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NUMERICAL SOLUTION OF AN OPTIMAL CONTROL PROBLEM WITH VARIABLE TIME POINTS IN THE OBJECTIVE FUNCTION

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Abstract

In this paper, we consider the numerical solution of a class of optimal control problems involving variable time points in their cost functions. The control enhancing transform is first used to convert the optimal control problem with variable time points into an equivalent optimal control problem with fixed multiple characteristic time (MCT). Using the control parametrization technique, the time horizon is partitioned into several subintervals. Let the partition points also be taken as decision variables. The control functions are approximated by piecewise constant or piecewise linear functions in accordance with these variable partition points. We thus obtain a finite dimensional optimization problem. The control parametrization enhancing control transform (CPET) is again used to convert approximate optimal control problems with variable partition points into equivalent standard optimal control problems with MCT, where the control functions are piecewise constant or piecewise linear functions with pre-fixed partition points. The transformed problems are essentially optimal parameter selection problems with MCT. The gradient formulae for the objective function as well as the constraint functions with respect to relevant decision variables are obtained. Numerical examples are solved using the proposed method.

1. Introduction

Optimal control theory has many successful practical applications in areas ranging from economics to various engineering disciplines. Since most practical problems are rather too complex to allow analytical solutions, numerical methods are unavoidable for solving these complex practical problems. There are numerous computational

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methods for solving various practical optimal control problems. For details, see [3, 10–12] and [9]. In particular, the control parametrization technique is used in [11], while the control parametrization enhancing technique (CPET) is introduced in [7].

In this paper, we consider the numerical solution of a optimal control problem involving variable time points. Its motivation comes from a situation in which a target is moving as a function of time in a space. A space-craft is launched into the space and its trajectory is maneuvered by certain control actions. The mission of the space-craft is to take measurements at various time points over a given mission period which is divided into a number of time subintervals. Each of the time points is to be selected from the respective time subinterval. Suppose we wish to take the measurement in each time subinterval at the time point at which the distance between the moving target and the space-craft is minimum. Let the sum of these distances be the cost function. Then we have an optimal control problem, where the control actions of the space-craft and the variable time points are to be chosen optimally with respect to the given cost function. A different problem, also involving variable time points, has been discussed in [2] and [1] from the theoretical point of view. In that problem, each equation in the dynamical system is defined on an interval with variable initial and termination time points which are decision variables. The dynamical system of the problem considered here has fixed initial and termination time points, but has some variable observation time points within the time interval. Also, the main focus of the present paper is to present some efficient techniques for the numerical solution of optimal control problems with variable time points, while [2] and [1] are only concerned with the theory of necessary optimality conditions for their problems. The rest of our paper is organized as follows.

A general class of optimal control problems containing the situation just mentioned above as an example is formulated in Section 2, where the cost function includes multiple variable time points. The control parametrization enhancing technique is used to transform the problem into a form solvable by the control parametrization technique in Section 3. More specifically, the control parametrization transform [7] is first used to convert the optimal control problem with variable time points to an equivalent optimal control problem with fixed multiple characteristic time (MCT) (*cf.* [8]). Using the control parametrization technique [11], the time horizon is partitioned into several subintervals. The control functions are approximated by piecewise constant or piecewise linear functions with pre-fixed partition points. We thus obtain a finite dimensional optimization problem. Clearly, a finer partition would produce a more accurate solution. It is also intuitively clear that the number of partitions can be much reduced if the partition points are taken as decision variables. However, it is pointed out in [12] that there are several numerical difficulties associated with such a direct approach. The control parametrization enhancing control transform (CPET) (*cf.* [12]) is used to convert approximate optimal control problems with variable partition points into equivalent standard optimal control problems with MCT, where the control functions are piecewise constant or piecewise linear continuous functions with pre-fixed partition points. The transformed problems are essentially optimal parameter selection problems with MCT. The gradient formulae for the objective function as well as the constraint functions with respect to relevant decision variables are derived in Section 4. With these gradient formulae, each of the transformed optimal control problems is solvable as an optimal parameter selection problem, and the software MISER 3.2 [6] can be modified for solving these optimal parameter selection problems. In Section 5, two examples are solved using the proposed method. Section 6 concludes the paper.

2. Problem formulation

Consider a process described by the following system of differential equations defined on $[0, T]$:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), z(t)), \tag{2.1}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2.2}$$

where T is a fixed terminal time, $\boldsymbol{x} = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, $\boldsymbol{u} = [u_1, \dots, u_m]^T \in \mathbb{R}^m$ and $z = [z_1, \dots, z_p]^T \in \mathbb{R}^p$ are, respectively, state, control and system parameters, while $\boldsymbol{f} = [f_1, \dots, f_n]^T \in \mathbb{R}^n$ is a continuously differentiable function with respect to all its arguments, and $\boldsymbol{x}^0$ is a given vector.

