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INTERNAL COMPLETENESS AND INJECTIVITY OF BOOLEAN ALGEBRAS IN THE TOPOS OF M-SETS

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In this paper we study internal completeness, injectivity and some related notions in the category **MBoo** of Boolean algebras in the topos **MEns** of M-sets, for a monoid M.

In Section 1, we deal with the notion of internal completeness in **MBoo** and show that an algebra A in **MBoo** is internally complete if and only if the embedding $[\cdot]\colon A \longmapsto N(A)$ of A into the algebra $N(A)$ of (internal) normal ideals of A is an isomorphism.

In Section 2, we study the notion of injectivity and essential extensions in **MBoo** and show that: injectivity implies internal completeness; the injective hull of 2 is $H(2)$, the algebra of all subsets of M, if an only if M is a finite group; for a finite monoid M, 2 is injective if and only if M has a right absorbing element; and for a finite and commutative monoid M, a subalgebra A of $H(2)$ is an essential estenstion of 2 if and only if A is generated by the blocks of a monoid congruence θ on M with M/θ being a group. Further, we give examples to show that the latter result is not true in general.

Finally, in Section 3, we characterise the subdirectly irreducible algebras in **MBoo**.

0. PRELIMINARIES

0.1 For a monoid M let **MEns** be the topos of all (left) M-sets (sets with a left M-action) and the equivariant maps between them. Considering M as a category with one object, **MEns** is the functor category $\mathbf{Ens}^{\mathbf{M}}$ where **Ens** is the category of sets. Hence, the subobject classifier Ω of this topos is the set of all left ideals of M (subsets of M which are closed under the left multiplication) together with the action of M on Ω given by division, that is for $t \in M$ and $S \in \Omega$, $tS = \{s \in M \mid st \in S\}$. The

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true map $1 \longrightarrow \Omega$, where $1 = \{0\}$ is the terminal object of **MEns**, takes 0 to M, the largest ideal of M. For any subobject $A \overset{\tau}{\rightarrowtail} B$, the classifying map $f_\tau : B \longrightarrow \Omega$ is defined by $f_\tau(b) = \{x \in M \mid xb \in A\}$. Hence $f_\tau(b) = M$ if and only if $b \in A$. Notice that $\Omega = \{\phi, M\}$ if and only if M is a group. For $A \in$ **MEns**, a global element $f : 1 \longrightarrow A$ is given by an element $f(0)$ of A which is fixed under the action of M. Since $tS = M$ if and only if $t \in S$, one can check easily that Ω has exactly two global elements. This shows that the topos **MEns** is bivalued. But **MEns** is a Boolean topos if and only if M is a group.

0.2 The power set $\mathcal{P}(M)$ of M is an M-set with the action given by division, and Ω is a pseudo-complemented subalgebra of $\mathcal{P}(M)$. The pseudo-complement of $S \in \Omega$ is

$$S^* = \{s \in M \mid (\forall t \in M)(ts \notin S)\}$$

It is easily checked that Ω is a Stone algebra, that is $S^* \cup S^{**} = M$, if and only if $S^* = \emptyset$ for all $S \neq \emptyset$ in Ω. Also, by [9], Ω is a Stone algebra if and only if M satisfies the (left) Ore condition, that is, for any a, b in M, there exist s, t in M such that $sa = ta$. In fact, if the Ore condition is satisfied, $S \neq \emptyset$ is in Ω and $a \in S$, then for any $b \in M$, there exist s, t in M with $tb = sa$. Now, since S is a left ideal of M, $tb = sa \in S$. This gives us that $b \notin S^*$, and hence $S^* = \emptyset$.

