THE PACKING OF SPHERES IN THE SPACE $l_p$
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1. Introduction. A point $x$ in the real or complex space $l_p$ is an infinite sequence
$(x_1, x_2, x_3, \ldots)$ of real or complex numbers such that $\sum_{r=1}^{\infty} |x_r|^p$ is convergent. Here $p \geq 1$ and
we write
$$||x|| = \left(\sum_{r=1}^{\infty} |x_r|^p\right)^{1/p}.$$ The unit sphere $S$ consists of all points $x \in l_p$ for which $||x|| \leq 1$. The sphere of radius $a \geq 0$
and centre $y$ is denoted by $S_a(y)$ and consists of all points $x \in l_p$ such that $||x-y|| \leq a$. The
sphere $S_a(y)$ is contained in $S$ if and only if $||y|| \leq 1 - a$, and the two spheres $S_a(y)$ and $S_b(z)$ do
not overlap if and only if $||y-z|| > 2a$.

The statement that a finite or infinite number of spheres $S_a(y)$ of fixed radius $a$ can be packed
in $S$ means that each sphere $S_a(y)$ is contained in $S$ and that no two such spheres overlap.

In a recent paper [3] the packing of spheres $S_a(y)$ in $S$ was considered for the case $p=2$.
It is the object of the present paper to extend this work to all $p \geq 1$. The results obtained are
of a different character according as $p<2$ or $p>2$, and in the latter case are somewhat
surprising.

We write
$$\lambda_p = \left\{1 + 2^{1-1/p}\right\}^{-1}, \quad \mu_p = \left\{1 + 2^{1/p}\right\}^{-1}.$$ Observe that $\lambda_p = \mu_p$ and that $\lambda_p < \mu_p$ when $p>2$. As usual, $\delta_{nr}$ is 1 or 0 according as $n=r$
or $n \neq r$.

**Theorem 1.** If $1 < p < 2$, an infinity of spheres $S_a(y)$ of fixed radius $a$ can be packed in $S$ if and only if $\lambda_p \leq a$. If (i) $a \leq \lambda_p$, the spheres may be centred at the points $y_n = (y_{1n}, y_{2n}, y_{3n}, \ldots)$, where $y_{nr} = (1 - a)\delta_{nr}$ $(n \geq 1, r \geq 1)$. If (ii) $\lambda_p < a \leq 1$, the maximum number of spheres $S_a(y)$ which can be packed in $S$ does not exceed $L_p(a)$, where $L_p(a) = 1$, and
$$L_p(a) = \left(1 - \frac{1}{2} \left(1 - \frac{a}{\lambda_p}\right)^{1/(p-1)}\right)^{-1} (1 < p \leq 2).$$

**Theorem 2.** If $p > 2$, an infinity of spheres $S_a(y)$ of fixed radius $a$ can be packed in $S$ if and only if $\mu_p \leq a$. If (i) $a \leq \mu_p$, the spheres may be centred at the points $y_n = (y_{1n}, y_{2n}, y_{3n}, \ldots)$, where $y_{nr} = (1 - a)\delta_{nr}$ $(n \geq 1, r \geq 1)$. If (ii) $\lambda_p < a \leq \mu_p$, any finite number, no matter how large, of spheres $S_a(y)$ can be packed in $S$, but an infinite number cannot. If (iii) $\mu_p < a \leq 1$, the maximum number of spheres $S_a(y)$ which can be packed in $S$ does not exceed
$$M_p(a) = \left(1 - \frac{1}{2} \left(1 - \frac{a}{\mu_p}\right)^{1/(p-1)}\right).$$

