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NOETHERIAN SPECTRUM ON RINGS AND MODULES

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Abstract. It is shown that the well-known characterizations of when a commutative ring R has Noetherian spectrum carry over to characterizations of when the set Spec(M) of prime submodules of a finitely generated module M is Noetherian. The symmetric algebra $S_R(M)$ of M is used to show that the Noetherian property of Spec(R), and some related properties, pass from the ring R to the finitely generated R-modules.

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1. The sets of prime and radical submodules. Each ring in this paper is assumed to be commutative with identity unless stated otherwise. A submodule $N \neq M$ of a finitely generated R-module M is said to be *primary* if $ax \in N$, $a \in R$, $x \in M \setminus N$ implies $a \in \mathrm{rad}(N :_R M)$. If $ax \in N$, $a \in R$, $x \in M \setminus N$ implies $a \in (N :_R M)$, N is said to be a *prime submodule* of M. It follows that if N is primary then $\mathrm{rad}(N :_R M)$ is prime, and that a submodule N is prime if and only if $(N :_R M) = \mathfrak{p}$ is prime and M/N is torsion-free over $R/\mathfrak{p}$. We denote the set of prime submodules of M by Spec(M). For a submodule N of M, $V(N)$ denotes $\{P \in \mathrm{Spec}(M) \mid N \subseteq P\}$, and the *$M$-radical* of N is $\mathrm{rad}_M(N) = \cap V(N)$.

A fairly large number of papers have considered the extent to which properties of the prime ideals of a ring have counterparts for the prime submodules of a module. For example, see [**2**, **3**, **5**, **9**, **11**–**26**, **28**, **30**, **32**, **33**]. A well-known result on Spec(R) states that if a ring R has the ascending chain condition (ACC) on radical ideals, then each radical ideal of R is a finite intersection of prime ideals (e.g. see [**10**, Theorem 87]). This work was partly motivated by the fact that the proof of this result does not immediately extend to modules because of the lack of a simple description of the set of elements of $\mathrm{rad}_M(N)$ for a submodule N of an R-module M. Recall that a ring R has the ACC on radical ideals if and only if Spec(R), with the Zariski topology, is a Noetherian topological space. Since any Noetherian topological space X is a union of finitely many maximal irreducible subspaces, called the irreducible components of X, [**4**, Section II.4.2, Proposition 10], one might expect that this topological result would imply that each radical submodule of a module M with ACC on radical submodules is a finite intersection of prime submodules, as in the case $M = R$. (e.g. see [**29**, p. 632].) However, as was pointed out in [**23**], the set $\{V(N) \mid N \text{ a submodule of } M\}$ need not be closed under finite unions, and thus this set is not the set of closed sets of a topology on Spec(M). Although Spec(M) is generally not a topological space, we will sometimes say Spec(M) is Noetherian when M has the ACC on radical submodules.

In Section 2, it is shown that the various characterizations, given in [**7**] and [**29**], for example, of the Noetherian spectrum property on a ring R, carry over to characterizations of when Spec(M) is Noetherian for a finitely generated R-module M. For example, M has the ACC on radical submodules if and only if (a) each submodule N of M has only finitely many minimal prime submodules containing it, and (b) M has the ACC on prime submodules. In particular, if Spec(M) is Noetherian and $N \subseteq M$ is a submodule, then $\mathrm{rad}_M(N)$ is an intersection of finitely many prime submodules. However, unlike the case for ideals, some of the minimal prime submodules of N may be redundant in this intersection.

In Section 3, the uniqueness properties are considered of a representation of a radical submodule N as a finite intersection of prime submodules. Since prime submodules are primary, such representations already come with the uniqueness properties of primary decomposition. However, there are some additions for decompositions into prime submodules, namely the associated prime ideals are of a more restrictive type and for each such associated prime $\mathfrak{p}$ there is a unique smallest choice for the $\mathfrak{p}$-prime component. It follows that in the case that R has Noetherian spectrum, describing the representation of the radical of a submodule N of a finitely generated R-module M as a finite intersection of prime submodules, is essentially equivalent to determining the associated prime ideals of $M/\mathrm{rad}_M(N)$.