0.3 For A, B in **MEns**, B^A is the set of all equivariant maps $f : M \times A \longrightarrow B$, together with the action of M defined by

$$(sf)(t,a) = f(ts,a)$$

for s, t in M and $a \in A$. It is easily sseen that $\Omega^A \cong$ Sub$(M \times A)$ subobjects of $M \times A$. For any subobject X of $M \times A$, we have

$$X = \bigcup_{s \in M} \{s\} \times X_s$$

where $X_s = \{a \in A \mid (s,a) \in X\}$. Hence we can identify X by a family $(X_s)_{s \in M}$, where for each $s \in M$, X_s is a subset of A with

$$(\forall t \in M)(a \in X_s \Rightarrow ta \in X_{ts}).$$

The action of M on Ω^A is then given by

$$tX = (X_{st})_{s \in M}.$$

0.4 In the following, **MLatt** will denote the category of lattices in **MEns**, with lattice maps preserving the M action, and **MBoo** is the category of Boolean algebras in **MEns**. For any algebra A, the underlying object of A is denoted by the same letter A.

1. Internal Completeness

1.1 For $A \in \mathbf{MLatt}$, $\mathrm{Id}(A)$ and $\mathrm{MId}(A)$ denote the set of ideals of (the lattice) and M-ideals of (the M-lattice) A, respectively. The action of M on $\mathrm{Id}(A)$ is given by $s \cdot J = [sJ]$, for $s \in M$ and $J \in \mathrm{Id}(A)$, where $[sJ]$ is the ideal of A generated by the set $sJ = \{sx \mid x \in J\}$. That is

$$s \cdot J = \{a \in A \mid (\exists x \in J)(a \leqslant sx)\}.$$

The action of M on $\mathrm{MId}(A)$ is defined in the same way.

It is clear that the map $\downarrow : A \to \mathrm{Id}(A)$ defined by a $\mapsto \downarrow a = \{x \in A \mid x \leqslant a\}$ is an equivariant map.

1.2 For $A \in \mathbf{MBoo}$, the *internal ideal lattice* $\mathcal{J}(A)$ of A in the topos **MEns** is given by

$$\mathcal{J}(A) = \{X = (X_s)_{s \in M} \mid \big(X \in \Omega^A\big) \,\&\, (\forall s \in M)(X_s \in \mathrm{Id}(A))\}.$$

For $X = (X_s)_{s\in M}$ and $Y = (Y_s)_{s \in M}$, their meet and join in $\mathcal{J}(A)$ is defined component-wise, that is

$$X \wedge Y = (X_s \wedge Y_s)_{s \in M} \text{ and } X \vee Y = (X_s \vee Y_s)_{s \in M},$$

where $X_s \wedge Y_s$ is their intersection and $X_s \vee Y_s$ is their join in $\mathrm{Id}(A)$, that is $X_s \vee Y_s = \{b \vee c \mid b \in X_s, c \in Y_s\}$. The action of M on $\mathcal{J}(A)$ is the same as the action on Ω^A. Notice that if M is a group, then $\mathcal{J}(A) = \mathrm{Id}(A)$.

One also defines a lattice embedding $[\cdot] \colon A \longmapsto \mathcal{J}(A)$ by $[a] = (\downarrow sa)_{s \in M}$, which preserves the action of M.

In addition, one defines a lattice embedding $(\,)^{\#} \colon \mathrm{Id}(A) \longmapsto \mathcal{J}(A)$, preserving the action, by $J^{\#} = ([sJ])_{s \in M}$, for $J \in \mathrm{Id}(A)$. Since $(\downarrow a)^{\#} = [a]$, the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\;[\cdot]\;} & \mathcal{J}(A) \\ {\scriptstyle\downarrow}\big\downarrow & & \big\uparrow{\scriptstyle (\,)^{\#}} \\ \mathrm{Id}(A) & = & \mathrm{Id}(A) \end{array}$$

1.3 Recall that a lattice A in **MEns** in *internally complete* (see [8], p.147) if there exists an order preserving equivariant map $V \colon \mathcal{J}(A) \longrightarrow A$ which is (internally) left adjoint to $[\cdot] \colon A \longmapsto \mathcal{J}(A)$. That is, for $X = (X_s)s \in M$ in $\mathcal{J}(A)$ and $c \in A$,

$$V X \leqslant c \iff X \leqslant [c] \iff (\forall s \in M)(X_s \leqslant \downarrow sc).$$

By Proposition 5.35 and 5.36 in [8], Ω and Ω^A are internally complete.

