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ON COUNTABILITY OF POINT-FINITE FAMILIES OF SETS

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1. Introduction. It is well known that in a separable topological space every point-finite family of open subsets is countable. In the following we are going to show that both in σ-finite measure-spaces and in topological spaces satisfying the countable chain condition, point-finite families consisting of "large" subsets are countable.

Notation and terminology. Let A be a set. The family consisting of all (finite) subsets of A is denoted by $\mathscr{P}(A)(\mathscr{F}(A))$. Let $\mathscr{L} \subset \mathscr{P}(A)$ be a family of subsets of A. The sets $\cup \{L | L \in \mathscr{L}\}$ and $\cap \{L | L \in \mathscr{L}\}$ are denoted by $\cup \mathscr{L}$ and $\cap \mathscr{L}$, respectively. We say that the family $\mathscr{L}$ is *point-finite* (*disjoint*) if for each $a \in A$, the family $\{L \in \mathscr{L} | a \in L\}$ has at most finitely many members (at most one member).

Throughout the following, we let X denote some set (the "basic" set).

For the meaning of terminology and notation used without definition in this paper, see [**7**] and [**3**].

2. Countability of point-finite families. We start by making the notion of a "large" set explicit; this is best done by considering systems of "small" sets.

Definition 1. A family $\mathscr{N} \subset \mathscr{P}(X)$ is a *σ-ideal* if the following conditions are satisfied: (i) if $N \in \mathscr{N}$ and $K \subset N$, then $K \in \mathscr{N}$; (ii) if $N_n \in \mathscr{N}$ for each $n \in \mathbf{N}$, then $\bigcup_{n \in \mathbf{N}} N_n \in \mathscr{N}$.

Let $(X, \mathscr{A}, \mu)$ be a measure-space. Denote by $\mathscr{N}_\mu$ the family formed by all those sets which are contained in some $A \in \mathscr{A}$ with $\mu(A) = 0$. Then $\mathscr{N}_\mu$ is the σ-ideal of μ-negligible sets. In this case the "large" subsets of X are the sets with positive outer measure with respect to μ.

Let (X, τ) be a topological space. Denote by $\mathscr{N}_\tau$ the family formed by all those sets which are contained in some countable union of nowhere-dense subsets of the space (X, τ). Then $\mathscr{N}_\tau$ is the σ-ideal of the subsets of 1. category and the "large" subsets of X are the sets of 2. category (with respect to τ).

To state our basic lemma, we need the concept of separability of a family of sets with respect to a σ-ideal.

Definition 2. Let $\mathscr{N} \subset \mathscr{P}(X)$ be a σ-ideal. We say that a family $\mathscr{L} \subset \mathscr{P}(X)$ is *$\mathscr{N}$-separable* if $\mathscr{L}$ has a countable subfamily $\mathscr{L}'$ such that $L \sim \cup \mathscr{L}' \in \mathscr{N}$

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for each $L \in \mathscr{L}$. If every subfamily of $\mathscr{L}$ is $\mathscr{N}$-separable, then we say that $\mathscr{L}$ is *hereditarily $\mathscr{N}$-separable.*

Countable families of sets are obviously hereditarily $\mathscr{N}$-separable with respect to any σ-ideal $\mathscr{N}$; the following result shows that the converse is true for point-finite families consisting of "large" sets.

LEMMA. *Let $\mathscr{N} \subset \mathscr{P}(X)$ be a σ-ideal. Then every hereditarily $\mathscr{N}$-separable and point-finite subfamily of $\mathscr{P}(X) \sim \mathscr{N}$ is countable.*

Proof. Let $\mathscr{L}$ be a hereditarily $\mathscr{N}$-separable and point-finite subfamily of $\mathscr{P}(X) \sim \mathscr{N}$. Let $\mathscr{L}_0 = \emptyset$ and define the subfamilies $\mathscr{L}_n$, $n \in \mathbf{N}$, of $\mathscr{L}$ inductively as follows: if the families $\mathscr{L}_k$ have been defined for $k < n$, then we take for $\mathscr{L}_n$ some countable subfamily $\mathscr{L}'$ of $\mathscr{L} \sim \bigcup_{k=0}^{n-1} \mathscr{L}_k$ such that $L \sim \bigcup \mathscr{L}' \in \mathscr{N}$ for each $L \in \mathscr{L} \sim \bigcup_{k=0}^{n-1} \mathscr{L}_k$; such countable subfamilies exist, since the family $\mathscr{L}$ is hereditarily $\mathscr{N}$-separable.

