ON A CONJECTURE OF G. HAJÓS

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1. Introduction. The purpose of this note is to provide by means of an example a negative answer to a conjecture of Hajó's [3] concerning the factorization of finite abelian groups. This question is also raised as Problem 81 in Fuchs [2].

If $S$, $T$ are subsets of an additive abelian group $G$ their sum $S + T$ is said to be direct if $s_1 + t_1 = s_2 + t_2$ implies $s_1 = s_2$, $t_1 = t_2$, where $s_i \in S$, $t_i \in T$. If the sum is direct and $S + T = G$, then we have a factorization of $G$. All sums considered in this note are direct. A subset $S$ of $G$ is said to be periodic if there exists $h \in G$, $h \neq 0$, with $S + h = S$. If $H = \{h \in G | S + h = S\}$, then $H$ is a subgroup of $G$ and we have $S = H + S_1$ for some subset $S_1$. When Hajó discovered that neither factor in a factorization of certain finite abelian groups $G$ need be periodic he asked the following weaker question. Is every factorization $G = S + T$ of a finite abelian group $G$ quasi-periodic in the sense that one factor, say $T$, is a disjoint union of subsets $T_i$ ($1 \leq i \leq m$, $m > 1$), such that there is a subgroup $H$ of $G$ of order $m$ with $S + T_1 = S + T_i + h_i$, where $H = \{h_i \mid 1 \leq i \leq m\}$? Clearly, if $T$ is periodic, the factorization is quasi-periodic with the set of periods of $T$, including 0, as the subgroup $H$.

2. Example. We give the following example of a non-quasi-periodic factorization. The construction is provided by a special case of a technique of de Bruijn [1], despite the closing remark of that paper.

Let $p$ be a prime, $p > 3$. Let $G$ be the direct sum of cyclic groups of orders $p^2$ and $p$. Let $a$ and $b$ of orders $p^2$ and $p$ generate $G$. We take

$$S = \{0, pa + 2b, 2pa + b, 3(pa + b), 4(pa + b), \ldots, (p-1)(pa + b)\},$$
$$T = V \cup W,$$

where

$$V = \{0, pa, 2pa, \ldots, (p-1)pa\},$$
$$W = \{0, b, 2b, \ldots, (p-1)b\} + \{a, 2a, \ldots, (p-1)a\}.$$

Then $G = S + T$, as is easily verified, and neither $S$ nor $T$ is periodic. This is essentially de Bruijn's construction of Theorem 2 of [1]. His notation is multiplicative and we have also multiplied his first factor by $st$ in order to put the identity into it, before changing to additive notation, replacing $s$ by $pa$ and $t$ by $b$ and using the particular set of coset representatives $0, a, \ldots, (p-1)a$, for $c_1, \ldots, c_m$.

If the factorization is quasi-periodic, one factor will split as a disjoint union of $m$ subsets of equal order, $m > 1$. These subsets can have order one only if one factor is periodic. Since neither $S$ nor $T$ is periodic and $S$ has prime order we see that the only possibility is that $T$ splits as a union of $p$ subsets each of order $p$. Let such a splitting occur and let $S + T_1 = S + T_i + h_i$. Then the subgroup $H$ has order $p$. Hence $H$ must be contained in the subgroup $K$ generated
by \(pa\) and \(b\). The sum \(S + H\) is direct and so has order \(p^2\). Now \(S, H \subseteq K\). It follows that \(S + H = K\). From \(S + T = G\) we have
\[
G = S + (UT_i) = \bigcup (S + T_i) = S + T_1 + H = S + T_1 + h_1 + H = S + T_1 + H = K + T_i.
\]
Hence each set \(T_i\) must be a set of coset representatives for \(G\) modulo \(K\). Therefore each set \(T_i\) contains one element from \(V\) and one element from \(\{0, b, \ldots, (p-1)b\} + ra\), for each \(r\) such that \(1 \leq r \leq p - 1\). Let \(x_1pa, y_1b + a \in T_1\) and \(x_2pa, y_2b + a \in T_2\). Then \(S + T_1 = S + T_2 + h_2\) implies that
\[
(S + T_1) \cap K = (S + T_2 + h_2) \cap K.
\]
Therefore \(S + x_1pa = S + x_2pa + h_2\). Since \(S\) is not periodic, we have \(h_2 = (x_1 - x_2)pa\).
Similarly \((S + T_1) \cap (K+a) = (S + T_2 + h_2) \cap (K+a)\) implies that \(S + y_1b + a = S + y_2b + a + h_2\).
Thus \(h_2 = (y_1 - y_2)b\). This gives \((x_1 - x_2)pa = (y_1 - y_2)b\). As \(G\) is a direct sum of the subgroups generated by \(a\) and \(b\) it follows that \(x_1pa = x_2pa\). This is impossible as \(T_1\) and \(T_2\) have empty intersection. Therefore the factorization \(G = S + T\) is not quasi-periodic.

3. Other related conjectures. Under certain conditions a factorization must be quasi-periodic. For example, let us assume that the factor \(S\) is contained in a proper subgroup \(K\) of \(G\) such that \(G\) is the direct sum of \(K\) and a subgroup \(H\). Then letting \(T_i = T \cap (K+h_i)\) for each \(h_i \in H\), from \(S + T = G\) and \(S \subseteq K\) we find that \(S + T_1 = K + h_1\). If \(H\) is listed so that \(h_1 = 0\), then \(S + T_1 = K\) and so \(S + T_1 = S + T_1 + h_1\) and the factorization is quasi-periodic. As we have seen, it need not be the case that such subgroups \(K\) and \(H\) exist. However the following weaker question is still open:

"If \(G\) is a nonzero additive finite abelian group and \(G = S + T\), where \(0 \in S\), \(0 \in T\), must one of the factors be contained in some proper subgroup \(K\) of \(G\)?"

There is another open question, which is weaker than the quasi-periodicity conjecture. If the factorization \(G = S + T\) is quasi-periodic, as above, then \(G = S + T_1 + H\) and \(T\) has been replaced by the periodic factor \(T_1 + H\). So we have the question as to whether it is always possible to replace one factor by a periodic factor. This question has already been suggested, in a letter to Fuchs, when a counterexample to problem 77 of [2] was given (see [5]), and is quoted by Fuchs in [4], p. 364. Thus this question is a possible replacement for both Problems 77 and 81 of [2].

REFERENCES