Let $\underline{\tau_i}$ and $\overline{\tau_i}$, $i = 0, 1, \dots, k$, be constants in the time interval $[0, T]$ such that

$$\underline{\tau_0} = \overline{\tau_0} = 0 < \underline{\tau_1} < \overline{\tau_1} < \underline{\tau_2} < \overline{\tau_2} < \cdots < \underline{\tau_k} < \overline{\tau_k} < T. \tag{2.3}$$

Furthermore, let a_i and b_i, $i = 1, \dots, s$, c_i, and d_i, $i = 1, \dots, m$, be fixed constants. Define

$$\begin{aligned}
\boldsymbol{V} &= \{\boldsymbol{t} = [t_1, \dots, t_k]^T \in \mathbb{R}^k : t_i \in [\underline{\tau_i}, \overline{\tau_i}],\ i = 1, \dots, k\},\\
\boldsymbol{Z} &= \{z = [z_1, \dots, z_r]^T \in \mathbb{R}^r : a_i \le z_i \le b_i,\ i = 1, \dots, r\},\\
\boldsymbol{U} &= \{\boldsymbol{u} = [u_1, \dots, u_m]^T \in \mathbb{R}^m : c_i \le u_i \le d_i,\ i = 1, \dots, m\}.
\end{aligned}$$

Any Borel measurable function $\boldsymbol{u} : [0, T] \to \boldsymbol{U}$ is called an admissible control. Let $\mathscr{U}$ be the class of all admissible controls. For each $(\boldsymbol{u}, z) \in \mathscr{U} \times \boldsymbol{Z}$, let $\boldsymbol{x}(\cdot \mid \boldsymbol{u}, z)$ denote the corresponding solution of the system (2.1)–(2.2).

Our optimal control problem may now be formally stated as: Given the dynamical system (2.1)–(2.2), find a $(\boldsymbol{t}, \boldsymbol{u}, z) \in \boldsymbol{V} \times \mathscr{U} \times \boldsymbol{Z}$ such that the cost function

$$g_0(\boldsymbol{t}, \boldsymbol{u}, z) = \sum_{i=1}^{k} \Phi_i(t_i, \boldsymbol{x}(t_i \mid \boldsymbol{u}, z)) \tag{2.4}$$

is minimized subject to the constraints

$$g_j(t_i, \boldsymbol{x}(t_i|\boldsymbol{u}, z), z) \le 0, \quad j = 1, \dots, q, \; i = 1, \dots, k, \tag{2.5}$$

$$a_i \le z_i \le b_i, \quad i = 1, 2, \dots, r, \tag{2.6}$$

$$c_i \le u_i(t) \le d_i, \quad i = 1, 2, \dots, m, \; t \in [0, T], \tag{2.7}$$

$$\underline{\tau_i} \le t_i \le \overline{\tau_i}, \quad i = 1, 2, \dots, k, \tag{2.8}$$

where $\Phi_i(t_i, \boldsymbol{x})$, $i = 1, 2, \dots, k$, and $g_j(t_i, \boldsymbol{x}, z) \le 0$, $j = 1, \dots, q$, $i = 1, \dots, k$, are continuously differentiable real valued functions on $[0, T] \times R^n$ and $[0, T] \times \mathbb{R}^n \times \mathbb{R}^r$, respectively. Let this optimal control problem be referred to as Problem (P).

3. Transformation

Let $s \in [0, k+1]$ be a new time variable and let $\boldsymbol{v}(s)$ be defined by

$$\boldsymbol{v}(s) = \sum_{i=1}^{k+1} v_i \cdot \chi_{[i-1,i)}(s), \tag{3.1}$$

where $\chi_{[i-1,i)}$ is the indicator function of the interval $[i-1, i)$ and v_i are nonnegative constants. Clearly, $v(s)$ is a nonnegative piecewise constant function, which is called the enhancing control, defined on $[0, k+1]$ with fixed switching points located at $\{1, 2, \dots, k\}$.

The control parametrization enhancing transform (CPET) maps $t \in [0, T]$ to $s \in [0, k+1]$ as follows:

$$\frac{dt}{ds} = \boldsymbol{v}(s), \qquad t(0) = 0,$$

where

$$\boldsymbol{v}(s) = \begin{cases} t_i - t_{i-1}, & s \in [i-1, i), \; i = 1, 2, \dots, k; \\ T - t_k, & s \in [k, k+1] \end{cases}$$

and satisfies

$$\underline{\tau_i} \le \int_0^i \boldsymbol{v}(s)\, ds \le \overline{\tau_i}, \quad i = 1, \dots, k, \quad \text{and} \tag{3.2}$$

$$\int_0^{k+1} \boldsymbol{v}(s)\, ds = T. \tag{3.3}$$

Such a function is called an enhancing control; let $\mathscr{V}^*$ denote the class of all such enhancing controls.

Under the CPET, the system dynamics change to

$$\frac{d}{ds}\begin{pmatrix} y(s) \\ t(s) \end{pmatrix} = v(s)\begin{pmatrix} f(t(s), y(s), w(s), z) \\ 1 \end{pmatrix}, \quad s \in [0, k+1], \tag{3.4}$$

with initial condition

$$\begin{pmatrix} y(0) \\ t(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \tag{3.5}$$

where $y(s) = x(t(s))$ and $w(s) = u(t(s))$. The constraints

$$c_i \le u_i(t) \le d_i, \quad i = 1, 2, \ldots, m, \ t \in [0, T], \tag{3.6}$$

reduce to

$$c_i \le w_i(s) \le d_i, \quad i = 1, 2, \ldots, m, \ s \in [0, k+1]. \tag{3.7}$$