To show that $\mathscr{L}$ is countable, it is enough to show that $\mathscr{L} = \bigcup_{n \in \mathbf{N}} \mathscr{L}_n$. Assume on the contrary that the family $\mathscr{L} \sim \bigcup_{n \in \mathbf{N}} \mathscr{L}_n$ is non-empty and let L be a set of this family. For each $n \in \mathbf{N}$, we have $L \in \mathscr{L} \sim \bigcup_{k=0}^{n-1} \mathscr{L}_k$ and it follows from the definition of the family $\mathscr{L}_n$ that the set $L_n = L \sim \bigcup \mathscr{L}_n$ belongs to the σ-ideal $\mathscr{N}$. We have $\bigcup_{n \in \mathbf{N}} L_n \in \mathscr{N}$ and it follows that we must have $L \sim \bigcup_{n \in \mathbf{N}} L_n \neq \emptyset$, since no set of $\mathscr{L}$ belongs to $\mathscr{N}$. Let $x \in L \sim \bigcup_{n \in \mathbf{N}} L_n$. Then we have $x \in \bigcup \mathscr{L}_n$ for each $n \in \mathbf{N}$; this, however, is a contradiction as the family $\mathscr{L}$ is point-finite and as $\mathscr{L}_n$, $n \in \mathbf{N}$, are mutually disjoint subfamilies of $\mathscr{L}$. It follows that $\mathscr{L} = \bigcup_{n \in \mathbf{N}} \mathscr{L}_n$; the family $\mathscr{L}$ is thus countable.

By using this lemma, we can show that certain countability conditions are equivalent for subfamilies of σ-algebras. Besides hereditary $\mathscr{N}$-separability, we are going to consider the following condition.

Definition 3. A family $\mathscr{L} \subset \mathscr{P}(X)$ is *chain-countable* if every disjoint subfamily of $\mathscr{L}$ is countable.

THEOREM 1. *Let $\mathscr{N} \subset \mathscr{P}(X)$ be a σ-ideal and let $\mathscr{A} \subset \mathscr{P}(X)$ be a σ-algebra. Then the following are equivalent:*
(i) *$\mathscr{A}$ is hereditarily $\mathscr{N}$-separable.*
(ii) *The family $\mathscr{A} \sim \mathscr{N}$ is chain-countable.*
(iii) *Every point-finite subfamily of $\mathscr{A} \sim \mathscr{N}$ is countable.*

Proof. (i) $\Rightarrow$ (iii) follows from the lemma and (iii) $\Rightarrow$ (ii) is trivially true. It remains to prove that (ii) $\Rightarrow$ (i).

Assume that the family $\mathscr{A} \sim \mathscr{N}$ is chain-countable. To show that $\mathscr{A}$ is hereditarily $\mathscr{N}$-separable, assume on the contrary that $\mathscr{A}$ has a subfamily, say $\mathscr{B}$, which is not $\mathscr{N}$-separable. We use transfinite induction to construct a subfamily $\{B_\alpha | \alpha < \Omega\}$ of $\mathscr{B}$, where Ω is the smallest uncountable ordinal. As $\mathscr{B}$ is not $\mathscr{N}$-separable, there exists $B_1 \in \mathscr{B}$ with $B_1 \notin \mathscr{N}$. Assume that for $\alpha < \Omega$, the sets B_β, $\beta < \alpha$, have been defined. The subfamily $\{B_\beta | \beta < \alpha\}$ is countable

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and it follows, since $\mathscr{B}$ is not $\mathscr{N}$-separable, that there exists $B_\alpha \in \mathscr{B}$ such that we have $B_\alpha \sim \bigcup_{\beta<\alpha} B_\beta \notin \mathscr{N}$. By the principle of transfinite induction, the foregoing defines the sets B_α, $\alpha < \Omega$. For each $\alpha < \Omega$, the set $C_\alpha = B_\alpha \sim \bigcup_{\beta<\alpha} B_\beta$ belongs to the family $\mathscr{A} \sim \mathscr{N}$; this, however, contradicts our assumption, since $\{C_\alpha | \alpha < \Omega\}$ is an uncountable disjoint family. It follows that the family $\mathscr{A}$ is hereditarily $\mathscr{N}$-separable.