Proposition 1.4. *If $A \in$ **MBoo** is internally complete, then A is complete in **Boo** and the action of **M** on A is complete (that is preserves the join).*

Proof: Since the underlying functor $U\colon \mathbf{MBoo} \longrightarrow \mathbf{Boo}$ has a left adjoint, which preserves pullbacks, A is complete as a Boolean algebra (see Proposition 5.6) [8]). In fact the join map $\widetilde{V}\colon \mathrm{Id}(A) \longrightarrow A$ is given by the following commutative diagram

$$\begin{array}{ccc} \mathcal{J}(A) & \xrightarrow{\;V\;} & A \\ {\scriptstyle (\,)^{\#}}\uparrow & & \uparrow{\scriptstyle \widetilde{V}} \\ \mathrm{Id}(A) & = & \mathrm{Id}(A) \end{array}$$

That is, $\widetilde{V}J = VJ^{\#}$, for $J \in \mathrm{Id}(A)$. Since the maps $(\,)^{\#}$ and V are both equivariant, so is $\widetilde{V}$.

The converse of the above proposition is not true in general. Consider the initial object 2 of **MBoo**. By [9], 2 is internally complete if an only if Ω is a stone algebra, that is if and only if M satisfies the left Ore condition (see 0.2).

If M is a group, then $A \in$ **MBoo** is internally complete if and only if $A \in$ **Boo** is complete. This is because $\mathcal{J}(A) = \mathrm{Id}(A)$, and the actions are isomorphism (onto) and hence complete, for the join in **Boo** is defined by means of the order and the actions preserve the order.

Let $A \in$ **MBoo** and $J \in \mathcal{J}(A)$. The (internal) pseudo-complement J^* of J is defined by

$$\begin{aligned} J_s^* &= \{x \in A \mid (\forall t \in M)(tx \in (J_{ts})^*)\} \\ &= \bigcap_{t \in M} t^{-1}(J_{ts})^* \end{aligned}$$

for each $s \in M$. We now prove that J^* is indeed the pseudo-complement of J. ∎

Lemma 1.5. *J^* is the pseudo-complement of J.*

Proof: J^* is in $\mathcal{J}(A)$, because each $(J_{ts})^*$ is in $\mathrm{Id}(A)$ and, since $t^{-1}(J_{ts})^*$ is in $\mathrm{Id}(A)$, $J_s^* = \bigcap_{t \in M} t^{-1}(J_{ts})^*$ belongs to $\mathrm{Id}(A)$. If $x \in J_s^*$ and $t_0 \in M$, then for any $t \in M$, $t(t_0 x) = (t t_0)x$ belongs to $(J_{tt_0 s})^*$, and hence $t_0 x \in J_{t_0 s}^*$. Now, $x \in J_s \cap J_s^*$ implies that $x \in J_s$ and $x \in J_s^*$. Hence $tx \in (J_{ts})^*$, for all $t \in M$, which implies that $x \in (J_s)^*$. Hence, by the definition of $(J_s)^*$, $x = x \wedge x = 0$.

Now let $J \wedge H = 0$, for some $H \in \mathcal{J}(A)$. Then $J_s \cap H_s = 0$, for all $x \in M$. To show that $H_s \subseteq J_s^*$, let $h \in H_s$. To see that $th \in (J_{ts})^*$, for all $t \in M$, let $b \in J_{ts}$. Now $th \wedge b \in J_{ts}$ and since $th \in H_{ts}$, $th \wedge b \in H_{ts}$. Then $th \wedge b$ being in $J_{ts} \cap H_{ts} = 0$ implies that $th \in (J_{ts})^*$. Hence J^* is indeed the pseudo-complement of J. ∎

Notice that, since $(t \cdot J)^* = t \cdot J^*$, the map $(\,)^*\colon \mathcal{J}(A) \longrightarrow \mathcal{J}(A)$ is equivariant.