Define $w(s) = (w_1(s), w_2(s), \ldots, w_m(s))$ where w_i, $i = 1, \ldots, m$, satisfy the constraints (3.7). Let $\mathscr{W}$ be the set of all such functions $w(s)$, and the Problem (P) is now transformed into the following optimal control problem: Given the dynamical system (3.4)–(3.5), find an admissible element $(v, w, z) \in \mathscr{V}^* \times \mathscr{W} \times Z$ such that the cost function:

$$\hat{g}_0(v, w, z) = \sum_{i=1}^{k} \hat{\Phi}_i \left(\sum_{j=1}^{i} v_j, y(i \mid w, z) z \right) \tag{3.8}$$

is minimized subject to the constraints:

$$\hat{g}_j \left(\sum_{j=1}^{i} v_j, y(i), z \right) \le 0, \quad j = 1, \ldots, l, \ i = 1, \ldots, k, \tag{3.9}$$

$$a_i \le z_i \le b_i, \qquad i = 1, 2, \ldots, r, \tag{3.10}$$

$$c_i \le w_i(s) \le d_i, \qquad i = 1, 2, \ldots, m, \ s \in [0, k+1], \tag{3.11}$$

$$v(s) \in \mathscr{V}^*. \tag{3.12}$$

This problem is referred to as Problem (P*).

An admissible element $(v, w, z) \in \mathscr{V}^* \times \mathscr{W} \times Z$ (respectively, (t, u, z)) is called a feasible element of Problem (P*) (respectively, Problem (P)) if the constraints (3.9)–(3.12) (respectively constraints (2.5)–(2.8)) are satisfied.

THEOREM 1. *Problem* (P*) *is equivalent to Problem* (P) *in the sense that* (v^*, w^*, z^*) *is a solution of Problem* (P*) *if and only if* (t^*, u^*, z^*) *is a solution of Problem* (P), *and* $\hat{g}_0(v^*, w^*, z^*) = g_0(t^*, u^*, z^*)$.

PROOF. Let $(\boldsymbol{t}_1, \boldsymbol{u}_1, \boldsymbol{z}_1) \in (\mathscr{V} \times \mathscr{U} \times Z)$ be a feasible element of Problem (P) and let $(\boldsymbol{v}_1, \boldsymbol{w}_1, \boldsymbol{z}_1) \in \mathscr{V}^* \times \mathscr{W} \times Z$ be the corresponding feasible element of problem (P*). Then it is easy to check that $\boldsymbol{x}(t)$ is the solution of (2.1)–(2.2) if and only if $\boldsymbol{y}(s)$ is the solution of (3.4)–(3.5), and

$$g_0(\boldsymbol{t}_1, \boldsymbol{u}_1, \boldsymbol{z}_1) = \sum_{i=1}^{k} \Phi(t_i, \boldsymbol{x}(t_i \mid \boldsymbol{u}_1, \boldsymbol{z}_1))$$
$$= \sum_{i=1}^{k} \hat{\Phi}_i \left(\sum_{j=1}^{i} v_j, \boldsymbol{y}(i \mid \boldsymbol{w}_1, \boldsymbol{z}_1), \boldsymbol{z}_1 \right) = \hat{g}_0(\boldsymbol{v}_1, \boldsymbol{w}_1, \boldsymbol{z}_1). \tag{3.13}$$

Hence the results follow readily.

In Problem (P) the cost function (2.4) is to be minimized with respect to $(\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{z}) \in (\mathscr{V} \times \mathscr{U} \times Z)$ where $\boldsymbol{t} = [t_1, \dots, t_k]$ and t_i, $i = 1, \dots, k$, are switching times. On the other hand, the cost function (3.8) in Problem (P*) is to be minimized with respect to $(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}) \in \mathscr{V}^* \times \mathscr{W} \times Z$, where $\boldsymbol{v}(t)$ is a nonnegative piecewise constant function. Since Problem (P) is equivalent to Problem (P*), we choose to solve Problem (P*) which is an optimal control problem with multiple characteristic times (see [8]). The main reason is that Problem (P*) is numerically more tractable, as it does not involve variable switching times.

In the classical control parametrization technique, each control function $\boldsymbol{w}_i(s)$ is approximated by a zeroth order or first order spline function (that is, a piecewise constant function or a piecewise linear continuous function) defined on a set of knots $\{0 = s_0^i, s_1^i, \dots, s_{p_i}^i = (k+1)\}$. Note that each component may have a different set of knots and the knots are not necessarily equally spaced. For the case of piecewise constant basis functions, we write the i-th control function as the sum of basis functions with coefficients or parameters $\{\sigma_{ij},\ j = 1, 2, \dots, p_i\}$:

$$\boldsymbol{w}_i(s) = \sum_{j=1}^{p_i} \sigma_{ij} B_{ij}^{(0)}(s),$$

where $B_{ij}^{(0)}(s)$ is the indicator function for the j-th interval of the i-th set of knots defined by

$$B_{ij}^{(0)}(s) = \begin{cases} 1, & s_{j-1}^i \le s \le s_j^i; \\ 0, & \text{otherwise.} \end{cases}$$

