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INTERPOLATION RESTRICTED TO DECREASING FUNCTIONS AND LORENTZ SPACES*

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For the real interpolation method, we identify the interpolated spaces of couples of classical Lorentz spaces through interpolation of the corresponding weighted L_p-spaces restricted to decreasing functions.

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1. Introduction

Throughout this paper a decreasing function is a non-increasing and non-negative function on $\mathbf{R}^+ = (0, \infty)$, which is endowed with Lebesgue measure, L_0 represents the vector lattice of (equivalence classes of) all measurable real functions on $\mathbf{R}^+$, and $L_p = L_p(\mathbf{R}^+)$.

We say that X is a *quasi-normed function space* if it is a quasi-normed space and a linear subspace of L_0, such that, if $|f| \le |g|$, $g \in X$ and $f \in L_0$, then $f \in X$ and $\|f\|_X \le \|g\|_X$. In our examples, X will have the *Fatou property*:

$$\text{If } f_n \in X, 0 \le f_n \uparrow f \text{ a.e. and } \sup \|f_n\|_X < \infty, \text{ then } f \in X \text{ and } \|f_n\|_X \uparrow \|f\|_X.$$

In this case, X is complete, i.e., it is a quasi-Banach function space.

The cone of all decreasing functions of X is denoted X^d. Here we are mainly interested with the case

$$X = L_p(\omega) = \{f \in L_0; \|f\|_{L_p(\omega)} = \|f\omega\|_p < \infty\} \qquad (0 < p \le \infty),$$

where $0 < \omega \in L_0$, a weight on $\mathbf{R}^+$, and $\|\cdot\|_p$ is the usual quasi-norm on $L_p(\mathbf{R}^+)$.

Operators on these spaces restricted to decreasing functions have been used by several authors (cf. [2], [19], [7], etc.) to characterize when a variety of classical operators are bounded on Lorentz spaces associated to pairs (u, ω) of weights,

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$$\Lambda_u^p(\omega) = \{f \in L_0;\ \|f\|_{\Lambda_u^p(\omega)} = \|f_u^*\|_{L_p(\omega)} < \infty\}, \tag{1}$$

where $f_u^*(x) = \inf\{\lambda > 0;\ \int_{\{|f|>\lambda\}} u(t)\,dt \le x\}$. We write $f^* = f_u^*$ and $\Lambda^p(\omega) = \Lambda_u^p(\omega)$ if $u(t) \equiv 1$.

Conversely, the boundedness of operators T acting on decreasing functions of $L_p(\omega)$ has been studied in [6] by considering the associate operator $\tilde{T}(f) = T(f_u^*)$ on $\Lambda_u^p(\omega)$.

On the other hand, interpolation of operators on weighted L_p-spaces restricted to monotone functions appear in [1], [11] and [21] in connection with some integral operators, and in [8, Theorem 2] we have seen how the boundedness of operators of Hardy type is enough to obtain interpolation results for cones of decreasing functions (cf. also Lemma 2 and Remark 3 below.) This type of interpolation was first used in [18] to present a unified account of some results about Fourier series with positive coefficients.

For a given pair $\overline{X} = (X_0, X_1)$ of quasi-normed function spaces, let $\overline{X}^d = (X_0^d, X_1^d)$, the corresponding pair of cones of decreasing functions, and $\Sigma(\overline{X}^d) = X_0^d + X_1^d$. For every $g \in \Sigma(\overline{X}^d)$, we denote

$$K^d(g, t) = K(g, t; \overline{X}^d) = \inf\{\|g_0\|_0 + t\|g_1\|_1;\ g = g_0 + g_1, g_j \in X_j^d (j = 0, 1)\}.$$

This K^d-functional is used to construct

$$(\overline{X}^d)_{f,q} = \left\{g \in \Sigma(\overline{X}^d);\ \|g\|_{f,q,d} = \left\|\frac{K^d(g, t)}{f(t)}\right\|_{L_q\left(\frac{dt}{t}\right)} < \infty\right\},$$

if $0 < q \le \infty$ and f a function parameter in the sense of [12]. We recall that this means that $f : \mathbf{R}^+ \to \mathbf{R}^+$ is continuous and $\bar{f}(t) = \sup f(ts)/f(s) < \infty$. As usual we write $(\overline{X}^d)_{\vartheta,q}$ for $(\overline{X}^d)_{f,q}$ when $f(t) = t^\vartheta (0 < \vartheta < 1)$.

To deal with these classes $(\overline{X}^d)_{f,q}$, a useful tool is the following elementary *decomposition lemma* (cf. [8, Lemma 1]).

