ON A CONJECTURE OF G. HAJÓS

by A. D. SANDS

(Received 6 March, 1973)

1. Introduction. The purpose of this note is to provide by means of an example a negative answer to a conjecture of Hajós [3] concerning the factorization of finite abelian groups. This question is also raised as Problem 81 in Fuchs [2].

If $S, T$ are subsets of an additive abelian group $G$ their sum $S+T$ is said to be direct if $s_1 + t_1 = s_2 + t_2$ implies $s_1 = s_2, t_1 = t_2$, where $s_i \in S, t_i \in T$. If the sum is direct and $S + T = G$, then we have a factorization of $G$. All sums considered in this note are direct. A subset $S$ of $G$ is said to be periodic if there exists $h \in G, h \neq 0$, with $S + h = S$. If $H = \{h \in G \mid S + h = S\}$, then $H$ is a subgroup of $G$ and we have $S = H + S_1$ for some subset $S_1$. When Hajós discovered that neither factor in a factorization of certain finite abelian groups $G$ need be periodic he asked the following weaker question. Is every factorization $G = S + T$ of a finite abelian group $G$ quasi-periodic in the sense that one factor, say $T$, is a disjoint union of subsets $T_i$ $(1 \leq i \leq m, m > 1)$, such that there is a subgroup $H$ of $G$ of order $m$ with $S + T_1 = S + T_i + h_i$, where $H = \{h_i \mid 1 \leq i \leq m\}$? Clearly, if $T$ is periodic, the factorization is quasi-periodic with the set of periods of $T$, including 0, as the subgroup $H$.

2. Example. We give the following example of a non-quasi-periodic factorization. The construction is provided by a special case of a technique of de Bruijn [1], despite the closing remark of that paper.

Let $p$ be a prime, $p > 3$. Let $G$ be the direct sum of cyclic groups of orders $p^2$ and $p$. Let $a$ and $b$ of orders $p^2$ and $p$ generate $G$. We take

$$S = \{0, pa + 2b, 2pa + b, 3(pa + b), 4(pa + b), \ldots, (p-1)(pa + b)\},$$

$$T = V \cup W,$$

where

$$V = \{0, pa, 2pa, \ldots, (p-1)pa\},$$

$$W = \{0, b, 2b, \ldots, (p-1)b\} + \{a, 2a, \ldots, (p-1)a\}.$$ 

Then $G = S + T$, as is easily verified, and neither $S$ nor $T$ is periodic. This is essentially de Bruijn's construction of Theorem 2 of [1]. His notation is multiplicative and we have also multiplied his first factor by $st$ in order to put the identity into it, before changing to additive notation, replacing $s$ by $pa$ and $t$ by $b$ and using the particular set of coset representatives $0, a, \ldots, (p-1)a$, for $c_1, \ldots, c_m$.

If the factorization is quasi-periodic, one factor will split as a disjoint union of $m$ subsets of equal order, $m > 1$. These subsets can have order one only if one factor is periodic. Since neither $S$ nor $T$ is periodic and $S$ has prime order we see that the only possibility is that $T$ splits as a union of $p$ subsets each of order $p$. Let such a splitting occur and let $S + T_1 = S + T_i + h_i$. Then the subgroup $H$ has order $p$. Hence $H$ must be contained in the subgroup $K$ generated
by $pa$ and $b$. The sum $S+H$ is direct and so has order $p^2$. Now $S, H \subset K$. It follows that $S+H = K$. From $S+T = G$ we have

$$G = S+(UT) = S+T_1 + H = S+T_1 + h_1 + H = S+T_1 + H = K + T_1.$$  

Hence each set $T_1$ must be a set of coset representatives for $G$ modulo $K$. Therefore each set $T_1$ contains one element from $V$ and one element from $\{0, b, \ldots, (p-1)b\} + ra$, for each $r$ such that $1 \leq r \leq p-1$. Let $x_1 pa, y_1 b+a \in T_1$ and $x_2 pa, y_2 b+a \in T_2$. Then $S+T_1 = S+T_2 + h_2$ implies that

$$(S+T_1) \cap K = (S+T_2 + h_2) \cap K.$$  

Therefore $S+x_1 pa = S+x_2 pa + h_2$. Since $S$ is not periodic, we have $h_2 = (x_1-x_2)pa$. Similarly $(S+T_1) \cap (K+a) = (S+T_2 + h_2) \cap (K+a)$ implies that $S+y_1 b+a = S+y_2 b+a + h_2$. Thus $h_2 = (y_1-y_2)b$. This gives $(x_1-x_2)pa = (y_1-y_2)b$. As $G$ is a direct sum of the subgroups generated by $a$ and $b$ it follows that $x_1 pa = x_2 pa$. This is impossible as $T_1$ and $T_2$ have empty intersection. Therefore the factorization $G = S+T$ is not quasi-periodic.

3. Other related conjectures. Under certain conditions a factorization must be quasi-periodic. For example, let us assume that the factor $S$ is contained in a proper subgroup $K$ of $G$ such that $G$ is the direct sum of $K$ and a subgroup $H$. Then letting $T_1 = T \cap (K+h)$ for each $h \in H$, from $S+T = G$ and $S \subset K$ we find that $S+T_1 = K+h_1$. If $H$ is listed so that $h_1 = 0$, then $S+T_1 = K$ and so $S+T_1 = S+T_1 + h_1$ and the factorization is quasi-periodic. As we have seen, it need not be the case that such subgroups $K$ and $H$ exist. However the following weaker question is still open:

"If $G$ is a nonzero additive finite abelian group and $G = S+T$, where $0 \in S, 0 \in T$, must one of the factors be contained in some proper subgroup $K$ of $G$?"

There is another open question, which is weaker than the quasi-periodicity conjecture. If the factorization $G = S+T$ is quasi-periodic, as above, then $G = S+T_1 + H$ and $T$ has been replaced by the periodic factor $T_1 + H$. So we have the question as to whether it is always possible to replace one factor by a periodic factor. This question has already been suggested, in a letter to Fuchs, when a counterexample to problem 77 of [2] was given (see [5]), and is quoted by Fuchs in [4], p. 364. Thus this question is a possible replacement for both Problems 77 and 81 of [2].

REFERENCES

2. L. Fuchs, Abelian Groups (Budapest, 1958).

UNIVERSITY OF DUNDEE