In Section 4, the symmetric algebra $S_R(M)$ of M is used to show that if Spec(R) is Noetherian, then Spec(M) is Noetherian for each finitely generated R-module M. It is also shown that $S_R(M)$ can be used to add to our understanding of the prime submodules of a module since, as shown in [**26**], for each prime submodule N of M there is a unique smallest prime ideal $\mathcal{E}_N$ of $S_R(M)$ such that $\mathcal{E}_N \cap M = N$.

2. Modules with ACC on radical submodules. In this section, it is shown that the characterizations of the Noetherian property of Spec(R) for a ring R, which are given in [**7**, **29**], carry over to characterizations of when Spec(M) is Noetherian for a finitely generated R-module M. Let R be a ring and M be an R-module. We denote the set of prime submodules of M by Spec(M) and the set of maximal proper submodules of M by Max(M). If $\mathcal{P} \subseteq$ Spec(M) and N is a submodule of M we let $V_{\mathcal{P}}(N) = \{P \in \mathcal{P} \mid N \subseteq P\}$ and $\mathcal{P}$-rad$(N) = \cap V_{\mathcal{P}}(N)$. In case $\mathcal{P} =$ Spec(M), we write $V(N)$ and $\mathrm{rad}_M(N)$ for $V_{\mathcal{P}}(N)$ and $\mathcal{P}$-rad(N), respectively. As with the prime spectrum of a ring R, one choice for $\mathcal{P}$ is the set Max(M) of maximal submodules of M.

DEFINITION 2.1. (2.1.1) If $\mathcal{P} \subseteq$ Spec(M) and N is a submodule of M, N is said to be a $\mathcal{P}$-*radical submodule* if it is an intersection of elements of $\mathcal{P}$. A $\mathcal{P}$-radical submodule of M which is prime is called a $\mathcal{P}$-*radical prime submodule* of M.
(2.1.2) We say that N is $\mathcal{P}$-radically finite if there exists a finitely generated submodule F of M such that $F \subseteq N$ and $V_{\mathcal{P}}(F) = V_{\mathcal{P}}(N)$.
(2.1.3) If $\mathcal{P} =$ Spec(M), we say *radical submodule* or *radically finite*, respectively, in place of $\mathcal{P}$-*radical submodule* or $\mathcal{P}$-*radically finite*. Similarly, if $\mathcal{P} =$ Max(M) we say J-*radical submodule* or J-*radically finite* respectively for $\mathcal{P}$-radical submodule or $\mathcal{P}$-radically finite.

A well-known result of I. S. Cohen states that a ring R is Noetherian if each prime ideal of R is finitely generated, and this is extended to finitely generated R-modules in [**11**, **13**]. In analogy with Cohen's result, it is shown in [**29**, Corollary 2.4] that a commutative ring R has Noetherian spectrum if and only if each prime ideal is an RFG-ideal, where an ideal A is called an RFG-ideal if the radical of A is the radical of a finitely generated ideal. It is pointed out in [**29**], however, that the corresponding notion of JRFG-ideals for J-radical ideals does not work in the same way to characterize Noetherian J-spectrum, or equivalently, Noetherian maximal spectrum. This was repaired in [**31**], by replacing RFG-ideals and JRFG-ideals by radically finite ideals and J-radically finite ideals, respectively. The following is the analogue of [**29**, Corollary 2.4] for modules. In the following, the hypothesis that M is finitely generated is used partly to ensure that $\mathrm{Spec}(M)$ is non-empty.

PROPOSITION 2.2. *Let R be a ring and let M be a finitely generated R-module. If $\mathcal{P} \subseteq \mathrm{Spec}(M)$ and M contains a submodule which is not $\mathcal{P}$-radically finite, then the set of non-$\mathcal{P}$-radically finite submodules contains maximal members, and any such maximal member is prime.*