Lemma 1. *Let $f, g, h \in L_0$ be decreasing functions such that $f \le g + h$. Then $f = f_0 + f_1$ with f_0 and f_1 decreasing, $f_0 \le g$ and $f_1 \le h$ (almost everywhere).*

There is no general way to identify $(\overline{X}^d)_{f,q}$, even if the usual interpolation space $\overline{X}_{f,p}$ by the K-method is known, but for a number of important cases $(\overline{X}^d)_{f,q} = (\overline{X}_{f,q})^d$.

Obviously this is true if $\overline{X}$ satisfies the following two conditions:

(a) $\Sigma(\overline{X})^d \subset \Sigma(\overline{X}^d)$, hence $\Sigma(\overline{X})^d = \Sigma(\overline{X}^d)$,

(b) $K^d(g, t) \le CK(g, t)$, hence $K^d(g, t) \simeq K(g, t)$, when $g \in \Sigma(\overline{X}^d)$ and $t > 0$,

for some constant $C = C(\overline{X}) > 0$. In this case $\overline{X}^d$ (or $\overline{X}$ as in [8]) is said to be a *Marcinkiewicz pair*.

We say that $T : \overline{X}^d \to \overline{X}$ is a bounded quasi-linear operator if it is an operator

$T:\Sigma(\overline{X})^d \to \Sigma(\overline{X})$ such that $|T(f+g)| \le C(|Tf|+|Tg|)$, and $T(X_j^d) \subset X_j$ with $\|Tf\|_{X_j} \le M_j \|f\|_{X_j}$ $(j=0,1)$.

Lemma 2. *If there exists a bounded quasi-linear operator $T:\overline{X}^d \to \overline{X}$ such that*

(i) $g \in \Sigma(\overline{X})^d$ implies $g \le Tg$, *and*

(ii) $g \ge 0$ *implies Tg decreasing,*

then $\overline{X}^d$ is a Marcinkiewicz pair.

Proof. Let $g = f_0 + f_1 \in \Sigma(\overline{X})^d$, with $0 \le f_j \in X_j$ $(j=0,1)$. Then $g \le C(Tf_0 + Tf_1)$ with $Tf_j \in X_j^d$ and from Lemma 1 we obtain $g = g_0 + g_1$ with $g_j \in X_j^d$ such that $g_j \le CTf_j$. Thus $g \in \Sigma(\overline{X}^d)$ and $K^d(g,t) \le CK(g,t)$. □

A typical choice for T is $T = P + Q = PQ = QP$, where P and Q are the Hardy operator and its adjoint

$$Pg(t) = \frac{1}{t}\int_0^t g(s)\,ds, \qquad Qg(t) = \int_t^\infty g(s)\,\frac{ds}{s}. \tag{2}$$

Remark 3. Let $D_2 f(t) = f(t/2)$. If D_2 and Q are both bounded in $\overline{X}$, then $\overline{X}^d$ is a Marcinkiewicz pair, since Qf is decreasing if $f \ge 0$ and, if f is decreasing, $f(2t) \le C\int_t^{2t} f(s)\frac{ds}{s} \le C\int_t^\infty f(s)\frac{ds}{s}$, whence $f \le Tf$ for $T = CD_2Q$ and Lemma 2 applies.

Remark 4. It is clear that a similar result to Lemma 1 holds for decreasing sequences instead of functions and, with the obvious changes, Lemma 2 is also true when $\overline{X}$ is a pair of quasi-normed lattices of sequences, such as weighted ℓ_p spaces.

Our aim is to study interpolation of classical Lorentz spaces $(\Lambda_u^{p_0}(\omega_0), \Lambda_u^{p_1}(\omega_1))$ through interpolation of the corresponding cones $(L_{p_0}(\omega_0)^d, L_{p_1}(\omega_1)^d)$ of decreasing functions of weighted L_p-spaces. The paper is organized as follows:

Since Lorentz spaces are symmetric spaces associated to weighted L_p-spaces, in Section 2 we consider interpolation of symmetric spaces. Next, in Section 3 we prove that, under suitable conditions and for increasing weights, the couples $(L_{p_0}(\omega_0)^d, L_{p_1}(\omega_1)^d)$ are Marcinkiewicz pairs. Section 4 deals with decreasing weights. In Section 6 we identify the real interpolated spaces of classical Lorentz spaces as function spaces of the same type.

We refer to [4] and [5] for undefined notation and general facts about interpolation spaces, and to [3] and [14] for interpolation properties of Banach function spaces.

2. Symmetric spaces

The quasi-normed function space X is said to be *symmetric* if and only if, whenever $g \in L_0$ and $f \in X$ are such that $g^* \le f^*$, then $g \in X$ and $\|g\|_X \le C\|f\|_X$, for some constant $C = C(X) \ge 1$.