Lemma 1.6. *If $A \in$ MBoo is internally complete and $J \in \mathcal{J}(A)$, then*

$$J^* = [(VJ)']$$

where $(\,)'$ is complementation in A.

Proof: Let $c = VJ$. To see that for each $s \in M$,

$$\mathcal{J}_s^* = \{x \mid tx \in (J_{ts})^*\} = \downarrow sc',$$

Let $x \in J_s^*$. Then we have

$$\begin{aligned} x \leqslant sc' &\iff sc \leqslant x' \\ &\iff sVJ \leqslant x' \\ &\iff Vs \cdot J \leqslant x' && \text{(by 1.4)} \\ &\iff s \cdot J \leqslant [x'] && \text{(definition of } V) \\ &\iff (\forall t \in M)((s \cdot J)_t \leqslant \downarrow tx') \\ &\iff (\forall t \in M)(J_{st} \leqslant \downarrow tx') \\ &\iff (\forall t \in M)\big((\downarrow tx')^* \leqslant (J_{st})^*\big) \\ &\iff (\forall t \in M)\big(\downarrow tx \leqslant (J_{st})^*\big) \\ &\iff (\forall t \in M)\big(tx \in (J_{st})^*\big) \\ &\iff (x \in J_s^*). \end{aligned}$$

□

1.7 Let $N(A) = \{J \in \mathcal{J}(A) \mid J = J^{**}\} = \{J^* \mid J \in \mathcal{J}(A)\}$. That is $N(A)$ is the equaliser of $(\,)^{**}\colon \mathcal{J}(A) \longrightarrow \mathcal{J}(A)$ and the identity map $\mathcal{J}(A) \longrightarrow \mathcal{J}(A)$. We call this the algebra of (internal) *normal ideals* of A. Since $[\cdot]^* = [(\,)']$, the embedding $[\cdot]\colon A \rightarrowtail \mathcal{J}(A)$ factors through $N(A)$. This shows that any algebra A in **MBoo** can be embedded into an internally complete one. That $N(A)$ is internally complete follows from the fact that the usual proof is constructively valid.

The above lemma shows that

Proposition 1.8. *$A \in$ MBoo in internally complete if and only if $[\cdot]\colon A \longrightarrow N(A)$ is an isomorphism.*

2. Injectivity in MBoo

2.1 Recall that an object A in a category is *injective* if and only if for any morphism $h\colon B \longrightarrow A$ and any monomorphism $g\colon B \rightarrowtail C$, there exists a morphism $f\colon C \longrightarrow A$ such that $fg = h$. Further, a monomorphism $h\colon A \rightarrowtail B$ is called *essential* if any

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$g\colon B \longrightarrow C$ for which gh is a monomorphism is itself a monomorphism. In **MBoo**, one checks easily that $h\colon A \rightarrowtail B$ is essential if and only if every M-ideal of B with zero inverse image by h is itself zero. The result of Ebrahimi [4] and the classical facts about Boolean algebras show that for any Grothendieck topos, and hence in particular for **MEns**, every algebra $A \in$ **MBoo** has an *injective hull* (that is an essential injective extension). That is, the category **MBoo** has enough injectives.

2.2 Consider the following adjointness

$$\mathbf{MEns} \underset{H}{\overset{U}{\rightleftarrows}} \mathbf{Ens}$$

with the underlying functor U a left adjoint of the functor H defined by : for any $X \in$ **Ens**, $H(X)$ is the set of *all* functions from the set M to the set X, with the action of M on $H(X)$ given by $(sf)(t) = f(ts)$, for $f \in H(X)$ and $s, t \in M$. This adjointness can be lifted to

$$\mathbf{MBoo} \underset{H}{\overset{U}{\rightleftarrows}} \mathbf{Boo}$$

denoted by the same letters. Since H has a left adjoint U which preserves finite limits, and hence monomorphisms, H preserves injective and complete Boolean algebras. In particular, for $2 \in$ **Boo**, the algebra $H(2)$ of all subsets of M is injective and internally complete.