2. The case $1 \leq p \leq 2$. Suppose that $m$ spheres $S_a(y_j)$ $(1 \leq j \leq m)$ can be packed in $S$, where $m \geq 1$, $a \leq 1$. By a simple extension of Lemma 1 of [2] from $n$-dimensional Euclidean
space to $l_p$, we find that
PACKING OF SPHERES IN $l_p$

$$\sum_{j=1}^{m} ||y_j||^p \geq 2^{p-1}m^{2-p}(m-1)^{p-1}a^p,$$

where the right-hand side denotes zero when $m = p = 1$. Hence, for at least one sphere $S_a(y_j)$,

$$||y_j|| \geq a\left(2\left(1 - \frac{1}{m}\right)^{1-1/p}\right),$$

and so

$$1 \geq a + ||y_j|| \geq a\left[1 + \left(\frac{2\left(1 - \frac{1}{m}\right)^{1-1/p}}{1-\frac{1}{m}}\right)\right].$$

If infinitely many spheres $S_a(y)$ can be packed in $S$ it follows that $1 \geq a(1 + 2^{1-1/p})$; i.e. $a \leq \lambda_p$.

If $\lambda_p < a < 1$ and $1 < p < 2$, we deduce from (1) that

$$\frac{1}{m} \geq 1 - \frac{1}{2}\left(1 - \frac{a}{m}\right)^{p(p-1)},$$

and, since the right-hand side is positive, $m \leq L_p(a)$. If $p = 1$, (1) shows that $m = 1 = L_1(a)$, because of the convention stated above; for otherwise we should have $a \leq \frac{1}{a} = \lambda_1$.

Finally, part (i) of Theorem 1 follows since, for $a \leq \lambda_p$ and $j \neq k$,

$$||y_j - y_k|| = 2^{1/p}(1 - a) \geq 2a,$$

and $||y_k|| = 1 - a$. Thus no two of the spheres $S_a(y_k)$ overlap and they are all contained in $S$. This completes the proof of Theorem 1.

3. The case $p > 2$. Suppose that $m$ spheres $S_a(y_j)$ ($1 \leq j \leq m$) can be packed in $S$, where $m \geq 1$, $a \leq 1$. By a simple extension of Lemma 2 of [1] (with $\beta = p$, $c_j = 1$), we find that

$$\sum_{j=1}^{m} ||y_j||^p \geq 2(m-1)a^p,$$

so that, for at least one sphere $S_a(y_j)$,

$$||y_j|| \geq a\left(2\left(1 - \frac{1}{m}\right)^{1/p}\right).$$

Hence

$$1 \geq a + ||y_j|| \geq a\left[1 + \left(2\left(1 - \frac{1}{m}\right)^{1/p}\right)\right],$$

from which we deduce that

$$\frac{1}{m} \geq 1 - \frac{1}{2}\left(1 - \frac{a}{m}\right)^p. \quad \text{...............(2)}$$

The right-hand side of (2) is positive if $\mu_p < a \leq 1$, so that we then obtain $m \leq M_p(a)$, which proves part (iii) of Theorem 2.

Part (i) of Theorem 2 can be proved as in §2. To prove part (ii) we suppose that $\lambda_p < a \leq \mu_p$ and take any positive integer $m$. For $1 \leq j \leq m$ and $n \geq 1$ put

$$y_{jn} = \varepsilon_{jn}2^{(1-m)/p}(1-a) = \varepsilon_{jn}b \quad (n \leq 2^{m-1}), \quad y_{jn} = 0 \quad (n > 2^{m-1}),$$

where $\varepsilon_{jn}$ is 1 or -1 according as the integral part of $(n-1)2^{-m}$ is even or odd. Then
$y_i = (y_{i1}, y_{i2}, y_{i3}, \ldots) \in l_p$. For example, when $m = 4$ the first 8 components of $y_1, y_2, y_3$ and $y_4$ are as follows:

(b b b b b b b b) 
(b b b b b b b b) 
(b b b b b b b b) 
(b b b b b b b b)

Clearly $\| y_j \| = 1 - a$ for $1 \leq j \leq m$ and, for $1 \leq j < k \leq m$,

$$\| y_j - y_k \| = 2^{1-1/p}(1-a) \geq 2a,$$

so that the $m$ spheres $S_a(y_j)$ are packed in $S$. Since $m$ can be any positive integer this proves part (ii).

