M}/G$ with $G=H_1/H_0$. With the definitions of § 2, the Frobenius class of the closed orbit τ, denoted $[\tau]$, is precisely the coset of H_0 in H_1 to which the homology class of τ belongs. Thus applying theorem 3 to this situation we have

PROPOSITION 9. *If H_0 is a cofinite subgroup of $H_1=H_1(M,\mathbb{Z})$ then*

$$\text{Card}\{\tau|[\tau]\in h+H_0,\ N(\tau)\leq x\}\sim\pi(x)/\text{Card}(H_1/H_0)$$

as $x\rightarrow\infty$.

J. Plante in [**15**], considered an Anosov flow which preserves a smooth volume and showed that the set of closed orbits generates $H_1(M, \mathbb{Z})$. He conjectured that this is true if one replaced the smooth volume preserving condition by the weaker condition that $\Omega = M$. As Sunada has observed, our theorem 3 can be used to answer Plante's question affirmatively. For if H_1 is not generated by the closed orbits then we can find $H_0 \neq H_1$, a cofinite subgroup of H_1, such that H_0 contains all the homology classes represented by closed orbits. Thus $[\tau]$ is the identity of $G = H_1/H_0$ and G is non-trivial. However, this contradicts proposition 9, which states that the Frobenius elements are equidistributed.

8. *Compact group extensions*

In this section we are concerned with compact (as opposed to finite) group extensions of Axiom A flows. Many of the ideas and proofs of earlier sections carry over, without serious modifications, to this situation. Where significantly new procedures are required we shall give, at least in outline, the necessary details. However, it will be apparent that we allow ourselves a degree of informality in this discussion.

Let $\tilde{\phi}_t$ be a C^1 flow on the compact Riemannian manifold $\tilde{M}$ and let G be a compact Lie group which acts differentiably and freely on $\tilde{M}$ and which commutes with $\tilde{\phi}$. We suppose that $\tilde{\Omega}$ is a ϕ, G invariant closed set. The quotients $M = \tilde{M}/G$, $\phi = \tilde{\phi}/G$, $\Omega = \tilde{\Omega}/G$ are well defined and we suppose that ϕ is an Axiom A flow on M with basic set Ω.

Except when G is finite, the flow $\tilde{\phi}$ will *not* be an Axiom A flow and *a priori* there is no guarantee that $\tilde{\phi}$ will have *any* closed orbits. Nevertheless each closed ϕ-orbit τ defines a Frobenius class $[\tau]$ in G as in § 2.

For simplicity we shall always assume that ϕ is (topologically) weak-mixing.

As before we define for each (finite dimensional) unitary representation R_χ with character χ the L-function

$$L(s, \chi) = \prod_\tau \det(I - N(\tau)^{-s} R_\chi(\tau))^{-1},$$

and our main effort is directed towards proving $L(s, \chi)$ analytic and nowhere zero in $\mathcal{R}(s) \geq 1$ when χ is not the trivial one dimensional character χ_0. However, we do not have Frobenius's result to help us reduce the problem to the cyclic or abelian case. Instead we 'lift' the problem using $\tilde{p}, p$ (of § 3) to the G extension $(X_A \times G)_{\tilde{f}}$ of X_f where $\tilde{f}(x, g) = f(x)$. These spaces support the suspension flows $\tilde{\sigma}_{\tilde{f}}, \sigma_f$ which are defined with respect to the maps $\tilde{\sigma}(x, g) = (\sigma x, \alpha(x) g)$ and σ respectively. In this case we cannot claim that $\alpha: X_A \to G$ is a function of a finite number of coordinates. Nevertheless, if one looks at Bowen's construction for the relationship between (Ω, ϕ) and (X_f, σ_f) one concludes that α is Hölder. The proof that $L(s, \chi)$ is non-zero and analytic in $\mathcal{R}(s) \geq 1$ therefore reduces to the problem of showing that $L_f(s, \chi)$ is non-zero and analytic in $\mathcal{R}(s) \geq 1$ when $\chi \neq \chi_0$, where

$$L_f(s, \chi) = \exp \sum_{n=1}^\infty \frac{1}{n} \sum_{\mathrm{Fix}_n} e^{-shf^n(x)} \chi(\alpha_n(x)).$$

The important data here is that f and α: $X_A \to G$ is Hölder and Pressure $(-hf) = 0$.