Let $(X, \mathscr{A}, \mu)$ be a measure-space. A set $A \in \mathscr{A}$ is said to be of σ-finite μ-measure if we have $A \subset \bigcup_{n\in\mathbf{N}} A_n$ for some sets $A_n \in \mathscr{A}$ with finite μ-measure; if the basic set X is of σ-finite μ-measure, then we say that the measure-space $(X, \mathscr{A}, \mu)$ is σ-finite. σ-finiteness of the space $(X, \mathscr{A}, \mu)$ is closely related to chain-countability of the family $\mathscr{A} \sim \mathscr{N}_\mu$. On the one hand we see, by considering a maximal disjoint subfamily of $\mathscr{A}$ consisting of sets of finite positive measure, that if $\mathscr{A} \sim \mathscr{N}_\mu$ is chain-countable, then there exists a set $A \in \mathscr{A}$ of σ-finite μ-measure such that for each $B \in \mathscr{A}$ with $B \cap A = \emptyset$, we have either $\mu(B) = 0$ or $\mu(B) = \infty$. On the other hand, it is well known and easily seen that if the measure-space $(X, \mathscr{A}, \mu)$ is σ-finite, then the family $\mathscr{A} \sim \mathscr{N}_\mu$ is chain-countable; hence we see that $(X, \mathscr{A}, \mu)$ is σ-finite if and only if $\mathscr{A} \sim \mathscr{N}_\mu$ is chain-countable and every set of infinite measure contains a set of positive finite measure.

The following is an immediate consequence of Theorem 1.

COROLLARY. *In a σ-finite measure space, every point-finite family consisting of measurable sets of positive measure is countable.*

For a related result, see [**4**], Lemma 3.

We now turn to consider countability of point-finite families of subsets of topological spaces. To be able to use the result of Theorem 1 in this setting, we have to consider some suitable σ-algebra of subsets of a topological space. Let (X, τ) be a topological space. For all $A \subset X$ and $B \subset X$, denote by $A \Delta B$ the symmetric difference $(A \sim B) \cup (B \sim A)$ of A and B. A subset A of X is said to have the Baire property if we have $A = O \Delta N$ for some $O \in \tau$ and $N \in \mathscr{N}_\tau$. The family consisting of all subsets of X with the Baire property is denoted by $\mathscr{B}_\tau$. It is well-known that the family $\mathscr{B}_\tau$ is a σ-algebra (see e.g. [**7**]).

THEOREM 2. *Let (X, τ) be a topological space such that the family $\tau \sim \mathscr{N}_\tau$ is chain-countable. Then every point-finite subfamily of $\mathscr{B}_\tau \sim \mathscr{N}_\tau$ is countable.*

Proof. By Theorem 1, it is enough to show that the family $\mathscr{B}_\tau \sim \mathscr{N}_\tau$ is chain-countable. Assume on the contrary that there exists a disjoint family $\mathscr{B} \subset \mathscr{B}_\tau \sim \mathscr{N}_\tau$ such that $\mathscr{B}$ is uncountable. We can represent some subfamily of $\mathscr{B}$ in the form $\{B_\alpha | \alpha < \Omega\}$ so that $B_\alpha \neq B_\beta$ whenever $\alpha \neq \beta$. For each $\alpha < \Omega$, there are sets $O_\alpha \in \tau$ and $N_\alpha \in \mathscr{N}_\tau$ so that $B_\alpha = O_\alpha \Delta N_\alpha$. Let $U_\alpha = \bigcup_{\beta<\alpha} O_\beta$ and $V_\alpha = O_\alpha \sim \bar{U}_\alpha$ for each $\alpha < \Omega$. Then $\{V_\alpha | \alpha < \Omega\}$ is an uncountable disjoint subfamily of τ. We show that none of the sets V_α, $\alpha < \Omega$, belong to the σ-ideal $\mathcal{N}_\tau$. Let α be an ordinal, $\alpha < \Omega$. Then we have

$$(1)\quad B_\alpha \subset O_\alpha \cup N_\alpha = V_\alpha \cup (O_\alpha \cap \bar{U}_\alpha) \cup N_\alpha \subset V_\alpha \cup (O_\alpha \cap U_\alpha) \cup (\bar{U}_\alpha \sim U_\alpha) \cup N_\alpha.$$