A symmetric space X will be a quasi-Banach function space. All *dilations* $D_\tau f(t) = f(t/\tau)$ $(\tau > 0)$ are bounded in X, since they are bounded on F-spaces (cf. [**13**, Proposition 2]).

Theorem 5. *If X_0 and X_1 are symmetric spaces, then $\Sigma(\overline{X})$, endowed with the usual sum quasi-norm, is also symmetric.*

Proof. Let $f = f_0 + f_1 \in \Sigma(\overline{X})$ with $f_j \in X_j$ $(j = 0, 1)$, $g \in L_0$, $g^* \leq f^*$. Then $f^*(t) \leq f_0^*(t/2) + f_1^*(t/2)$, and from Lemma 1 we obtain

$$g^* = g_0 + g_1, g_j \in X_j^d, g_j \leq D_2(f_j)^* = D_2(f_j^*) \qquad (j = 0, 1).$$

Thus

$$\|g^*\|_{\Sigma(\overline{X})} \leq C(\|g_0\|_{X_0} + \|g_1\|_{X_1}) \leq C\|D_2\|(\|f_0\|_{X_0} + \|f_1\|_{X_1}) \tag{3}$$

with $\|D_2\| = \max(\|D_2\|_{X_0, X_0}, \|D_2\|_{X_1, X_1})$, and

$$\|g^*\|_{\Sigma(\overline{X})} \leq C\|D_2\|\|f\|_{\Sigma(\overline{X})}. \tag{4}$$

If $\lim_{t\uparrow\infty} g^*(t) = 0$, there exists a measure-preserving transform $\sigma : \mathbf{R}^+ \to \mathbf{R}^+$ such that $g = g^* \circ \sigma$ (see [**3**, Corollary II.7.6]). Let $g^* = h_0 + h_1, 0 \leq h_j \in X_j$ $(j = 0, 1)$, such that

$$\|h_0\|_{X_0} + \|h_1\|_{X_1} \leq \|g^*\|_{\Sigma(\overline{X})} + \varepsilon.$$

Since $g = h_0 \circ \sigma + h_1 \circ \sigma$ and $(h_j \circ \sigma)^* = h_j^*$ $(j = 0, 1)$, from (3) and (4) we obtain

$$\|g\|_{\Sigma(\overline{X})} \leq \|h_0^*\|_{X_0} + \|h_1^*\|_{X_1} \leq C(\|h_0\|_{X_0} + \|h_1\|_{X_1}) \leq C(\|D_2\|\|f\|_{\Sigma(\overline{X})} + \varepsilon)$$

and it follows that $\|g\|_{\Sigma(\overline{X})} \leq C\|D_2\|\|f\|_{\Sigma(\overline{X})}$.

If $\lim_{t\uparrow\infty} g^*(t) > 0$, then $\lim_{t\uparrow\infty} f^*(t) > 0$ and $L_\infty \subset X_0 + X_1$. For a given $\varepsilon > 0$, we choose $|\beta(t)| \leq 1$, $|z(t)| \leq \varepsilon$ and a measure-preserving transform $\sigma : \mathbf{R}^+ \to \mathbf{R}^+$ such that $g = (f \circ \sigma)\beta + z$ (see [**14**, Theorem II.2.1]). Then

$$g = (f_0 \circ \sigma)\beta + (f_1 \circ \sigma)\beta + z \in X_0 + X_1$$

with $\|g\|_{\Sigma(\overline{X})} \leq C_0(\|f_0\|_{X_0} + \|f_1\|_{X_1} + \|z\|_{\Sigma(\overline{X})})$, hence $\|g\|_{\Sigma(\overline{X})} \leq C(\|f\|_{\Sigma(\overline{X})} + \varepsilon)$. □

Corollary 6. *If $\overline{X}$ is a couple of symmetric spaces, $0 < q \leq \infty$ and f a function parameter, then $\overline{X}_{f,q}$ is also a symmetric space.*

Remark 7. If $\overline{X}$ is a couple of symmetric spaces, it follows from (3) that $\overline{X}^d$ is a Marcinkiewicz pair (see also [**8**, Theorem 4]) since obviously $\Sigma(\overline{X})^d \subset \Sigma(\overline{X}^d)$ and, for any $g \in \Sigma(\overline{X})^d$, we can write $g = g_0 + g_1$ with $0 \leq g_j \in X_j$ $(j = 0, 1)$ and

$$\|g_0\|_{X_0} + \|g_1\|_{X_1} \le \|D_2\|(K(g,t)+\varepsilon),$$

thus $K^d(g,t) \le CK(g,t)$.