The flow $\tilde{\phi}$ is said to be *topologically mixing* if for each pair of non-empty open subsets U, V of $\tilde{\Omega}$ there exists t_0 such that $U \cap \tilde{\phi}_t V \neq \emptyset$ for all $t > t_0$. As a consequence ϕ is *totally topologically transitive* (t.t.t.) which means that for each $t \neq 0$, $\tilde{\phi}_t$ is topologically transitive. (It is well known that a homeomorphism is topologically transitive if it is topologically mixing). From this fact it is easy to deduce that $\tilde{\sigma}_{\tilde{f}}$ is t.t.t. since $\tilde{p}\tilde{\sigma}_{\tilde{f},t} = \tilde{\phi}_t \tilde{p}$, where $\tilde{p}$ is continuous and surjective and one-to-one on a dense G_δ set. We conclude, therefore, that $\tilde{\sigma}_{\tilde{f}}$ has no non-constant continuous eigenfunction, when $\tilde{\phi}$ is topologically mixing.

If $\tilde{\phi}$ is not topologically mixing then it is not mixing with respect to the measure $\tilde{m}$, which is the Haar (G) extension of the maximal measure m for ϕ, in which case $\tilde{\sigma}_{\tilde{f}}$ is not mixing for the measure $\tilde{m} \circ \tilde{p}$, which is the Haar extension for the measure of maximal entropy of σ_f. Therefore, $\tilde{\sigma}_{\tilde{f}}$ has a non-constant measurable eigenfunction F'' and

$$F''\tilde{\sigma}(x, g) = e^{-iaf(x)} F''(x, g) \quad \text{a.e., } a \neq 0.$$

From this we deduce that

$$F'(\sigma x) = e^{-iaf(x)} F'(x) \chi(\alpha(x)) \quad \text{a.e.}$$

for some non-trivial irreducible 1-dimensional character χ. A slightly involved argument enables us to conclude that

$$F(\sigma x) = e^{-iaf(x)} F(x) \chi(\alpha(x))$$

for some $F \in C(X_A)$. Hence $\tilde{\sigma}_{\tilde{f}}$ has a non-constant continuous eigenfunction. In short, we have sketched a proof of:

LEMMA. *$\tilde{\phi}$ is topologically mixing if and only if $\tilde{\sigma}_{\tilde{f}}$ has no non-constant continuous eigenfunction.*

Remarks (1). The topological mixing and topological weak-mixing conditions coincide for ϕ, σ_f and $\tilde{\sigma}_{\tilde{f}}$. They probably coincide for $\tilde{\phi}$ as well but the argument is likely to be involved. (See [**6**], for the methods which are likely to be useful).

(2) If μ is an equilibrium state for ϕ and $\tilde{\mu}$ is its Haar extension then $\tilde{\phi}$ is weakly-mixing with respect to $\tilde{\mu}$ or even Bernoulli if $\tilde{\phi}$ is topologically mixing. An analogous statement holds for the flows σ_f, $\tilde{\sigma}_{\tilde{f}}$. The proof uses a result of Rudolph [**17**].

9. *Ruelle operators and analyticity*

Let $M: X_A \to U(d)$ (the unitary group of $d \times d$ matrices) be continuous and let $|\ |$ denote the Euclidean norm on $\mathbb{C}^d$. Define

$$\text{var}_n M = \sup\{|M(x) - M(y)| : x_i = y_i, |i| \leq n\}.$$

We say that $M \in U_\theta(X_A, d)$ if $\sup_n (\text{var}_n M/\theta^{2n+1}) < +\infty$. When $M': X_A^+ \to U(d)$ is continuous, where X_A^+ is the corresponding one-sided shift space, and $\sup_n (\text{var}_n M'/\theta^n) < +\infty$ we write $M' \in U_\theta(X_A^+, d)$.

It is possible to show that if $M \in U_\theta(X_A, d)$ then there exist $M' \in U_\theta(X_A^+, d)$ and $N \in U_{\theta^{1/2}}(X_A, d)$ such that $M = N^{-1} M' N \circ \sigma$, in other words M and M' are cohomologous. (Here we consider $U_\theta(X_A^+, d) \subset U_{\theta^{1/2}}(X, d)$.)

The proof is entirely analogous to the proof of the more familiar fact [4] that $f=f'+k\sigma-k$ where $f'\in F_\theta(X_A^+)$ and $k\in F_{\theta^{1/2}}(X_A)$. Since χ is Hölder we can find a common $0<\theta<1$ such that $f\in F_\theta$ and $R_\chi(\alpha(x))\in U_\theta(X_A, d)$ and by our remarks above there is no loss in generality if we assume that $f\in F_\theta(X_A^+)$ and $M\in U_\theta(X_A^+, d)$ where M is cohomologous to $R_\chi(\alpha)$.

Let $\mathscr{L}_{f,M}$ be the Ruelle operator acting on the Banach space $F_\theta(X_A^+, d)$ of $\mathbb{C}^d$ valued functions defined by

$$(\mathscr{L}_{f,M}w)(x)=\sum_{\sigma y=x} e^{f(y)}M(y)w(y),$$

where $F_\theta(X_A^+, d)$ is provided with the norm $\|w\|=\sup_x|w(x)|+\sup_n(\mathrm{var}_n\, w/\theta^n)$.