For piecewise linear continuous basis functions, we write the i-th control function as:

$$\boldsymbol{w}_i(s) = \sum_{j=0}^{p_i} \sigma_{ij} B_{ij}^{(1)}(s),$$

where $B_{ij}^{(1)}(s)$ are the witch's hat functions defined by

$$B_{ij}^{(1)}(s) = \begin{cases} (s - s_1^i)/(s_0^i - s_1^i), & s \in [s_0^i, s_1^i]; \\ 0, & \text{otherwise}, \end{cases}$$

$$B_{ij}^{(1)}(s) = \begin{cases} (s - s_{j-1}^i)/(s_j^i - s_{j-1}^i), & s \in [s_{j-1}^i, s_j^i]; \\ (s - s_{j+1}^i)/(s_j^i - s_{j+1}^i), & s \in [s_j^i, s_{j+1}^i]; \\ 0, & \text{otherwise}, \end{cases}$$

$$B_{ip_i}^{(1)}(s) = \begin{cases} (s - s_{k_i-1}^i)/(s_{k_i}^i - s_{k_i-1}^i), & s \in [s_{p_i-1}^i, s_{p_i}^i]; \\ 0, & \text{otherwise}. \end{cases}$$

Thus $\boldsymbol{w}(s)$ can be uniquely identified with a control parameter vector $\boldsymbol{\sigma}$ and vice versa with:

$$\boldsymbol{\sigma} = \left[(\boldsymbol{\sigma}^1)^T, (\boldsymbol{\sigma}^2)^T, \ldots, (\boldsymbol{\sigma}^m)^T\right]^T, \quad \boldsymbol{\sigma}^i = [\sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,p_i}]^T$$

which satisfy conditions:

$$c_i \le \sigma_{ij} \le d_i, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, p_i. \tag{3.14}$$

Let $\boldsymbol{\Sigma}$ denote the set of all such control parameter vectors $\boldsymbol{\sigma}$.

We now apply the Control Parametrization Enhancing Transform (CPET) to Problem (P*). Let q be the second new time scale which varies from 0 to $k+1$. Then the transformation from $s \in [0, k+1]$ to $q \in [0, k+1]$ can be defined by the differential equation:

$$\frac{ds(q)}{dq} = \eta(q), \quad s(0) = 0,$$

where the scaling function $\eta(q)$ is called the enhancing control. It is a piecewise constant function with possible discontinuities at the pre-fixed knots $\xi_0, \ldots, \xi_M$, that is, $\eta(q) = \sum_{i=1}^M \eta_i \chi_i(q)$, where $\chi_i(q)$ is the indicator function defined by

$$\chi_i(q) = \begin{cases} 1, & \text{if } q \in [\xi_{i-1}, \xi_i]; \\ 0, & \text{otherwise}. \end{cases}$$

Clearly,

$$s(q) = \int_0^q \eta(\nu)\, d\nu = \sum_{j=1}^{i-1} \eta_j (\xi_j - \xi_{j-1}) + \eta_i (q - \xi_{i-1}).$$

Let $\boldsymbol{\Theta}$ denote the class of all such enhancing controls $\eta(q)$ satisfying

$$\eta(i) = \int_0^i \eta(\nu)\, d\nu = i, \quad i = 1, \ldots, k+1. \tag{3.15}$$

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Under the CPET, the system dynamics change to:

$$\frac{d}{dq}\begin{pmatrix}\hat{y}(q)\\ s(q)\end{pmatrix} = \eta(q)\begin{pmatrix}v(s(q))f(t(s(q)), \hat{y}(q), \sigma, z)\\ v(s(q))\end{pmatrix}, \quad q \in [0, k+1], \tag{3.16}$$

with initial conditions

$$\begin{pmatrix}\hat{y}(0)\\ s(0)\end{pmatrix} = \begin{pmatrix}x_0\\ 0\end{pmatrix}, \tag{3.17}$$

where $\hat{y}(s) = x(t(s(q)))$.

The constraints (3.7) are reduced to (3.14). Problem (P*) is now transformed into the following optimal control problem: Given the dynamical system (3.16)–(3.17) find an admissible element $(v, \eta, \sigma, z) \in \mathcal{V}^* \times \Theta \times \Sigma \times Z$ such that the cost function

$$\hat{\hat{g}}_0(v, \eta, \sigma, z) = \sum_{i=1}^{k} \hat{\hat{\Phi}}_i\left(\sum_{j=1}^{i} v_j, \hat{y}(i \mid \eta, \sigma, z)z\right) \tag{3.18}$$

is minimized subject to the constraints:

$$\hat{\hat{g}}_j\left(\sum_{j=1}^{i} v_j, \hat{y}(i \mid \eta, \sigma, z), z\right) \leq 0, \qquad j = 1, \dots, l,\ i = 1, \dots, k, \tag{3.19}$$

$$a_i \leq z_i \leq b_i, \qquad i = 1, \dots, r, \tag{3.20}$$

$$c_i \leq \sigma_{ij} \leq d_i, \qquad i = 1, \dots, k,\ j = 1, \dots, p_i, \tag{3.21}$$

$$v(s) \in \mathcal{V}^*, \quad \eta(q) \in \Theta. \tag{3.22}$$

This problem is referred to as Problem (P**). Problem (P**) is an approximate optimal control problem with MCT, where the control functions are piecewise constant or piecewise continuous functions with pre-fixed partition points. Hence it can be viewed as an optimal parameter selection problem with MCT.