Proof. The set of non-$\mathcal{P}$-radically finite elements is easily seen to be inductive. Let N be maximal in this set. If N is not prime, there exists $x \in M \setminus N$ and $a \in R$ such that $ax \in N$ and $a \notin (N :_R M)$. Then $(N :_M a)$ and $N + aM$ are $\mathcal{P}$-radically finite. Let $H \subseteq N + aM$ be a finitely generated submodule such that $V_{\mathcal{P}}(H) = V_{\mathcal{P}}(N + aM)$. Writing the finitely many generators h_i of H as $h_i = x_i + am_i$, $x_i \in N$, $m_i \in M$ and letting F_1 be the submodule generated by the x_i, we see that there are finitely generated submodules $F_1 \subseteq N$, $F_2 \subseteq (N :_M a)$ such that

$$V_{\mathcal{P}}(F_1 + aM) = V_{\mathcal{P}}(N + aM) \text{ and } V_{\mathcal{P}}(F_2) = V_{\mathcal{P}}(N :_M a).$$

Then $F_1 + aF_2 \subseteq N$ and $F_1 + aF_2$ is finitely generated. Suppose $P \in V_{\mathcal{P}}(F_1 + aF_2)$. Since $aF_2 \leq P$, either $F_2 \subseteq P$ or $aM \subseteq P$. In case $F_2 \subseteq P$, we get $N \subseteq (N :_M a) \subseteq P$ by the choice of F_2. In case $aM \subseteq P$, we get $F_1 + aM \subseteq P$. Thus $N \subseteq N + aM \subseteq P$. Thus $V_{\mathcal{P}}(F_1 + aF_2) = V_{\mathcal{P}}(N)$, and therefore $\mathcal{P}\text{-rad}(N) = \mathcal{P}\text{-rad}(F_1 + aF_2)$. Thus $\mathcal{P}\text{-rad}(N)$ is $\mathcal{P}$-radically finite, contradicting the choice of N. $\square$

The following theorem extends results in [**7**, **29**] on ideals to submodules, and refines these results even in the case of ideals by choosing an arbitrary subset $\mathcal{P}$ of $\mathrm{Spec}(M)$. For this, we note that if M is an R-module and $\mathcal{P} \subseteq \mathrm{Spec}(M)$, the intersection of a chain of $\mathcal{P}$-radical prime submodules is easily seen to be a $\mathcal{P}$-radical prime submodule. Thus if N is a submodule of M, each $\mathcal{P}$-radical prime submodule in $V(N)$ contains a $\mathcal{P}$-radical prime submodule which is minimal among $\mathcal{P}$-radical prime submodules in $V(N)$. Let $\min_{\mathcal{P}}(N)$ denote the set of minimal $\mathcal{P}$-radical primes $P \in V(N)$. Observe that for submodules N_1 and N_2 of M, $\mathcal{P}\text{-rad}(N_1) = \mathcal{P}\text{-rad}(N_2)$ if and only if $\min_{\mathcal{P}}(N_1) = \min_{\mathcal{P}}(N_2)$.

THEOREM 2.3. *Let R be a commutative ring and let M be a finitely generated R-module. The following statements are equivalent:*

(1) *M has ACC on $\mathcal{P}$-radical submodules.*
(2) *Each submodule of M is $\mathcal{P}$-radically finite.*
(3) *Each prime submodule of M is $\mathcal{P}$-radically finite.*
(4) *M has ACC on $\mathcal{P}$-radical prime submodules and $\min_{\mathcal{P}}(N)$ is finite for each submodule N of M.*
(5) *M has ACC on $\mathcal{P}$-radical prime submodules and $\min_{\mathcal{P}}(N)$ is finite for each finitely generated submodule N of M.*

Proof. For (1) $\Rightarrow$ (2), suppose there exists a submodule N of M that is not $\mathcal{P}$-radically finite. Let $x_1 \in N$. Then $N \nsubseteq \mathcal{P}\text{-rad}(Rx_1)$. There exists $x_2 \in N$ with $x_2 \notin \mathcal{P}\text{-rad}(Rx_1)$. Then $\mathcal{P}\text{-rad}(Rx_1) \subsetneq \mathcal{P}\text{-rad}(Rx_1 + Rx_2)$ and $N \nsubseteq \mathcal{P}\text{-rad}(Rx_1 + Rx_2)$, and so on, a contradiction to M having ACC on $\mathcal{P}$-radical submodules.