LEMMA 2.3. *$[\cdot]\colon A \rightarrowtail N(A)$ is essential.*

PROOF: Let the composite $A \xrightarrow{[\cdot]} N(A) \xrightarrow{\varphi} B$ be a monomorphism; that is, for $x \in A$, $\varphi[x] = \varphi[(\downarrow sx)_{s\in M}] = 0$ implies that $x = 0$. Let $X = (X_s)_{s\in M}$ be in $N(A)$ and $\varphi(X) = 0$. Let $X_s \neq 0$, for some $s \in M$. Then $X_s = (s \cdot X)_e \neq 0$, where e is the identity of M. Let $x \in (s \cdot X)_e$, then, for any $t \in M$, $tx \in (s \cdot X)_{te}$, and hence $\downarrow tx \subseteq (s \cdot X)_t$, for $(s \cdot X)_t$ in an ideal of A. Thus $[x] \leqslant s \cdot X$, and hence $\varphi[x] \leqslant \varphi(s \cdot X) = s\varphi(X) = 0$. Thus $x = 0$, and hence $X_s = 0$, which proves the lemma. ∎

PROPOSITION 2.4. *If $A \in$ MBoo is injective, then it is internally complete.*

PROOF: By the above lemma, $N(A)$ is essential over A and since A is injective $A \cong N(A)$. Hence, by 1.8, A is internally complete. ∎

The converse of the above proposition is not true in general: for a nontrivial (finite) group M, 2 is internally complete in **MBoo** but it is not injective, since $H(2)$ is an essential extension of 2. In fact we have the following proposition which is a special case of Lemma 1.9 of [3].

PROPOSITION 2.5. *$H(2)$ is an essential extension of 2 if and only if M is a finite group.*

PROOF: It is clear that an algebra E in **MBoo** is essential over 2 if and only if it is simple, that is $M\,\mathrm{Id}(E) \cong 2$. Let M be a finite group, and $I \neq 0$ be an M-ideal of $H(2)$. Let $\emptyset \neq K \in I$, and $s \in K$. Since I is an M-ideal, $sK = \{t \mid ts \in K\}$ in I, and hence $\{e\} \subseteq sK$ is in I. Now, for any $m \in M$, $m^{-1}\{e\} = \{m\}$ is in I. Since M is finite, $M = \bigcup_{m \in M} \{m\}$ belongs to I. This shows that $H(2)$ is simple, and hence essential over 2. Conversely, let $H(2)$ be essential over 2. Let $S \subseteq M$ be the set of all right invertible elements of M. For $K \subseteq S$ and $s \in M$, $sK = \{t \mid ts \in K\}$ is a subset of S. Hence the set of all subsets of S is a (nontrivial) M-ideal of $H(2)$. By essentialness, we get that $M = S$. This shows that M is a group. Now, since M is a group, it is checked easily that the set $P_f(M)$ of all finite subsets of M is a (nontrivial) M-ideal of $H(2)$. By essentialness, $M \in P_f(M)$ and hence M is finite. ∎

PROPOSITION 2.6. *The injective hull of 2 in* **MBoo** *is $H(2)$ if and only if M is a finite group.*

REMARK 2.7. For any monoid M, the injective hull of 2 in **MBoo** is $H(2)/\mathcal{J}$ for a maximal M-ideal J of $H(2)$.

PROPOSITION 2.8. *For a finite monoid M, 2 is injective in* **MBoo** *if and only if M has a "right absorbing" element a (that is, $as = a$ for all $s \in M$).*