4. Gradient formulae

To solve Problem (P**), we need to derive the gradient formulae of the cost function as well as the constraint functions $\hat{\hat{g}}_j\left(\sum_{j=1}^{i} v_j, \hat{y}(i \mid \eta, \sigma, z), z\right)$, $j = 0, 1, \dots, q$, $i = 1, \dots, k$, (3.9). Let us do this for a slightly more general problem, and hence the notation used in this section is applicable only to this section. For example, $\tau = (\tau_1, \tau_2, \dots, \tau_k)$ is used to denote the vector of multiple characteristic times. The parameter set (v, η, σ, z) is replaced by z, and the cost functional is generalized to

$$g_0(z) = \sum_{i=1}^{k} \Phi_{0,i}\big(\tau_i, x(\tau_i \mid z), z\big) + \int_0^T h_0\big(t, x(t), z\big)\, dt, \tag{4.1}$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_k < \tau_{k+1} = T$. The state is described by the following system of differential equations:

$$\dot{x}(t) = f(t, x(t), z) \tag{4.2}$$

with the initial conditions

$$x(0) = x^0(z). \tag{4.3}$$

Let $\mathscr{Z}$ replace $(\mathscr{V}^* \times \Theta \times \Sigma \times Z)$. The optimal parameter selection problem (P^{**}) is to find an admissible element $z \in \mathscr{Z}$ such that the cost function (4.1) is minimized subject to the constraints:

$$g_j(z) = \sum_{i=1}^{k} \Phi_{j,i}\big(\tau_i, x(\tau_i \mid z), z\big) + \int_0^T h_j\big(t, x(t), z\big)\, dt, \quad j = 1, \ldots, l. \tag{4.4}$$

For each $j = 0, 1, \ldots, l$, the corresponding costate system is given by the following system of differential equations:

$$\dot{\lambda}_j^T(t) = -\frac{\partial H_j(t, x(t), z, \lambda_j)}{\partial x(t)}, \quad j = 0, 1, \ldots, l, \tag{4.5}$$

where $t \in (\tau_{i-1}, \tau_i)$, $i = 1, 2, \ldots, k+1$, and, for each $j = 0, 1, \ldots, l$, $H_j = h_j + \lambda_j^T f$ is the corresponding Hamiltonian.

The costate system is subject to the boundary conditions:

$$\lambda_j^T(\tau_i^+) + \frac{\partial \Phi_{j,i}(\tau_i, x(\tau_i), z)}{\partial x(\tau_i)} = \lambda_j^T(\tau_i^-), \quad i = 1, \ldots, k, \tag{4.6}$$

$$\lambda_j^T(T) = 0. \tag{4.7}$$

For the gradient of the cost function as well as the constraint functions $g_j(z)$, $j = 0, 1, \ldots, l$, with respect to the system parameter z, we have the following theorem.

THEOREM 2. *The gradient of the cost function* (4.1) *or each of the constraint functions* (4.4) *with respect to the system parameter z is*

$$\Delta_z g_j(z) = \int_0^{k+1} \frac{\partial H_j}{\partial z}\, dt + \lambda_j^T(0) \frac{\partial x^0(z)}{\partial z} + \sum_{i=1}^{k} \frac{\partial \Phi_{j,i}(\tau_i, x(\tau_i \mid z), z)}{\partial z},$$

for $j = 0, 1, \ldots, l$.

PROOF. Let us re-write (4.1) and (4.4) as

$$g_j(z) = \int_0^T \left[h_j(t, x(t), z) + \sum_{i=1}^{k} \delta(t - \tau_i) \Phi_{j,i}(t, x(t \mid z), z) \right] dt,$$

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and the dynamical system (4.2) as $\dot{x}(t) = \delta(t)x^0(z) + f(t, x(t), z)$ with the initial conditions $x(0) = 0$. Here the delta-functions, by convention, have their mass within $[0, T]$.

Define the corresponding Hamiltonian

$$H_j^* = \left[h_j + \sum_{i=1}^{k} \delta(t-\tau_i)\Phi_{j,i}(t, x(t \mid z), z) \right] + \lambda_j^T(t)\left[\delta(t)x^0(z) + f\big(t, x(t), z\big)\right]$$

for each $j = 0, 1, \ldots, l$, and the corresponding costate system is given by the following system of differential equations:

$$-\dot{\lambda}_j^T(t) = \frac{\partial[h_j + \lambda_j^T f(t, x(t), z)]}{\partial x(t)} + \sum_{i=1}^{k} \delta(t-\tau_i)\frac{\partial \Phi_{j,i}(t, x(t \mid z), z)}{\partial x(t)} \tag{4.8}$$

subject to the boundary conditions:

$$\lambda_j^T(T) = 0. \tag{4.9}$$

It is clear that the costate system (4.8)–(4.9) is equivalent to the costate system (4.5)–(4.7). Using the usual formula, the gradient of the functional $g_j(z)$ with respect to z is given by