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For (2) $\Rightarrow$ (1), let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain of $\mathcal{P}$-radical submodules. Let $N = \cup_{i=1}^{\infty} N_i$. Then $\mathcal{P}$-rad$(N) = \mathcal{P}$-rad(C) for some finitely generated submodule $C \subseteq N$, since N is $\mathcal{P}$-radically finite. Then $C \subseteq N_j$ for some j and $\mathcal{P}$-rad$(C) \subseteq N_j \subseteq \mathcal{P}$-rad$(N) = \mathcal{P}$-rad$(C)$. Therefore $N_j = N_k$ for all $k \geq j$.

The implication (2) $\Rightarrow$ (3) is clear, and (3) $\Rightarrow$ (2) follows from Proposition 2.2.

For (1) $\Rightarrow$ (4), since $\min_{\mathcal{P}}(N) = \min_{\mathcal{P}}(\mathcal{P}\text{-rad}(N))$ for each submodule N of M, then if $\min_{\mathcal{P}}(N)$ is infinite for some submodule N of M, then by (1) there exist a $\mathcal{P}$-radical submodule N of M which is maximal among such $\mathcal{P}$-radical submodules N of M with $\min_{\mathcal{P}}(N)$ infinite. Then N is not prime. Otherwise, $\min_{\mathcal{P}}(N) = \{N\}$ is finite.

So there exists $h \in M \setminus N$ and $a \in R$ such that $ah \in N$ and $aM \nsubseteq N$. Also, if $P \in \mathrm{Spec}(M)$ satisfies $N \subseteq P$ and $h \notin P$, then since $ah \in N \subseteq P$, $aM \subseteq P$. Therefore $V(N) = V(N + Rh) \cup V(N + aM)$. Thus $\min_{\mathcal{P}}(N) \subseteq \min_{\mathcal{P}}(N + Rh) \cup \min_{\mathcal{P}}(N + aM)$, and by the choice of N, $\min_{\mathcal{P}}(N + Rh)$ and $\min_{\mathcal{P}}(N + aM)$ are finite, a contradiction to the assumption that $\min_{\mathcal{P}}(N)$ is infinite.

The implication (4) $\Rightarrow$ (5) is clear.

For (5) $\Rightarrow$ (3), let $N \in \mathrm{Spec}(M)$. Suppose that N is not $\mathcal{P}$-radically finite. Then $N \neq 0$. So there exists a non-zero $x \in N$. For at least one of the finitely many $Q \in \min_{\mathcal{P}}(Rx)$, we have $N \nsubseteq Q$. Indeed, if $N \subseteq Q$ for each such Q, we would have $\min_{\mathcal{P}}(Rx) = \min_{\mathcal{P}}(N)$. Then $V_{\mathcal{P}}(N) = V_{\mathcal{P}}(Rx)$, contradicting the assumption that N is not $\mathcal{P}$-radically finite. Then for each of the finitely many $Q \in \min_{\mathcal{P}}(Rx)$ such that $N \nsubseteq Q$, there exists an element $x_Q \in N \setminus Q$. Let A_1 be the submodule of N generated by $\{x\} \cup \{x_Q \mid Q \in \min_{\mathcal{P}}(Rx),\ N \nsubseteq Q\}$.

Then A_1 is a finitely generated submodule of N and $A_1 \neq N$ since N is not $\mathcal{P}$-radically finite. As above, there exists at least one $Q_1 \in \min_{\mathcal{P}}(A_1)$ with $N \nsubseteq Q_1$. Further, each member Q_1 of $\min_{\mathcal{P}}(A_1)$ with $N \nsubseteq Q_1$ contains a member Q of $\min_{\mathcal{P}}(Rx)$. Since $N \nsubseteq Q_1$, then $N \nsubseteq Q$. But then since $x_Q \in Q_1 \setminus Q$, $Q \subsetneq Q_1$.