The next theorem will be useful when applied to classical Lorentz spaces $\Lambda^p(\omega)$, since they appear when we "symmetrize" weighted L^p-spaces in the following way:

If X is quasi-normed function space such that D_2 is bounded on X, then we define

$$X^s = \{g \in L_0;\ g^* \in X\} \text{ with } \|g\|_{X^s} = \|g^*\|_X.$$

Remark that $\|\cdot\|_{X^s}$ is a quasi-norm and, if X has Fatou property, then X^s has also this property, and it is complete.

Theorem 8. *If D_2 is bounded on X_0 and X_1, and if $\overline{X}^d$ is a Marcinkiewicz pair, then*

$$(X_0^s, X_1^s)_{f,q} = (X_0, X_1)_{f,q}^s,$$

for any function parameter f and any $0 < q \le \infty$.

Proof. From $X_j^d = X_j^{sd}$ and from Remark 6 we get

$$(X_0, X_1)_{f,q}^d = (X_0^d, X_1^d)_{f,q} = (X_0^s, X_1^s)_{f,q}^d$$

and $(X_0^s, X_1^s)_{f,q} = (X_0, X_1)_{f,q}^s$, since they are both symmetric with the same decreasing functions. □

The boundedness of D_2 on $L_p(\omega)$ is equivalent to the Δ_2-condition,

$$\omega(2t) \le C\omega(t). \tag{5}$$

As easy examples show (cf. [**8**, Examples 1 and 3] and Examples 9 and 15 below), some restrictions on the weights are needed for $(L_{p_0}(\omega_0)^d, L_{p_1}(\omega_1)^d)$ to be a Marcinkiewicz pair. It will be also convenient to consider weighted L_p-spaces for the measure $t^{-1}dt$:

$$L_p^*(\omega) = L_p(\omega(t)t^{1/p}) = \left\{ g \in L_0;\ \int_0^\infty |g(t)\omega(t)|^p \frac{dt}{t} < \infty \right\}.$$

Example 9. For the increasing weights $\omega_0(s) = s$ and $\omega_1(s) = \max(1,s)$,

$$L_1^*(\omega_0)^d = L_1^d,\ L_1^*(\omega_1)^d = L_1(\max(1/s,1))^d$$

is not a Marcinkiewicz pair, since

$$(L_1^*(\omega_0), L_1^*(\omega_1))^d_{9,1} = L_1(\max(1/s, 1)^9)^d \neq \{0\}$$

and $(L_1^*(\omega_0)^d, L_1^*(\omega_1)^d)_{9,1} = \{0\}$.

3. Weighted L_p-spaces with increasing weights

Lemma 10. *If ω is an increasing weight and $0 \leq f \in L_0$, then*

(1) $\sup_{s>0} f^*(s)\omega(s) \leq \sup_{s>0} f(s)\omega(s)$, *and*

(2) $\int_0^\infty f^*(s)\omega(s)\,ds \leq \int_0^\infty f(s)\omega(s)\,ds$.

Proof. For (2) we refer to [**14**, (2.39), p. 74].
For (1) assume that f is a simple function,

$$f = s_n = \sum_{i=1}^{n} a_i \chi_{[\alpha_i,\beta_i)} \qquad (a_i > 0, \beta_i \leq \alpha_{i+1}).$$

If $n = 1$ the result is obvious. Let $n > 1$ and

$$s_n^* = s_{n-1}^* + a_k \chi_{[m,m+\beta_k-\alpha_k)}$$

with $a_k = \min_{1\leq i\leq n} a_i$, $m = \sum_{i\neq k}(\beta_i - \alpha_i)$, $s_n = s_{n-1} + \alpha_k \chi_{[\alpha_k,\beta_k}$.
Now all we need is to consider the following possible situations:

(a) $a_k\omega(\beta_k) \geq \sup s_{n-1}\omega$.

(b) $\sup s_n\omega = \sup s_{n-1}\omega$.

(c) $a_k\omega(m + \beta_k - \alpha_k) \geq \sup s_{n-1}^*\omega$.

(d) $\sup s_n^*\omega = \sup s_{n-1}^*\omega$.