$$\begin{aligned}
\Delta_z g_j(u, z) &= \int_0^T \frac{\partial H_j^*}{\partial z}\, dt \\
&= \int_0^T \left\{ \frac{\partial[h_j + \lambda_j^T f(t, x(t), z)]}{\partial z} + \lambda_j^T(t)\delta(t)\frac{\partial x^0(z)}{\partial z} \right. \\
&\qquad \left. + \sum_{i=1}^{k} \delta(t-\tau_i)\frac{\partial \Phi_{j,i}(t, x(t|z), z)}{\partial z} \right\} dt \\
&= \int_0^T \frac{\partial[h_j + \lambda_j^T f(t, x(t), z)]}{\partial z}\, dt + \lambda_j^T(0)\frac{\partial x^0(z)}{\partial z} \\
&\qquad + \sum_{i=1}^{k} \frac{\partial \Phi_{j,i}(\tau_i, x(\tau_i \mid z), z)}{\partial z} \\
&= \int_0^T \frac{\partial H_j}{\partial z}\, dt + \lambda_j^T(0)\frac{\partial x^0(z)}{\partial z} + \sum_{i=1}^{k} \frac{\partial \Phi_{j,i}(\tau_i, x(\tau_i \mid z), z)}{\partial z}.
\end{aligned}$$

This completes the proof.

5. Examples

EXAMPLE 1. A minimal approach of a dynamical system to a curve.

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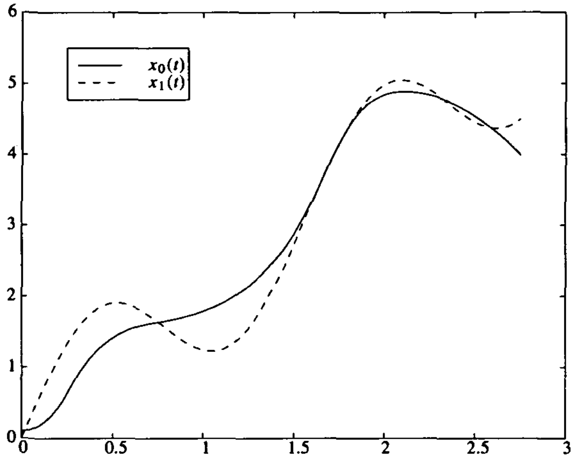

FIGURE 1. The target trajectory $x_0(t)$ and the optimal trajectory $x_1^*(t)$

Let x_0 be a function of time given by $x_0(t) = \sin(4t) + 2t$. Consider a process described by the following differential equations:

$$\dot{x_1} = x_2, \quad \dot{x_2} = -u_1(t)x_2 - x_1 + u_2(t), \tag{5.1}$$

with initial conditions

$$x_1(0) = 0.1, \quad x_2(0) = 0.2. \tag{5.2}$$

Suppose $\underline{\tau_i}$, $\overline{\tau_i}$, $i = 1, \ldots, 5$, are given by $\underline{\tau_i} = (5i-2)T/30$, $\overline{\tau_i} = (5i+2)T/30$ and $T = 3$. Our objective is to find observation times t_i, $i = 1, \ldots, k$, and the controls $u_1(t)$ and $u_2(t)$ such that the cost function

$$\sum_{i=1}^{5} \left(\sin 4t_i + 2t_i - x_1(t_i)\right)^2 \tag{5.3}$$

is minimized subject to the constraints: $\underline{\tau_i} \le t_i \le \overline{\tau_i}$, $i = 1, \ldots, 5$, $-3.0 \le u_1(t) \le 4.0$ and $-3.0 \le u_2(t) \le 11.5$.

Define the CPET transform which maps from t to s to be $dt/ds = v(s)$. Let $v(s) = \sum_{i=1}^{6} v_i \chi_{[i-1,i)}(s)$, where v_i, $i = 1, \ldots, 6$, are collectively referred to as the parameter vector v. The equivalent transformed problem is: Given the dynamical system

$$\dot{y}_1 = v y_2, \quad \dot{y}_2 = v(-w_1(s)y_2 - y_1 + w_2(s)),$$

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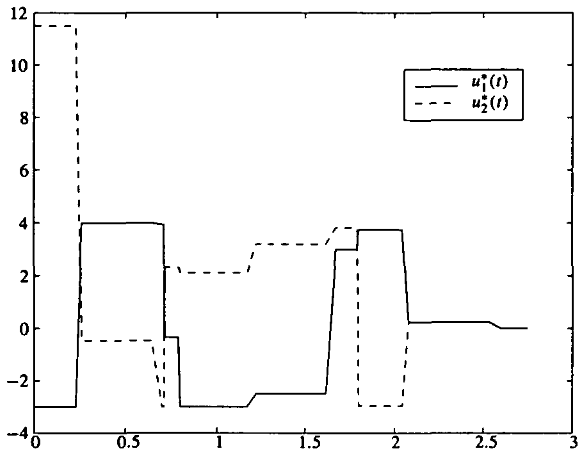

FIGURE 2. The optimal control $(u_1^*(t), u_2^*(t))$

with initial conditions $y_1(0) = 0.1$ and $y_2(0) = 0.2$, find a parameter vector $\boldsymbol{\upsilon}$ and control functions $w_1(s) = u_1(t(s))$, $w_2(s) = u_2(t(s))$ such that the cost function

$$\sum_{i=1}^{5}\left(\sin 4\sum_{j=1}^{i}\upsilon_j + 2\sum_{i=1}^{i}\upsilon_j - y_1(i)\right)^2$$

is minimized subject to

$$\underline{\tau_j} \le \sum_{i=1}^{j}\upsilon_i \le \overline{\tau_j}, \quad j = 1, 2, \ldots, 5, \quad \sum_{i=1}^{6}\upsilon_i = 3,$$
$$-3.0 \le w_1(s) \le 4.0, \quad -3.0 \le w_2(s) \le 11.5, \quad \forall s \in [0, 6).$$