For each of the finitely many $Q_1 \in \min_{\mathcal{P}}(A_1)$ with $N \nsubseteq Q_1$, there exists an element $x_{Q_1} \in N \setminus Q_1$. Let A_2 be the submodule of N generated by $A_1 \cup \{x_{Q_1} \mid Q_1 \in \min_{\mathcal{P}}(A_1),\ N \nsubseteq Q_1\}$. Then $A_2 \subsetneq N$ and there exists a $Q_2 \in \min_{\mathcal{P}}(A_2)$ such that $N \nsubseteq Q_2$. For each such $Q_2 \in \min_{\mathcal{P}}(A_2)$, we have $Q \subsetneq Q_1 \subsetneq Q_2$ for some $Q_1 \in \min_{\mathcal{P}}(A_1)$ with $N \nsubseteq Q_1$ and $Q \in \min_{\mathcal{P}}(Rx)$ with $N \nsubseteq Q$.

If this process did not stop, then since $\min_{\mathcal{P}}(Rx)$ is finite, we would generate infinitely many chains of $\mathcal{P}$-radical primes, all starting with the same $Q \in \min_{\mathcal{P}}(Rx)$ with $N \nsubseteq Q$. But then we would have infinitely many chains of $\mathcal{P}$-radical primes, all starting with the same two $\mathcal{P}$-radical prime submodules $Q \subsetneq Q_1$, with $Q \in \min_{\mathcal{P}}(Rx)$ with $N \nsubseteq Q$, $Q_1 \in \min_{\mathcal{P}}(A_1)$ with $N \nsubseteq Q_1$ and so on, contradicting the hypothesis that M has ACC on $\mathcal{P}$-radical prime submodules. Thus the process must stop. Therefore for some k, each $Q_k \in \min_{\mathcal{P}}(A_k)$ contains N. Thus $\min_{\mathcal{P}}(A_k) = \min_{\mathcal{P}}(N)$. Then $V_{\mathcal{P}}(N) = V_{\mathcal{P}}(A_k)$, contradicting the assumption that N is not $\mathcal{P}$-radically finite. $\square$

In the case $\mathcal{P} = \mathrm{Spec}(M)$, the above result is related to [**15**, Corollary to Theorem 5] which assumes the module M is Laskerian and that each prime ideal $\mathfrak{p}$ of R containing the annihilator of M has finite height.

In the case $\mathcal{P} = \mathrm{Spec}(M)$, the following corollary is given in [**22**, Theorem 4.2] under the hypothesis that M is Noetherian, but without the assumption that R is commutative.

COROLLARY 2.4. *Let R be a commutative ring, let M be a finitely generated R-module and let $\mathcal{P} \subseteq \mathrm{Spec}(M)$. If M has ACC on $\mathcal{P}$-radical submodules, then* $\min_{\mathcal{P}}(N)$ *is finite for*

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each submodule N of M. In particular, each $\mathcal{P}$-radical submodule of M is an intersection of finitely many $\mathcal{P}$-radical prime submodules.

If in the above corollary we take $M = R$, $\mathcal{P} = \mathrm{Spec}(R)$ and I is an ideal of R, it is immediate that each of the finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of I is irredundant in the representation $\mathrm{rad}_R(I) = \cap_{i=1}^n \mathfrak{p}_i$. If we replace R with a general finitely generated R-module M and I with a submodule N of M, this no longer holds. That is, some of the prime submodules $P_1, \ldots, P_n$ of M which are minimal over N can be redundant in the representation $\mathrm{rad}_M(N) = \cap_{i=1}^n P_i$ [**24**, Example 1.9].

3. Uniqueness properties of finite prime decompositions. In this section, the uniqueness properties are considered of a representation of a radical submodule N of M as a finite intersection of prime submodules of M. For simplicity, we restrict to the case that $\mathcal{P} = \mathrm{Spec}(M)$, and thus drop the $\mathcal{P}$. Before considering these, we recall the definitions of the two types of associated prime ideals that are relevant to our discussion. For further properties of these and some other types of associated prime ideals see, for example, [**8**].

DEFINITION 3.1. Let M be an R-module and $\mathfrak{p} \in \mathrm{Spec}(R)$.
(3.1.1) If $\mathfrak{p} = \mathrm{rad}(0 :_R m)$ for some $m \in M$, $\mathfrak{p}$ is called a Z-S *associated prime* of M (after Zariski–Samuel).
(3.1.2) If $\mathfrak{p} = (0 :_R m)$ for some $m \in M$, $\mathfrak{p}$ is called a *Bourbaki associated prime* of M.