For each $\beta < \alpha$, we have

$$O_\alpha \cap O_\beta \subset (B_\alpha \cup N_\alpha) \cap (B_\beta \cup N_\beta) \subset N_\alpha \cup N_\beta.$$

It follows that $O_\alpha \cap U_\alpha \subset \bigcup_{\beta \leqq \alpha} N_\beta$. As there are at most countably many ordinals less than α, it follows from the foregoing that $O_\alpha \cap U_\alpha \in \mathcal{N}_\tau$. On the other hand, as U_α is an open set, the set $\bar{U}_\alpha \sim U_\alpha$ is closed and nowhere-dense so that we have $\bar{U}_\alpha \sim U_\alpha \in \mathcal{N}_\tau$. It now follows by using (1) that we have $B_\alpha \subset V_\alpha \cap N$ for some $N \in \mathcal{N}_\tau$; since $B_\alpha \notin \mathcal{N}_\tau$, it follows that $V_\alpha \notin \mathcal{N}_\tau$. We have shown that $V_\alpha \in \sim\mathcal{N}_\tau$ for each $\alpha < \Omega$; this, however, is a contradiction since the family $\mathcal{T} \sim \mathcal{N}_\tau$ was assumed to be chain-countable. It follows that the family $\mathcal{B}_\tau \sim \mathcal{N}_\tau$ is chain-countable.

We say that a topological space (X, τ) satisfies the countable chain condition (ccc) if the family τ is chain-countable. Theorem 2 generalizes some results in [**1**] and [**10**] on countability of point-finite families of open subsets of spaces satisfying ccc.

3. Uncountability of point-finite families. In this section we give an example of a metacompact topological space which satisfies ccc but is not Lindelöf; such a space obviously has uncountable point-finite families of open subsets.

Example. A normal metacompact σ-space (X, τ) which satisfies ccc but is not Lindelöf.

(Metacompact spaces are called weakly paracompact in [**3**]. A topological space is a σ-space if it has a σ-discrete family of closed subsets such that every open set is the union of some sets of this family.)

Construction. Let A_1 be some uncountable set. Define the sets A_n, $n = 2, 3, \ldots$, inductively by the formula $A_{n+1} = \mathcal{F}(\mathcal{P}(A_n))$, $n \in \mathbf{N}$. Denote the set $\bigcup_{n \in \mathbf{N}} A_n$ by X. For every $x \in X$, denote by $n(x)$ that number $n \in \mathbf{N}$ for which we have $x \in A_n$. For all $x \in X$ and $z \in A_{n(x)+1}$, let

$$z_x = \{a \in z | x \in a\}$$

and let $P_z(x) = \{u \in A_{n(x)+1} | u \cap z = z_x\}$. It is not difficult to show that for each $n \in \mathbf{N}$ and for all $x, y \in A_n$ and $z, u \in A_{n+1}$, we have

(1) $P_z(x) \cap P_u(x) = P_{z \cup u}(x)$;

(2) $P_z(x) \cap P_u(y) = \emptyset$ if and only if either $z_x \cap (u \sim u_y) \neq \emptyset$ or $u_y \cap (z \sim z_x) \neq \emptyset$.

We define a topology τ on X as follows: a set $O \subset X$ belongs to τ if and only if for each $x \in O$, we have $P_z(x) \subset O$ for some $z \in A_{n(x)+1}$. Condition (1) above shows that the family τ is indeed a topology.