We can now use proposition 3 of [13] and theorem 3 of [16] and their arguments to prove theorem 4. Let χ be a non-trivial irreducible character then with M cohomologous to $R_\chi(\alpha(x))$ we have

$$L(k, M)=\exp\sum_{n=1}^{\infty} 1/n \sum_{\mathrm{Fix}_n} e^{k^n(x)}\mathrm{Tr}\, M_n(x),$$

(where $M_n(x)=M(\sigma^{n-1}x)\dots M(x)$) is non-zero and analytic in an $F_\theta(X_A^+, d)$ neighbourhood of f when $P(Rf)\le 0$ unless $P(Rf)=0$, $d=1$, $\mathscr{L}_{f,M}w=w$, $w\in F_\theta(X_A^+)$ in which case $L(k, M)(1-e^{p(k-iv)})$ is non-zero and analytic in a neighbourhood of $f=u+iv$.

THEOREM 4. *If $\tilde{f}(x, g)=f(x)$ is strictly positive, $f\in F_\theta(X_A^+)$ and $\sigma_{\tilde f}$ is topologically weak-mixing and if χ is a non-trivial irreducible character then*

$$L_f(s,\chi)=\exp\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\mathrm{Fix}_n} e^{-shf^n(x)}\chi(\alpha_n(x))$$

is non-zero analytic in $\mathscr{R}(s)\ge 1$.

To prove the theorem we have to exclude the possibility that $\mathscr{L}_{-(1+it_0)hf,M}w=w$ for some $w\in F(X_A^+)$. But this eigenfunction equation implies

$$e^{-it_0hf}\chi(\alpha(x))\,W'(x)=W'(\sigma x)$$

for some $W'\in F_\theta(X_A)$ where χ is 1-dimensional. Defining $W^*(x,g)=\chi^{-1}(g)\,W'(x)$ we have

$$e^{-it_0hf}\,W^*=W^*\circ\tilde\sigma,$$

which leads to

$$e^{-it_0ht}\,\bar W=\bar W\circ\tilde\sigma_{\tilde f,t},$$

i.e. a continuous eigenfunction $\bar w$ for the flows $\tilde\sigma_{\tilde f}$. Hence $\bar w$ is constant and $t_0=0$ so that W^* is constant and χ is trivial. This is a contradiction.

We have therefore proved:

THEOREM 5. *If $\tilde\phi$ is topologically mixing and χ is a non-trivial irreducible character then $L(s,\chi)$ is non-zero and analytic in $\mathscr{R}(s)\ge 1$.*

From this we deduce the uniform distribution of Frobenius classes in the conjugacy classes of G.

THEOREM 6. *If $\tilde{\phi}$ is topologically mixing and $F \in C(G)$ is a class function ($F(g) = F(hgh^{-1})$ for all $g, h \in G$) then*

$$\sum_{N(\tau)\le x} F([\tau]) \sim \frac{x}{\log x} \int F(g)\, dg$$

where dg is Haar measure.

The proof is analogous to the proof of theorem 3 now that we know that $L(s, \chi)$ is non-zero analytic for $\mathcal{R}(s) \ge 1$ when χ is irreducible and non-trivial. For the trivial character χ_0, of course, $L(s, \chi_0) = \zeta(s)$.

Finally, we should note the situation where $\tilde{\phi}$ is not topologically mixing. In this case there exists $a \ne 0$ and a non-trivial one-dimensional irreducible character χ such that $F(\sigma x) = e^{iaf(x)}\chi(\alpha(x))F(x)$ for some $F \in C(X_A)$. Hence $\chi[\tau] = e^{ia\lambda(\tau)}$ for each σ_f closed orbit. We can therefore deduce a similar equation for almost all closed ϕ-orbits i.e.

$$\text{Card}\,\{\tau\colon N(\tau) \le x, \chi(\tau) \ne e^{ia\lambda(\tau)}\} = o(\pi(x))$$

where

$$\pi(x) \sim \frac{x}{\log x} \sim lix = \int_2^x \frac{dy}{\log y}.$$

Using Stieljes integration with respect to $\pi(x)$, it is possible to show that for closed ϕ-orbits τ

LEMMA. $\sum_{N(\tau)\le x} e^{ib\lambda(\tau)h} \sim x^{ib}/(1+ib)$.

From this we conclude:

THEOREM 7. *If ϕ is not topologically mixing then there exists $a \ne 0$, and an irreducible non-trivial 1-dimensional character χ such that*

$$\sum_{N(\tau)\le x} \chi(\tau) \sim \frac{x^{ia/h}}{1 + ia/h}\,\pi(x).$$

Thus, in this case, the Frobenius classes are *not* uniformly distributed.