If N is a submodule of the R-module M and S is a multiplicative subset of R, we denote the *isolated S-component* $\cup_{s \in S}(N :_M s)$ of N by $N_{(S)}$. If $S = R \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of R, we write $N_{(\mathfrak{p})}$ instead of $N_{(S)}$.

THEOREM 3.2. *Let R be a commutative ring and let N be a submodule of the R-module M such that M/N is finitely generated. Then the following hold:*

(1) $\mathrm{rad}_M(N) = \cap\{(N + \mathfrak{p}M)_{(\mathfrak{p})} \mid \mathfrak{p} \in V(N :_R M)\}$ *by* [**30**, Corollary 1.2].
(2) *For each $\mathfrak{p} \in V(N :_R M)$, $(N + \mathfrak{p}M)_{(\mathfrak{p})}$ is the unique smallest $\mathfrak{p}$-prime submodule of M containing N by* [**17**, Corollary 4.4(2)].
(3) *If $\mathfrak{p}$ is minimal in $V(N :_R M)$, then the $\mathfrak{p}$-prime submodule $(N + \mathfrak{p}M)_{(\mathfrak{p})}$ is the unique $\mathfrak{p}$-prime submodule of M which is minimal over N* by [**17**, Corollary 4.4(3)].

The following theorem summarizes the uniqueness conditions of finite prime decompositions of a radical submodule of a finitely generated R-module M. Its proof is similar to the well-known uniqueness result on primary decomposition. See, for example, [**1**, p. 57] or [**34**, p. 253]. The existence of the representation of $\mathrm{rad}_M(N)$ in Theorem 3.3 holds if M has ACC on radical submodules by Corollary 2.4.

THEOREM 3.3. *Let N be a submodule of a finitely generated R-module M and suppose* $\mathrm{rad}_M(N) = N_1 \cap \ldots \cap N_k$ *for finitely many prime submodules N_i. Then this representation can be reduced to a normal decomposition, that is one in which the prime ideals $\mathfrak{p}_i = (N_i :_R M)$ are distinct and no N_i contains the intersection of the others. In this case, the following hold:*

(a) *The prime ideals $(N_i :_R M) = \mathfrak{p}_i$ are uniquely determined as the set* $\mathrm{Ass}_B(M/\mathrm{rad}_M(N)) = \mathrm{Ass}_{ZS}(M/\mathrm{rad}_M(N))$.
(b) *The prime ideals $\mathfrak{p}$ which are minimal in* $\mathrm{Supp}(M/\mathrm{rad}_M(N)) = \mathrm{Supp}(M/N)$ *are in the set* $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$.

(c) *For each such* $\mathfrak{p}_i$*, which is minimal in* $\mathrm{Supp}(M/N)$*, the* $\mathfrak{p}_i$*-prime component* N_i *is uniquely determined as* $(N + \mathfrak{p}_i M)_{(\mathfrak{p}_i)}$.

(d) *For each* $\mathfrak{p}_i \in \mathrm{Ass}_B(M/\mathrm{rad}_M(N)) = \mathrm{Ass}_{ZS}(M/\mathrm{rad}_M(N))$*, which is not minimal in* $\mathrm{Supp}(M/N)$*,* $(N + \mathfrak{p}_i M)_{(\mathfrak{p}_i)}$ *is the unique smallest choice for the* $\mathfrak{p}_i$*-prime component* N_i.

The following result follows from Corollary 2.4, Theorem 3.3 and Theorem 3.2.

PROPOSITION 3.4. *Assume* M *is finitely generated with ACC on radical submodules. Then* $\mathfrak{p} \in \mathrm{Ass}_{ZS}(M/\mathrm{rad}_M(N))$ *if and only if a* $\mathfrak{p}$*-prime submodule* P *of* M *occurs in a normal prime decomposition of* $\mathrm{rad}_M(N)$*. In this case,* $\mathfrak{p} \in \mathrm{Ass}_B(M/\mathrm{rad}_M(N))$ *and we may choose* P *to be* $(N + \mathfrak{p}M)_{(\mathfrak{p})}$.