Using the CPET transform again, which maps from s to q, with pre-fixed knots $\xi_0, \xi_1, \ldots, \xi_{10}$,

$$\frac{ds}{dq} = \eta(q), \quad s(0) = 0,$$

where

$$\eta(q) = \sum_{i=1}^{10}\eta_i\chi_i(q), \quad \chi_i(q) = \begin{cases} 1, & q \in [\xi_{i-1}, \xi_i),\ i = 1, 2, \ldots, 10; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly

$$\int_0^i \eta(q)\,dq = i, \quad i = 1, 2, \ldots, 6. \tag{5.4}$$

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We thus obtain the approximate optimal control problem with MCT: Given the dynamical system

$$\dot{\hat{y}}_1 = \eta\upsilon y_2, \quad \dot{\hat{y}}_2 = \eta\upsilon\left(-\sigma^1\hat{y}_2 - \hat{y}_1 + \sigma^2\right)$$

with initial conditions $\hat{y}_1(0) = 0.1$ and $\hat{y}_2(0) = 0.2$, find (υ, σ, η) such that the cost function

$$\sum_{i=1}^{5}\left(\sin 4\sum_{j=1}^{i}\upsilon_j + 2\sum_{i=1}^{i}\upsilon_j - \hat{y}_1(i)\right)^2$$

is minimized subject to

$$\underline{\tau_j} \le \sum_{i=1}^{j}\upsilon_i \le \overline{\tau_j}, \quad j = 1, 2, \ldots, 5, \quad \sum_{i=1}^{6}\upsilon_i = 3,$$
$$-3.0 \le \sigma_{1,j} \le 4.0, \quad -3.0 \le \sigma_{2,j} \le 11.5, \quad j = 1, \ldots, 11,$$
$$\int_0^i \eta(q)\,dq = i, \quad i = 1, \ldots, 6.$$

This is an optimal control problem with MCT cost function. Using the gradient formulae obtained in Section 4, the optimal control software package MISER 3.2 can be adapted to solve this optimal control problem. Figure 1 shows the optimal trajectory of $x_1^*(t)$ and the trajectory for $x_0(t) = \sin(4t) + 2t$. The optimal observation times are $t_1 = 0.7$, $t_2 = 0.8$, $t_3 = 1.67$, $t_4 = 1.81$, $t_5 = 2.59$, $t_6 = 2.75$ with a minimum cost function value of 0.027889. Figure 2 shows the optimal control functions $u_1^*(t)$, $u_2^*(t)$ respectively.

EXAMPLE 2. A three dimensional optimal control problem with variable characteristic times in the cost function.

Consider the dynamical system

$$\dot{x}_1 = x_3 + z_1, \quad \dot{x}_2 = z_2 x_3 + z_3, \quad \dot{x}_3 = z_4 x_1 + x_2 + z_5 x_3 \tag{5.5}$$

with initial conditions

$$x_1(0) = 0.1, \quad x_2(0) = 0.2, \quad x_3(0) = 0.1. \tag{5.6}$$

Let the target trajectory be specified as follows:

$$x_{01}(t) = \sin(4t) + 2t, \quad x_{02}(t) = t^3 - 3t^2 + 3t.$$

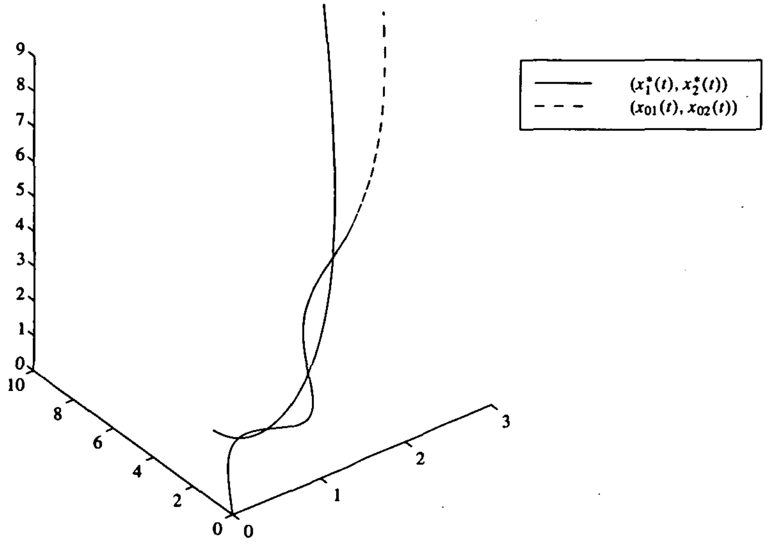

FIGURE 3. The optimal trajectory $(x_1^*(t), x_2^*(t))$ and target trajectory $(x_{01}(t), x_{02}(t))$