10. *Frame bundle flows*

Let ϕ_t be the geodesic flow on the unit tangent bundle $M = T_1M_0$ of Riemannian manifold M_0 of (sectional) negative curvature. This provides a prototypical example of an Axiom A (or even Anosov) flow. Moreover $\Omega = M$ is a basic set itself. We shall suppose, in addition, that M_0 is orientable and consider $\tilde{M}$, the manifold of positively oriented orthonormal frames (bases). If x is the first vector of such a frame F_x, it determines a geodesic and $\phi_t x$. The frame bundle flow $\tilde{\phi}$ is defined on $\tilde{M}$ by carrying the frame F_x along the geodesic to $\phi_t x$ by parallel translation. The group $G = \mathrm{SO}\,(d-1)$ acts freely on $\tilde{M}$ in a natural way leaving each first vector fixed. In this way we see that $\pi_G\colon \tilde{M} \to M$, G commuting with $\tilde{\phi}$, so that $\phi = \tilde{\phi}/G$, $\tilde{\Omega} = \tilde{M}$, $\tilde{M}/G = M$.

In addition we should note that ϕ preserves Liouville measure l and $\tilde{\phi}$ preserves the Haar extension $\tilde{l}$ of l. We refer the reader to Brin [**6**], who together with various colleagues (Pesin, Gromov) has investigated ergodicity and mixing problems associated with frame bundle flows (always with respect to $\tilde{l}$).

In this situation we can interpret the Frobenius class $[\tau]$ of a closed orbit τ in M as the conjugacy class in G of the holonomy associated with a closed geodesic. We are therefore entitled to conclude:

THEOREM 8. *If the frame bundle flow ϕ is topologically mixing (in particular if it is weak-mixing and therefore Bernoulli with respect to $\tilde{l}$) then the holonomy class associated with the closed geodesic is uniformly distributed in the conjugacy classes of* SO$(d-1)$ *as lengths tend to infinity.*

REFERENCES

[1] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** (1971), 401-414.

[2] R. Bowen. Periodic orbits for hyperbolic flows. *Amer. J. Math.* **94** (1972), 1-30.

[3] R. Bowen. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.* **95** (1973), 429-459.

[4] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lecture Notes in Maths. 470. Springer: Heidelberg, 1975.

[5] R. Brauer. On Artin's *L*-series with general group characters. *Annals of Math.* **48** (1947), 502-514.

[6] M. Brin. Ergodic theory of frame flows. In *Ergodic theory and dynamical systems, II* (ed. A. Katok). Birkhauser: Basel, 1982.

[7] R. Gangolli. Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one. *Illinois J. Math.* **21** (1977), 1-41.

[8] D. A. Hejhal. The Selberg trace formula and Riemann zeta function. *Duke Math. J.* **43** (1976), 441-482.

[9] H. Huber. Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen, II. *Math. Ann.* **142** (1961), 385-398.

[10] A. Manning. Axiom A diffeomorphisms have rational zeta functions. *Bull. London Math. Soc.* **3** (1971), 215-220.

[11] G. A. Margulis. Applications of ergodic theory to the investigation of manifolds of negative curvature. *Funkt. Anal. i Ego Pril.* **3** (1969), 89-90.

[12] W. Parry. An analogue of the prime number theorem for closed orbits of shifts of finite type and their suspensions. *Israel J. Math.* **45** (1983), 41-52.

[13] W. Parry. Bowen's equidistribution theory and the Dirichlet density theorem. *Ergod. Th. & Dynam. Sys.* **4** (1984), 117-134.

[14] W. Parry & M. Pollicott. An analogue of the prime number theorem for closed orbits of Axiom A flows. *Annals. of Math.* **118** (1983), 573-591.

[15] J. F. Plante. Homology of closed orbits of Anosov flows. *Proc. Amer. Math. Soc.* **37** (1973), 297-300.

[16] M. Pollicott. A complex Ruelle-Perron-Frobenius theorem and two counter-examples. *Ergod. Th. & Dynam. Sys.* **4** (1984), 135-146.

[17] D. J. Rudolph. Classifying the isometric extensions of a Bernoulli shift. *J. d'Analyse Mathematique* **34** (1978), 36-60.

[18] D. Ruelle. *Thermodynamic Formalism.* Addison-Wesley: Reading, Mass., 1978.

[19] P. Sarnak. Class numbers of indefinite binary quadratic forms. *J. Number Theory* **15** (1982), 229-247.

[20] A. Selberg. Harmonic analysis and discontinuous subgroups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc.* **20** (1956), 47-87.

[21] T. Sunada. Geodesic flows and geodesic random walks. Preprint.

[22] N. Wiener. *The Fourier integral and certain of its applications.* C.U.P., Cambridge, 1967.