4. Radicals of submodules and the symmetric algebra. Let M be an R-module and let $S_R(M) = \oplus_{i\geq 0} S^i(M)$ be the symmetric algebra of M. In this section, $S_R(M)$ is used to show that if $\mathrm{Spec}(R)$ is Noetherian, then $\mathrm{Spec}(M)$ is Noetherian for each finitely generated R-module M. For this we use some results from [**26**].

DEFINITION 4.1. [**26**, Definition 2.1] Let N be a $\mathfrak{p}$-prime submodule of the finitely generated R-module M and let $T = R \setminus \mathfrak{p}$. The $\mathfrak{p}$-component $((\mathfrak{p}, N)T^{-1}S_R(M)) \cap S_R(M) = \mathcal{E}_N$ of $(\mathfrak{p}, N)S_R(M)$ in $S_R(M)$ will be called the *expansion* of N. Let $\mathcal{E}_N^i = \mathcal{E}_N \cap S^i(M)$.

PROPOSITION 4.2. [**26**, Proposition 2.1 and Theorem 2.2] *Let* N *be a* $\mathfrak{p}$*-prime submodule of* M *and* $T = R \setminus \mathfrak{p}$*. Then*

(1) $\mathcal{E}_N$ *is a homogeneous ideal.*

(2) $\mathcal{E}_N^0 = \mathfrak{p}$ *and* $\mathcal{E}_N^1 = N$.

(3) *If* M *is a finitely generated* R*-module and* N *is a prime submodule, then* $\mathcal{E}_N$ *is a prime ideal of the symmetric algebra* $S_R(M)$.

In the following proposition we collect some simple results relating prime submodules of a finitely generated R-module M to the prime ideals of $S_R(M)$.

PROPOSITION 4.3. *Let* M *be a finitely generated* R*-module:*

(1) *If* $\mathcal{P}$ *is a prime (resp. primary) ideal of* $S_R(M)$*, then* $\mathcal{P} \cap M = N$ *is either* M *or a* $\mathfrak{p}$*-prime (resp.* $\mathfrak{p}$*-primary) submodule of* M *where* $\mathfrak{p} = \mathcal{P} \cap R$.

(2) *If* $\mathcal{P}$ *is a prime ideal of* $S_R(M)$ *and* $\mathcal{P} \cap M = N \neq M$*, then* $\mathcal{E}_N \subseteq \mathcal{P}$.

(3) *The mapping* $N \mapsto \mathcal{E}_N$ *is an injection from* $\mathrm{Spec}(M)$ *to* $\mathrm{Spec}(S_R(M))$ [**26**, Corollary 2.3].

(4) *If* $N \neq M$ *is a submodule of* M*,* $\mathrm{rad}_M(N) = M \cap \mathrm{rad}(N \cdot S_R(M))$ [**26**, Corollary 2.4].

Proof

(1) Assume that $N \neq M$ and let $rm \in N$, $r \in R$, $m \in M \setminus N$. Then by the definition of prime (resp. primary) ideal of $S_R(M)$, $r \in \mathcal{P}$, (resp. $r^n \in \mathcal{P}$ for some $n \in \mathbb{N}$). Thus $rM \subseteq N$ (resp. $r^n M \subseteq N$ for some n). Thus N is a prime (resp. primary) submodule of M. If $r \in \mathfrak{p} = \mathcal{P} \cap R$, then $\mathfrak{p}M \subseteq \mathcal{P} \cap M = N$. So $\mathfrak{p} \subseteq (N :_R M)$. Conversely, if $r \in (N :_R M)$, then $rM \subseteq \mathcal{P} \cap M = N$, and since $N \subsetneq M$ and $\mathcal{P}$ is a prime ideal of $S_R(M)$, then $r \in \mathcal{P} \cap R = \mathfrak{p}$. So $\mathfrak{p} = (N :_R M)$.