Verification of properties. (i) (X, τ) is normal. Let S and F be closed subsets of X such that $S \cap F = \emptyset$. For every $n \in \mathbf{N}$, let $S_n = S \cap A_n$ and $F_n = F \cap A_n$. We define inductively mutually disjoint subsets R_n and Q_n of A_n for $n \in \mathbf{N}$ as follows. We let $R_1 = S_1$ and $Q_1 = F_1$. If the sets R_{n-1} and Q_{n-1} have been defined, then we let $z = \{R_{n-1}, Q_{n-1}\}$,

$$R_n' = \cup \{P_z(x) | x \in R_{n-1}\},$$

$Q_n' = \cup \{P_z(x) | x \in Q_{n-1}\}$ and, finally, $R_n = (R_n' \sim F_n) \cup S_n$ and

$$Q_n = (Q_n' \sim S_n) \cup F_n.$$

To show that $R_n \cap Q_n = \emptyset$, let v be a point of R_n. If $v \in S_n$, then $v \notin Q_n$, since $S_n \cap F_n = \emptyset$. Assume that $v \in R_n' \sim F_n$. Then we have $v \in P_z(x)$ for some $x \in R_{n-1}$ and it follows that we have $R_{n-1} \in v$. As $R_{n-1} \cap Q_{n-1} = \emptyset$, we have $R_{n-1} \in \sim w$ for each $w \in Q_n'$; hence we have $v \notin Q_n'$ and thus, further, $v \notin Q_n$. We have shown that $R_n \cap Q_n = \emptyset$. When we let $O = \bigcup_{k \in \mathbf{N}} R_k$ and $U = \bigcup_{k \in \mathbf{N}} Q_k$, O and U are open sets containing S and F, respectively, and as we have $R_k \cap Q_k = \emptyset$ for each $k \in \mathbf{N}$, the sets O and U are disjoint.

(ii) (X, τ) is a metacompact σ-space. To show this, we construct a sequence $(V_n(x))_{n \in \mathbf{N}}$ of neighborhoods for each $x \in X$. Let $x \in X$. The set $A_{n(x)}$ is infinite and hence we can find a sequence $(B_n(x))_{n \in \mathbf{N}}$ of distinct subsets of $A_{n(x)}$ such that $B_1(x) = \{x\}$ and $x \in B_n(x)$ for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ let $z(x, n) = \{B_1(x), \ldots, B_n(x)\}$ and further, let $R_{n,1}(x) = \{x\} \cup P_{z(x,n)}(x)$. For all $y \in X$ and $n \in \mathbf{N}$, we now define the sets $R_{n,k}(y)$, for $k > 1$, inductively by the formula

$$R_{n,k}(y) = \cup \{R_{n,1}(x) | x \in R_{n,k-1}(y)\}$$

and, further, we denote by $V_n(y)$ the open neighborhood $\bigcup_{k \in \mathbf{N}} R_{n,k}(y)$ of y.

To show that (X, τ) is metacompact, we show first that for each $x \in X$, the set $C(x) = \{y \in X | x \in V_1(y)\}$ is finite. Let $n \in \mathbf{N}$ and assume that we have shown that the sets $C(x)$, $x \in A_n$, are finite. Let $x \in A_{n+1}$ and let $C'(x) = \{r \in A_n | x \in R_{1,1}(r)\}$. For each $r \in C'(x)$, we have $\{r\} \in x$ and it follows, since x is a finite set, that the set $C'(x)$ is finite. It is easily seen that we have $V_1(y) \cap C'(x) \neq \emptyset$ for each $y \in C(x) \sim \{x\}$; hence, we have $C(x) \sim \{x\} \subset \cup \{C(r) | r \in C'(x)\}$. By the assumption that we have made, each of the sets $C(r)$, $r \in C'(x)$, is finite; from this it follows by the finiteness of the set $C'(x)$ that the set $C(x)$ is finite. We have shown that if the sets $C(x)$, $x \in A_n$, are finite, then so are the sets $C(x)$, $x \in A_{n+1}$. As we have $C(x) = \{x\}$ for each $x \in A_1$, it follows that $C(x)$ is a finite set for every $x \in X$. Now, if $\mathcal{U}$ is an open cover of (X, τ) and for each $x \in X$, $U(x)$ a set of $\mathcal{U}$ containing x, then it follows from the foregoing that the open refinement

$$\{U(x) \cap V_1(x) | x \in X\}$$

of $\mathscr{U}$ is point-finite; the space (X, τ) is thus metacompact.