E.g., if (a) and (b) hold true, in the case $\sup s_n^*\omega = \sup s_{n-1}^*\omega$, then by induction,

$$\sup s_n^*\omega = \sup s_{n-1}^*\omega \leq \sup s_{n-1}\omega \leq \sup s_n\omega.$$

In the other case

$$\sup s_n^*\omega \leq a_k\omega(m + \beta_k - \alpha_k) \leq a_n\omega(m + \beta_k - \alpha_k) \leq a_n\omega(\beta_n) \leq \sup s_n\omega. \qquad \square$$

Theorem 11. *Let $0 < p_0, p_1 \leq \infty$. If ω_0, ω_1 are two increasing weights that satisfy the Δ_2-condition* (5), *then $(L_{p_0}(\omega_0)^d, L_{p_1}(\omega_1)^d)$ is a Marcinkiewicz pair.*

Proof. Let $f = f_0 + f_1 \in (L_{p_0}(\omega_0) + L_{p_1}(\omega_1))^d$ with $f_j \in L_{p_j}(\omega_j)$. We apply Lemma 1 to $f \leq D_2 f_0^* + D_2 f_1^*$ and we obtain $f = g_0 + g_1$ with $g_j \in L_{p_j}(\omega_j)^d$ such that $g_j \leq D_2 f_j^*$. We apply Lemma 10 to get

$$\int_0^\infty (D_2 f_j^*)^p \omega_j \le C \int_0^\infty (f_j^*)^p \omega_j \le C \int_0^\infty |f_j|^p \omega_j.$$

Whence, $f \in L_{p_0}(\omega_0)^d + L_{p_1}(\omega_1)^d$ and

$$K^d(f,t) \le \|g_0\|_{L_{p_0}(\omega_0)} + t\|g_1\|_{L_{p_1}(\omega_1)} \le C(\|f_0\|_{L_{p_0}(\omega_0)} + t\|f_1\|_{L_{p_1}(\omega_1)}).$$

Thus $K^d(f,t) \le CK(f,t)$. □

Remark 12. With the obvious changes in Lemma 10 and Theorem 11 we obtain the corresponding results for sequence spaces $\ell_{p_0}(\omega_0)$ and $\ell_{p_1}(\omega_1)$ with $\omega_j = \{\omega_j^n\}_{n=1}^\infty$ increasing weights which satisfy the Δ_2-condition.

Theorem 13. *Let $0 < p_0, p_1 \le \infty$. If ω_0, ω_1 are two increasing weights that satisfy the Δ_2-condition* (5) *and if there exists $a > 1$ such that*

$$\inf_{x>0} \frac{\omega_j(ax)}{\omega_j(x)} = \frac{1}{r} > 1, \tag{6}$$

then $(L_{p_0}^*(\omega_0)^d, L_{p_1}^*(\omega_1)^d)$ is a Marcinkiewicz pair.

Proof. Assume $p_0, p_1 < \infty$. For any $f \in L_{p_j}^*(\omega_j)^d$ we write

$$\int_0^\infty |f(x)\omega_j(x)|^{p_j} \frac{dx}{x} = \sum_{n\in Z} \int_{a^n}^{a^{n+1}} |f(x)\omega_j(x)|^{p_j} \frac{dx}{x}$$

to obtain from the hypotheses

$$\int_0^\infty |f(x)\omega_j(x)|^{p_j} \frac{dx}{x} \simeq \sum_{n\in Z} |f(a^n)\omega_j(a^n)|^{p_j}$$

and it follows that

$$K(f,t; L_{p_0}^*(\omega_0)^d, L_{p_1}^*(\omega_1)^d) \simeq K(\{f(a^n)\}, t; \ell_{p_0}(\omega_0(a^n))^d, \ell_{p_1}(\omega_1(a^n))^d).$$

Now, to apply the version of Lemma 2 for sequences (Remark 4) to prove that $(\ell_{p_0}(\omega_0(a^n))^d, \ell_{p_1}(\omega_1(a^n))^d)$ is a Marcinkiewicz pair, we define $(\tau_n\alpha)(k) = \alpha(k-n)$ and

$$T(\alpha) = \left\{\sum_{k\ge n} |\alpha_k|\right\}_{n\in Z} = \left\{\sum_{m\ge 0} |\tau_{-m}\alpha(n)|\right\}_{n\in Z}.$$

Then

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$$\|T(\alpha)\|_{\ell_{p_j}(\{\omega_j(a^n)\})} \leq \left(\sum_{m\geq 0} \|\tau_{-m}\|\right) \|\alpha\|_{\ell_{p_j}(\{\omega_j(a^n)\})},$$

with $\|\tau_{-m}\| \leq r^{-m}$, since $\omega_j(a^{k-n}) \leq r^{-n} w_j(a^k)$ and

$$\|\tau_{-n}\alpha\|^{p_j}_{\ell_{p_j}(\{\omega_j(a^n)\})} \leq \sum_{k\in\mathbf{Z}} |\alpha(k)\omega_j(a^k)|^{p_j} r^{-np_j}.$$

Thus we have a bounded sublinear operator T on $\ell_{p_j}(\{\omega_j(a^n)\})$ $(j = 0, 1)$ such that $T(\alpha)$ is always decreasing and $|\alpha_n| \leq T(\alpha)_n$, and $(\ell_{p_0}(\omega_0(a^n))^d, \ell_{p_1}(\omega_1(a^n))^d)$ is a Marcinkiewicz pair.