It remains to prove that if an infinity of spheres $S_a(y)$ can be packed in $S$, then $a \leq \lambda_p$. We therefore suppose that the spheres $S_a(y_n)$ $(n = 1, 2, 3, \ldots)$ can be packed in $S$, where $y_n = (y_{n1}, y_{n2}, y_{n3}, \ldots)$. By considering each coordinate $y_{nr}$ $(r = 1, 2, 3, \ldots)$ in turn, picking out convergent subsequences, and renumbering the spheres, we may suppose that, for each fixed $r \geq 1$, $y_{nr} \to y_r$, say, as $n \to \infty$.

Since, for every positive integer $N$ and all $n \geq 1$,

$$\sum_{r=1}^{N} |y_{nr}|^p \leq \| y_n \|^p \leq (1-a)^p,$$

we have

$$\sum_{r=1}^{N} |y_r|^p \leq (1-a)^p,$$

and it follows that $y = (y_1, y_2, y_3, \ldots) \in l_p$ and

$$\| y \| \leq 1 - a.$$

Now take any positive integer $n$ and any $\varepsilon > 0$ and choose an integer $N$, depending on $n$ and $\varepsilon$, such that

$$\sum_{r>N} |y_{nr}|^p < \varepsilon^p.$$

Then, for $m > n$, since $S_a(y_m)$ and $S_a(y_n)$ do not overlap,

$$(2a)^p \leq \sum_{r=1}^{\infty} |y_{mr} - y_{nr}|^p \quad \text{..................................................(3)}$$

$$= \sum_{r=1}^{N} |y_{mr} - y_{nr}|^p + \sum_{r>N} |y_{mr} - y_{nr}|^p. \quad \text{..................................................(4)}$$

Now by Minkowski's inequality,

$$\left( \sum_{r>N} |y_{mr} - y_{nr}|^p \right)^{1/p} \leq \left( \sum_{r>N} |y_{mr}|^p \right)^{1/p} + \left( \sum_{r>N} |y_{nr}|^p \right)^{1/p}$$

$$\leq (1-a) + \varepsilon.$$

Hence, by (4),

$$(2a)^p - (1-a+\varepsilon)^p \leq \sum_{r=1}^{N} |y_{mr} - y_{nr}|^p.$$
PACKING OF SPHERES IN $l_p$

If we let $m \to \infty$ in this inequality, we obtain

$$(2a)^p - (1 - a + \varepsilon)^p \leq \sum_{r=1}^{N} |y_r - y_{nr}|^p \leq \sum_{r=1}^{\infty} |y_r - y_{nr}|^p,$$

from which, since $\varepsilon$ is arbitrary, we deduce that

$$(2a)^p - (1 - a)^p \leq \sum_{r=1}^{\infty} |y_r - y_{nr}|^p. \quad \ldots \quad (5)$$

We now apply to (5) an argument similar to that applied to (3). For any $\varepsilon > 0$, choose a positive integer $N$, depending on $\varepsilon$, such that

$$\sum_{r>N} |y_r|^p < \varepsilon^p.$$ 

In place of (4) we have

$$(2a)^p - (1 - a)^p \leq \sum_{r=1}^{N} |y_r - y_{nr}|^p + \sum_{r>N} |y_r - y_{nr}|^p \leq \sum_{r=1}^{N} |y_r - y_{nr}|^p + (1 - a + \varepsilon)^p.$$

On letting $n \to \infty$ we get, since $\varepsilon$ is arbitrary,

$$(2a)^p \leq 2(1 - a)^p$$

which is equivalent to $a \leq \lambda_p$. This completes the proof of Theorem 2.

4. The case $p = \infty$. In this case $S$ can be interpreted as the "cube" consisting of points $x = (x_1, x_2, x_3, \ldots)$ for which $|x_r| \leq 1$ for $r = 1, 2, 3, \ldots$, and similarly for $S_a(y)$. It is clear that for $\frac{1}{2} < a \leq 1$, only one cube $S_a(y)$ can be packed in $S$, while, for $a \leq \frac{1}{2}$, infinitely many can; for we may take their centres at the points $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \ldots)$.

REFERENCES


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