Let $\underline{\tau_i} = (5i-2)T/30$, $\overline{\tau_i} = (5i+2)T/30$, $i = 1, \ldots, 5$, and $T = 3$. Then we formulate the following problem: Given the dynamic system (5.5)–(5.6), find observation times t_i, $i = 1, \ldots, 5$, and system parameters z_i, $i = 1, \ldots, 5$, such that the cost function

$$\sum_{i=1}^{5} \left\{ \left(\sin 4t_i + 2t_i - x_1(t_i) \right)^2 + \left(t_i{}^3 - 3t_i{}^2 + 3t_i - x_2(t_i) \right)^2 \right\} \tag{5.7}$$

is minimized subject to the constraints

$$\begin{gathered} \underline{\tau_i} \le t_i \le \overline{\tau_i}, \quad i = 1, 2, \ldots, 5, \\ -1.0 \le z_1 \le 2.0, \quad -1.0 \le z_2 \le 10.0, \\ -1.0 \le z_3 \le 1.0, \quad -1.0 \le z_4 \le 1.0, \quad -4.0 \le z_5 \le 1.0. \end{gathered}$$

Use the CPET transform to map t to s: $dt/ds = v(s)$. Let $v(s) = \sum_{i=1}^{6} \sigma_i \chi_{[i-1,i)}(s)$, where σ_i, $i = 1, \ldots, 6$, are collectively referred to as the parameter vector $\boldsymbol{\sigma}$. Furthermore, let z_i, $i = 1, \ldots, 5$, be collectively referred to as the system parameter z. The equivalent transformed problem is: Given the dynamical system

$$\begin{aligned} \dot{y}_1 &= v(y_3 + z_1), & y_1(0) &= 0.1, \\ \dot{y}_2 &= v(z_2 y_3 + z_3), & y_2(0) &= 0.2, \\ \dot{y}_3 &= v(z_4 y_1 + y_2 + z_5 y_3), & y_3(0) &= 0.1, \end{aligned}$$

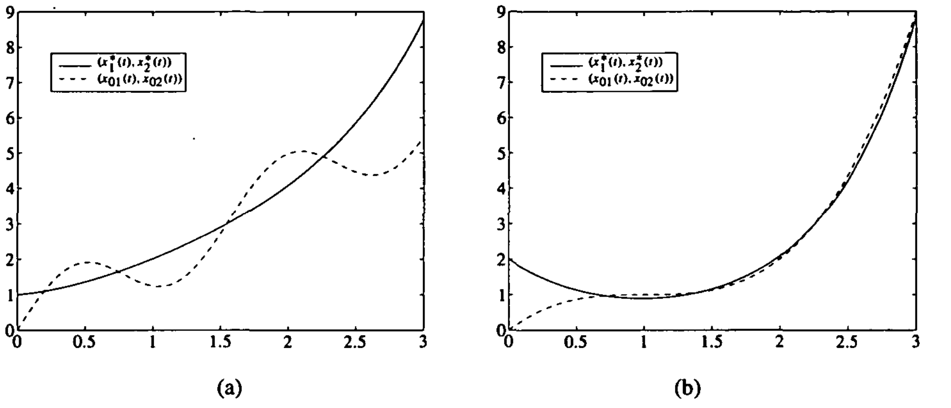

FIGURE 4. The projections of the trajectories depicted in Figure 3 onto the planes (a) $X_1 \times [0, T]$ and (b) $X_2 \times [0, T]$

find (v, z) such that

$$\sum_{i=1}^{5} \left\{ \left(\sin 4 \sum_{j=1}^{i} v_j + 2 \sum_{i=1}^{i} v_j - y_1(i) \right)^2 + \left(\left(\sum_{j=1}^{i} v_j \right)^3 - 3 \left(\sum_{j=1}^{i} v_j \right)^2 + 3 \sum_{j=1}^{i} v_j - y_2(t_i) \right)^2 \right\}$$

is minimized, subject to

$$\underline{\tau_j} \leq \sum_{i=1}^{j} v_i \leq \overline{\tau_j}, \quad j = 1, 2, \ldots, 5, \qquad \sum_{i=1}^{6} v_i = 3,$$
$$-1.0 \leq z_1 \leq 2.0, \quad -1.0 \leq z_2 \leq 10.0,$$
$$-1.0 \leq z_3 \leq 1.0, \quad -1.0 \leq z_4 \leq 1.0, \quad -4.0 \leq z_5 \leq 1.0.$$

The gradient formulae obtained in Section 4 are applicable. Thus MISER 3.2 can be adapted to solve this optimal control problem with MCT cost function. Figure 3 shows the optimal trajectory of $(x_1^*(t), x_2^*(t))$ and the trajectory for $x_{01}(t) = \sin(4t) + 2t$ and $x_{02}(t) = t^3 - 3t^2 + 3t$, and Figures 4 are projections of Figure 3 onto the planes $X_1 \times [0, T]$ and $X_2 \times [0, T]$. The optimal observation times are $t_1 = 0.7$, $t_2 = 0.8$, $t_3 = 1.544678$, $t_4 = 2.2$, $t_5 = 2.3$ with a minimum cost function value of 0.1915 and optimal system parameters of $z_1 = 1.32478$, $z_2 = 2.13784$, $z_3 = -0.43432$, $z_4 = -0.00573633$, $z_5 = 0.073166$.

6. Conclusion

A computational method was obtained for solving the optimal control problem with time variables in the objective function. The method is based on the combination of the control enhancing transform and the control parametrization technique. The method is efficient and supported by rigorous mathematical analysis. Two numerical examples are solved using this method.

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