Now, as in [5, Example 2.3.22], we obtain a bounded linear operator

$$R : L^*_{p_j}(\omega_j) \rightarrow \ell_{p_j}(\{\lambda^n_j\}) \qquad (j = 0, 1)$$

if $Rf = \{(\log a)^{-1} \int_{a^n}^{a^{n+1}} f(s)\frac{ds}{s}\}_{n\in\mathbf{Z}}$ and $\lambda^n_j = (\int_{a^n}^{a^{n+1}} \omega_j(s)^{p_j}\frac{ds}{s})^{1/p_j}$, since the weights ω_j are increasing and satisfy the Δ_2-condition. Remark that $\lambda^n_j \simeq \omega_j(a^n)$ and $\ell_{p_j}(\{\lambda^n_j\}) = \ell_{p_j}(\{\omega_j(a^n)\})$.

If f is decreasing, $f(a^{n+1}) \leq (\log a)^{-1} \int_{a^n}^{a^{n+1}} f(s)\frac{ds}{s} \leq f(a^n)$ and then

$$K(Rf, t; \ell_{p_0}(\omega_0(a^n)), \ell_{p_1}(\omega_1(a^n))) \leq CK(f, t; L^*_{p_0}(\omega_0), L^*_{p_1}(\omega_1)),$$

with $\tau_{-1}(\{f(n)\}) \leq Rf$ and τ_{-1} bounded on $\ell_{p_j}(\omega_j(a^n))$. Hence

$$K(\{f(n)\}, t; \ell_{p_0}(\omega_0(a^n)), \ell_{p_1}(\omega_1(a^n))) \leq CK(f, t; L^*_{p_0}(\omega_0), L^*_{p_1}(\omega_1)).$$

To prove the theorem in the case $(L^*_{p_0}(\omega_0)^d, L_\infty(\omega_1)^d)$ $(0 < p_0 < \infty)$, consider the operator $P : L_\infty(\omega_1) \rightarrow \ell_\infty(\omega_1(a^n))$ such that

$$Pf = \left\{\sup_{a^n\leq s<a^{n+1}} |f(a^n)|\right\}_{n\in\mathbf{Z}}. \qquad \square$$

Remark 14. With a similar proof, $(L^*_{p_0}(\omega_0)^d, L^*_{p_1}(\omega_1)^d)$ is still a Marcinkiewicz pair if ω_0 and ω_1 are two weights which satisfy the Δ_2-condition and with properties

$$\inf_{x>0} \frac{\omega_j(ax)}{\omega_j(x)} > 1, \sup_{n\in\mathbf{Z}} \sup_{s,t\in[a^n,a^{n+1}]} \frac{\omega_j(s)}{\omega_j(t)} < \infty,$$

without any monotonicity condition on the weights.

4. Weighted L_p-spaces with decreasing weights

For decreasing weights it is easy to produce examples of couples of weighted L_p-spaces whose cones of decreasing functions are not Marcinkiewicz pairs.

Example 15. Let $0 < \vartheta < 1$ and ω any integrable decreasing weight such that $\omega^\vartheta \in L_1$ (e.g., $\omega(t) = e^{-t}$). Then

$$(L_1^d, L_1(\omega)^d)_{\vartheta,1} \neq (L_1, L_1(\omega))_{\vartheta,1}^d,$$

since in this case $K^d(1,t) = t\|1\|_{L_1(\omega)}$ and $1 \notin (L_1^d, L_1(\omega)^d)_{\vartheta,1}$, but $1 \in L_1(\omega^\vartheta)^d = (L_1, L_1(\omega))_{\vartheta,1}^d$.

Remark 16. For increasing weights we have used a condition on the growth, the Δ_2-condition. For a decreasing weight we shall consider the condition

$$\frac{1}{s}\int_0^s \omega(t)\,dt \simeq \omega(s) \qquad \left(\text{i.e., } \frac{1}{s}\int_0^s \omega(t)\,dt \leq C\omega(s)\right) \tag{7}$$

which is a decrease property, since, as shown in [9], it is equivalent to

$$\inf_{x>0} \frac{\omega(rx)}{\omega(x)} > \frac{1}{r} \text{ for some constant } r > 1.$$

Theorem 17. *Let $0 < p_0, p_1 < \infty$ and let ω_0, ω_1 be two decreasing weights such that $\omega_0^{p_0}$ and $\omega_1^{p_1}$ satisfy the decrease condition* (7). *Then $(L_{p_0}(\omega_0)^d, L_{p_1}(\omega_1)^d)$ is a Marcinkiewicz pair.*