To show that (X, τ) is a σ-space, it is clearly enough to show that X can be represented in the form $X = \bigcup_{n \in \mathbf{N}} F_n$ so that the sets F_n are closed and discrete in (X, τ). Let $F_1 = A_1$ and for each $n > 1$, denote by F_n the set consisting of all those elements of $X \sim A_1$ which have less than n elements. As the elements of $X \sim A_1$ are finite sets, we see that $X = \bigcup_{n \in \mathbf{N}} F_n$. That the sets F_n are closed and discrete is seen by observing that for every $x \in X$, we have $V_n(x) \cap F_n \subset \{x\}$ for each $n \in \mathbf{N}$; this is clear for $n = 1$, since

$$V_1(x) \cap \bigcup_{n \leqq n(x)} A_n = \{x\}$$

and for $n > 1$, it is a consequence of the fact that we have

$$V_n(x) \sim \{x\} = \bigcup \{P_{z(y,n)}(y) | y \in V_n(x)\}$$

(note that for each $y \in X$, we have $P_{z(y,n)}(y) \cap F_n = \emptyset$, since every element of the set $P_{z(y,n)}(y)$ contains the n-element set $z(y, n)$). It follows from the foregoing that (X, τ) is a σ-space (and also a T_1-space).

(iii) (X, τ) satisfies ccc. Assume on the contrary that there exists an uncountable disjoint family $\mathscr{O}$ of open subsets of X. Choose a point from each set of the family $\mathscr{O}$ and denote the set of these points by D. To every $x \in D$, there corresponds an element $z(x)$ of the set $A_{n(x)+1}$ such that the set $P_{z(x)}(x)$ is contained in that set of the family $\mathscr{O}$ to which x belongs. As the set D is uncountable, we see that there exist $n \in \mathbf{N}$, $k \in \mathbf{N}$ and an uncountable subset H of D such that $H \subset A_n$ and such that for each $x \in H$, the set $z(x)$ has k elements. By using a result in [**5**] (vii, p. 235), we see that there exists $z \in A_{n+1}$ and an uncountable set $I \subset H$ such that for any two distinct elements x and y of I, we have $z(x)_x \cap z(y)_y = z$; further, by the same result, there exists $u \in A_{n+1}$ and an uncountable set $J \subset I$ such that we have $(z(x) \sim z(x)_x) \cap (z(y) \sim z(y)_y) = u$ for all $x \in J$ and $y \in J$, $x \neq y$. Now, let x be a point of J. We have $P_{z(y)}(y) \cap P_{z(x)}(x) = \emptyset$ for each $y \in J \sim \{x\}$ and it follows from condition (2) above, since the set $z(x)$ is finite and the set J uncountable, that for some $B \in z(x)$ and for some uncountable $K \subset J$, we have either $B \in z(x)_x$ and $B \in \bigcap_{y \in K} (z(y) \sim z(y)_y)$ or $B \in z(x) \sim z(x)_x$ and $B \in \bigcap_{y \in K} z(y)_y$. The first case is impossible since $B \in z(x)_x$ would imply that $B \notin u$ and

$$B \in \bigcap_{y \in K} (z(y) \sim z(y)_y)$$

would imply that $B \in u$. Similarly, we see that the second case is impossible. This contradiction shows that (X, τ) must satisfy ccc.

(iv) (X, τ) is not a Lindelöf-space. This is clear, since the uncountable subset A_1 of X is closed and discrete.

Remarks. (i) For some other examples of metacompact, non-Lindelöf spaces satisfying ccc, see [**8**] and [**9**].

(ii) In example G of [**2**], R. H. Bing constructed a normal, non-collectionwise normal space F, starting from an arbitrary uncountable set P. If we take $P = A_n$ for some $n \in \mathbf{N}$, then it is easily seen that the subspace $A_n \cup A_{n+1}$ of the space (X, τ) in the example above is homeomorphic with the subspace of F considered in Example 2 of [**6**].

(iii) By observing that we can consider $(A_1, \mathscr{P}(A_1))$ as a discrete topological space, we see that the construction of the above example can be modified so as to yield for each T_1-space (Y, σ) an imbedding of (Y, σ) as a closed G_δ-subspace in a space (X, τ) satisfying ccc; this space (X, τ) can moreover be chosen so that the set $X \sim Y$ is a countable union of closed and discrete sets and so that if (Y, σ) is normal (metacompact), then (X, τ) is also normal (metacompact).

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