(2) Let $\mathcal{P} \cap R = \mathcal{P} \cap S^0(M) = \mathfrak{p}$ and $T = R \setminus \mathfrak{p}$. Then, clearly, $(T^{-1}((\mathfrak{p}, N)S_R(M)) \cap S_R(M) \subseteq \mathcal{P}$. □

COROLLARY 4.4. *If R has ACC on radical ideals, then each finitely generated R-module M has ACC on radical submodules.*

Proof. Since $\mathrm{Spec}(R)$ is Noetherian and M is finitely generated, $\mathrm{Spec}(S_R(M))$ is Noetherian [**29**, Corollary 2.6]. Thus $S_R(M)$ has ACC on radical ideals. But then M has ACC on radical submodules by Proposition 4.3(4). □

In [**29**], a ring R is said to be an *FC-ring* if each ideal I of R has only finitely many minimal prime ideals. By [**29**, Example 2.9], R can be an FC-ring without the polynomial ring $R[X]$ in one variable being an FC-ring. In [**27**] a ring R is said to be an *FGFC-ring* if each finitely generated ideal I of R has only finitely many minimal prime ideals, and it was shown in [**27**, Theorem 2.3] that a ring R is an FGFC-ring if and only if $R[X]$ is.

COROLLARY 4.5. *Let R be an FGFC-ring, let M be a finitely presented R-module and let N be a finitely generated submodule of M. Then there are only finitely many prime submodules of M which are minimal over N. In particular,* $\mathrm{rad}_M(N)$ *is a finite intersection of prime submodules.*

Proof. Since M is finitely presented, by writing $M = F/H$ where F is free of rank n and H is a finitely generated submodule of F, we get $S_R(M) \cong S_R(F/H) \cong R[X_1, \ldots, X_n]/I$ where I is a finitely generated ideal of $R[X_1, \ldots, X_n]$. Thus $S_R(M)$ is an FGFC-ring by [**27**, Theorem 2.3 and Proposition 2.2(b)]. If P is a prime submodule of M which is minimal over N, then by Proposition 4.2 and Proposition 4.3(2), $\mathcal{E}_P$ is the unique smallest prime ideal H of $S_R(M)$ with $H \cap M = P$. Since there are only finitely many prime ideals of $S_R(M)$ which are minimal over $NS_R(M)$, the result follows. □

Recall that an R-module M is said to be *Laskerian* if it is finitely generated and each submodule of M is a finite intersection of primary submodules of M [**4**, Section IV.2, Exercise 23], and a ring R is Laskerian if it is Laskerian as an R-module. It is shown in [**6**, Theorem 4] that a Laskerian ring has Noetherian spectrum. Although the somewhat intricate proof given there does not appear to readily adapt to give the analogous result for modules, the module result does hold, as we show next.

COROLLARY 4.6. *If the R-module M is Laskerian, then* $\mathrm{Spec}(M)$ *is Noetherian.*

Proof. Let I be the annihilator of M. Then R/I is a submodule of a direct sum of finitely many copies of M, and hence R/I is a Laskerian ring. Thus by [**6**, Theorem 4], $\mathrm{Spec}(R/I)$ is Noetherian, and then by Corollary 4.4, $\mathrm{Spec}(M)$ is Noetherian. □

PROPOSITION 4.7. *Let $N \neq M$ be a submodule of a finitely generated R-module M. Then* $\mathrm{rad}_M(N)$ *is a prime submodule of M if and only if* $\mathrm{rad}(N \cdot S_R(M))$ *is a prime ideal of $S_R(M)$.*

Proof. $(\Rightarrow)$ Suppose $\mathrm{rad}_M(N) = P$ is prime. Then P is the smallest prime submodule of M containing N. By parts (2) and (4) of Proposition 4.3, $\mathcal{E}_P$ is the smallest prime ideal of $S_R(M)$ containing $N \cdot S_R(M)$. Thus $\mathcal{E}_P = \mathrm{rad}(N \cdot S_R(M))$.

$(\Leftarrow)$ If $N \cdot S_R(M)$ has prime radical $\mathcal{P}$, then by Proposition 4.3(4), $\mathrm{rad}_M(N) = \mathcal{P} \cap M \neq M$. Thus $\mathrm{rad}_M(N) = \mathcal{P} \cap M$ is a a prime submodule of M by Proposition 4.3(1). □

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