Proof. For $\Phi_j(x) = (\int_0^x \omega_j^{p_j}(s)\,ds)^{1/p_j}$ we have $L_{p_j}(\omega_j) = L_{p_j}^*(\Phi_j)$ with equivalent norms and we know from Theorem 13 that $(L_{p_0}^*(\Phi_0)^d, L_{p_1}^*(\Phi_j)^d)$ is a Marcinkiewicz pair. □

Remark 18. In the case $p_j \geq 1$, in Theorem 17 we are under the conditions of Remark 3, since, as shown in [9], for any decreasing weight ω and $p \geq 1$, the following properties are equivalent:

(a) $Q : L_p(\omega)^d \to L_p(\omega)$, bounded.

(b) $Q : L_p(\omega) \to L_p(\omega)$, bounded.

(c) ω^p satisfies condition (7).

5. Interpolation of Lorentz spaces

In general it is not true that the interpolated space by the real K-method of a couple $(\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))$ of Lorentz spaces is the Lorentz space associated to the interpolated space of corresponding L_p-spaces. In Example 15 of a couple $(L_1(\omega_0)^d, L_1(\omega_1)^d)$ which is not a Marcinkiewicz pair,

$$(\Lambda^1(\omega_0), \Lambda^1(\omega_1))_{\vartheta,1} \neq \Lambda^1(\omega_0^{1-\vartheta}\omega_1^\vartheta).$$

The results of the previous sections allow to give sufficient conditions to identify the interpolated spaces of couples of Lorentz spaces Λ_u^p defined as in (1).

Theorem 19. *Let (ω_0, ω_1) be a couple of Δ_2-weights and $0 < p_0, p_1 \leq \infty$, and assume that $(L_{p_0}(\omega_0)^d, (L_{p_1}(\omega_1)^d))$ is a Marcinkiewicz pair. Then, for any weight u,*

$$(\Lambda_u^{p_0}(\omega_0), \Lambda_u^{p_1}(\omega_1))_{\vartheta,q} = \Lambda_u^q(\omega_0^{1-\vartheta}\omega_1^{\vartheta})$$

if $1/q = (1-\vartheta)/p_0 + \vartheta/p_1$ and $0 < \vartheta < 1$.

Proof. Denote

$$K(f,t) = K(f,t; \Lambda_u^{p_0}(\omega_0), \Lambda_u^{p_1}(\omega_1))$$

and

$$K(f_u^*, t) = K(f_u^*, t; L_{p_0}(\omega_0), L_{p_1}(\omega_1)).$$

We only need to prove that $K(f,t) \simeq K(f_u^*, t)$ (cf. [**4**, Theorem 5.5.1]).

Obviously, $K(f_u^*, t) \leq cK(f,t)$ for any $f \in \Lambda_u^{p_0}(\omega_0) + \Lambda_u^{p_1}(\omega_1)$, since we can consider $f = f_0 + f_1$ with

$$\|f_0\|_{\Lambda_u^{p_0}(\omega_0)} + t\|f_1\|_{\Lambda_u^{p_1}(\omega_1)} \leq 2K(f,t)$$

and then $f_u^* \leq D_2 f_{0u}^* + D_2 f_{1u}^*$.

For the converse we consider two cases. First assume that $f_u^* \in L_{p_0}(\omega_0) + L_{p_1}(\omega_1)$ is such that $\lim_{x\to\infty} f_u^*(x) = 0$.

Since $(L_{p_0}(\omega_0)^d, L_{p_1}(\omega_1)^d)$ is a Marcinkiewicz pair, we can consider a decomposition $f_u^* = f_0 + f_1$ with $f_j \in L_{p_j}(\omega_j)^d$ with

$$\|f_0\|_{L_{p_0}(\omega_0)} + t\|f_1\|_{L_{p_1}(\omega_1)} \leq 2K^d(f_u^*, t).$$

In our case there exists a measure-preserving transformation σ (cf. [**3**, Corollary II.7.6]) such that

$$f = f_u^* \circ \sigma = f_0 \circ \sigma + f_1 \circ \sigma$$

with $f_j \circ \sigma$ and f_j equimeasurable. So $(f_j \circ \sigma)_u^* = f_j^* = f_j$ (a decreasing function), $\|f_j \circ \sigma\|_{\Lambda_u^{p_j}(\omega_j)} = \|f_j\|_{L_{p_j}(\omega_j)}$ and

$$K(f,t) \leq \|f_0 \circ \sigma\|_{\Lambda_u^{p_0}(\omega_0)} + t\|f_1 \circ \sigma\|_{L_u^{p_1}(\omega_1)} \leq \|f_0\|_{L_{p_0}(\omega_0)} + t\|f_1\|_{L_{p_1}(\omega_1)} \leq 2K^d(f_u^*, t).$$