PROOF: Let 2 be injective. Since $H(2)$ is injective, there exists $h\colon H(2) \longrightarrow 2$. Now $h^{-1}\{1\}$ is an M-ultrafilter on M(in $H(2)$). Since M is finite, $h^{-1}\{1\}$ is generated by $\{a\}$, for some $a \in M$, that is

$$\bigcap_{\substack{h(X)=1 \\ X \subseteq M}} X = \{a\}.$$

Now, for any $s \in M$, $h(s\{a\}) = sh(\{a\}) = s \cdot 1 = 1$. This implies that $\{a\} \subseteq s\{a\}$. But $s\{a\} = \{x \in M \mid xs = a\}$, hence $as = a$ for all $s \in M$. Conversely, let $a \in M$ be a right absorbing element. Consider $\mathcal{A} = \{X \mid a \in X \subseteq M\} = \uparrow \{a\}$ in $H(2)$. For any $s \in M$, $a \in sX$, because $sX = \{x \in M \mid xs \in X\}$ and $as = a$ is in X. Hence, for $X \in \mathcal{A}$, $as = a \in X$ implies that $a \in sX$. Thus $\mathcal{A}$ is an M-ultrafilter on M. Thus, there exists $h\colon H(2) \longrightarrow 2$ given by

$$h(X) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{if } a \notin X \end{cases}$$

which is an **MBoo** morphism. Hence 2 retracts the injective algebra $H(2)$. This shows that 2 is injective. ∎

Notice that the finiteness of M is not needed to prove the converse of the above proposition.

PROPOSITION 2.9. *Let M be a finite and commutative monoid. Then a subalgebra A of $H(2)$ is an essential extension of 2 if and only if A is generated by (the blocks) of a monoid congruence θ on M with M/θ being a group.*

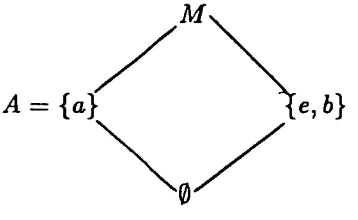
PROOF: Since M is commutative, the maps $\lambda_t : A \longrightarrow A$ induced by the action of $t \in M$ are endomorphisms. Since A is essential over 2, and hence simple, the λ_t's are one to one. By finiteness of A, the λ_t's are isomorphisms. Now, for $\theta = \{(s, t) \mid \lambda_s = \lambda_t\}$, M/θ is a group.

To see that A is generated by the θ-blocks, let $E \in A$ be the atom of A containing e, which exists because A is finite. If $\lambda_s = \mathrm{Id} = \lambda_e$, that is $s \in \theta[e]$, the θ block of e, then $sE = E$ and hence $s \in E$. This shows that $\theta[e] = E$. On the other hand, for $s \in E$, $E \subseteq sE$, because $e \in sE$ and E is an atom. Since the λ_t's are automorphisms, sE is an atom, and hence $E = sE$. Further $s(tE) = tE$, by commutativity, and thus $\lambda_s = \mathrm{Id}$, leaving all the atoms of A fixed; every atom of A is of the form tE for some $t \in M$, because $M = t_1E \cup \ldots \cup t_kE$ for suitable $t_1, \ldots, t_k$.

Finally, if $\lambda_{\bar{s}} = \lambda_s^{-1}$, then $sE = \theta[\bar{s}]$, because $(xs, e) \in \theta$ if and only if $(x, \bar{s}) \in \theta$. This shows that A is generated by the blocks of a congruence θ on M with M/θ being a group.

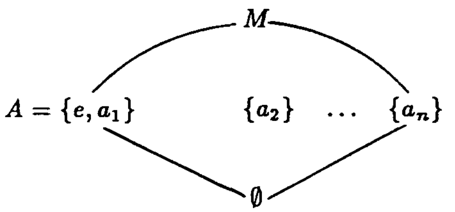
Conversely if θ is a monoid congruence on M with M/θ being a group, then $s\theta[e] = \theta[\bar{s}]$ for $s\bar{s} = e$ and the θ-blocks generate a subalgebra A of $H(2)$ which is an essential extension of 2. ∎

EXAMPLE 2.10. *Consider the monoid $M = \{e, a, b\}$ with $xy = y$, for x, y in M. The algebra*



in **MBoo** *is an essential extension (in fact the injective hull) of 2, but the partition $\{\{a\}, \{e, b\}\}$ which generates A is not a monoid congruence.*

In fact, for the monoid $M = \{e, a_1, \ldots, a_n\}$ with $xy = y$, for x, y in M, the algebra



is the injective hull of 2.

3. Subdirect irreduciblity

Recall that an algebra A is a category is *subdirectly irreducible* if and only if for any monomorphism $f: A \rightarrowtail \prod_{i \in I} A_i$, there exists an $i \in I$ with $p_i f: A \longrightarrow A_i$ a monomorphism, where p_i is the i-th projection. For $A \in \mathbf{MBoo}$, this is equivalent to A having a smallest nonzero M-ideal. For a different proof of the following see [**7**].

Lemma 3.1. *If $A \in \mathbf{MBoo}$ is subdirectly irreducible, then A can be embedded into $H(2)$.*

Proof: For $A \in \mathbf{MBoo}$, $U(A)$ is in **Boo** (see 2.2) and by the Representation Theorem in **Boo**, there exists a set S with a monomorphism $U(A) \longrightarrow 2^S$. Hence we have

$$A \rightarrowtail H\,U(A) \rightarrowtail H(2^S) \simeq (H2)^S.$$

Now, since A is subdirectly irreducible, $A \rightarrowtail H(2)$ is an embedding. ∎

Lemma 3.2. *Every subalgebra A of $H(2)$ is subdirectly irreducible.*

Proof: Suppose A is not subdirectly irreducible. Thus there exists a family $I_\lambda (\lambda \in \Lambda)$ of nontrivial M-ideals of A whose intersection is trivial. Since the I_λ's are nontrivial, there are nonempty sets X_λ in I_λ, for each λ. For each λ, take $s_\lambda \in X_\lambda$. Since the I_λ's are M-ideals, $s_\lambda X_\lambda \in I_\lambda$, hence the intersection X of the sets $s_\lambda X_\lambda (\lambda \in \Lambda)$ belongs to the intersection of the M-ideals $I_\lambda (\lambda \in \Lambda)$ which is trivial. Now, $X = \{x \in M \mid (\forall \lambda)(x s_\lambda \in X_\lambda)\}$, and hence $e \in X$, that is $X \neq \emptyset$ which is a contradiction. ∎

By the last two lemmas we get

Proposition 3.3. *An algebra in $\mathbf{MBoo}$ is subdirectly irreducible if and only if it is isomorphic to a subalgebra of the algebra $H(2)$.*

References

[1] R. Balbes and P. Dwinger, *Distributive Lattices* (University of Missouri Press, 1975).

[2] B. Banaschewski and K.R. Bhutani, 'Boolean algebras in a localic topos', *Math. Proc. Cambridge Philos. Soc.* **100** (1986), 43-55.

[3] S. Burris and M. Valeriote, 'Expanding varieties by monoids of endomorphisms', *Algebra Universalis* **17** (1983), 150–169.

[4] M.M. Ebrahimi, 'Algebra in a Grothendieck topos: injectivity in quasi-equational classes.', *J. Pure. Appl. Algebra* **26** (1982), 269–280.

[5] R. Goldblatt, *Topoi: The Categorical Analysis of Logic* (North-Holland, Amsterdam, 1979).

[6] P. Halmos, *Lectures on Boolean Algebras* (van Nostrand, Princeton, 1963).

[7] J. Ježek, 'Subdirectly irreducible and simple Boolean algebras with endomorphisms', in *Universal Algebra and Lattice Theory*: *Lecture Notes in Math.* 1149, Proc., Charleston, 1984, pp. 150–162 (Springer-Verlag, Berlin, Heidelberg, New York, 1985).

[8] P.T. Johnston, *Topos Theory* (Academic Press, 1977).

[9] P.T. Johnstone, 'Conditions related to de Morgan's law': *Lectuure Notes in Math.* 753, Proc., Durham 1977, pp. 479–491 (Springer-Verlag, Berlin, heidelberg, New York).

[10] S. MacLane, *Categories for the Working Mathematician* (Graduate Texts in Mathematics, No 5, Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[11] R. Sikorski, *Boolean Algebras* (Ergebnisse der Math. Band 25 Springer-Verlag, Berlin, Heidelberg, New York, 1964).

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