Assume now that $f_u^* \in L_{p_0}(\omega_0) + L_{p_1}(\omega_1)$ is such that $\lim_{x\to\infty} f_u^*(x) = f_u^*(\infty) > 0$ (thus $L_{p_0}(\omega_0) + L_{p_1}(\omega_1)$ contains L_∞) and let $f_u^* = f_0 + f_1$ as above. In this case we consider any measure-preserving transformation σ between $\mathbf{R}^+$ with the measure u and $\mathbf{R}^+$ with Lebesgue measure with $\operatorname{supp}\sigma = \mathbf{R}^+$, and define $h = f_u^* \circ \sigma = f_0 \circ \sigma + f_1 \circ \sigma$. Again we obtain

$$K(h, t) \leq 2K^d(f_u^*, t).$$

Since $h_u^* = f_u^*$,

$$K(h, t) \simeq K(f_u^*, t) = K(h_u^*, t)$$

and we observe that $K(f, t) \leq CK(h, t)$ by considering a measure-preserving transformation ω such that $|f(t)| \leq |h \circ \omega(t)| + \varepsilon$ (cf. [**14**, Theorem II.2.1]), hence

$$K(f, t) \leq K(h \circ \omega, t) + \varepsilon K(1, t) \leq K(h, t) + \varepsilon \min(\|1\|_{L^{p_0}(\omega_0)}, t\|1\|_{L^{p_1}(\omega_1)}),$$

for every $\varepsilon > 0$.

Since $f_u^*(\infty) \in L_{p_0}(\omega_0)^d + L_{p_1}(\omega_1)^d$ and a decomposition of a constant function into sum of two decreasing functions necessarily gives constant terms, and $(L_{p_0}(\omega_0)^d, L_{p_1}(\omega_1)^d)$ is a Marcinkiewicz pair, one of the integrals $\|1\|_{L^{p_0}(\omega_0)}$, $\|1\|_{L^{p_1}(\omega_1)}$ is finite.

Hence $\|1\|_{L^{p_0}(\omega_0)} < \infty$ or $\|1\|_{L^{p_1}(\omega_1)} < \infty$ respectively. □

Corollary 20. *Let ω be any Δ_2-weight and $0 < p_0, p_1 \leq \infty$. Then*

$$(\Lambda_u^{p_0}(\omega), \Lambda_u^{p_1}(\omega))_{\vartheta,q} = \Lambda_u^q(\omega)$$

if $1/q = (1-\vartheta)/p_0 + \vartheta/p_1$, for any weight u.

Proof. In [**8**, Remark 3] we show that, for any quasi-Banach function lattice X, (X^d, L_∞^d) is a Marcinkiewicz pair. Thus, so is $(L_r(\omega)^d, L_\infty^d)$ and, if we consider $0 < r < p_j$ $(j = 0, 1)$, by reiteration (cf. [**8**, Corollary 2]) it follows that $(L_{p_0}(\omega)^d, L_{p_1}(\omega)^d)$ is also a Marcinkiewicz pair, and we can apply Theorem 19. □

Observe that Theorem 19 holds also for Lorentz spaces on $\mathbf{R}^n$.

Moreover, as an application of Theorem 8 to the case of weighted L_p spaces, since the interpolation of weighted L_p spaces is well known (cf. [**10**]), we can state another description of the interpolated spaces of couples of Lorentz spaces. If ω, u and v are three weights and $0 < p \leq \infty$, define

$$\Lambda_{u,v}^p(\omega) = \{f \in L_0;\ (f^*v)_u^* \in L_p(\omega)\}$$

with $\|f\| = \|(f^*v)_u^*\|_{L_p(\omega)}$. In the case $u = v = 1, \Lambda_{u,v}^p(\omega) = \Lambda^p(\omega)$ and, when v is decreasing, if $U(t) = \int_0^t u(s)\,ds$, $\Lambda_{u,v}^p(\omega) = \Lambda_u^p(\omega)^s = \Lambda^p(v\omega(U)u^{1/p})$, which is space $\Lambda^p(\omega')$ for a suitable weight w'. Then we have:

Remark 21. Let (ω_0, ω_1) be a couple of Δ_2-weights, f a function parameter, $0 < q < \infty$ and $0 < p_0, p_1 \le \infty$, and assume that $(L_{p_0}(\omega_0)^d, L_{p_1}(\omega_1)^d)$ is a Marcinkiewicz pair. In this case, the interpolation results for weighted L^p spaces can be used (see [15] and [16]) and we obtain:

(a) If $p_0 \ne p_1$, then

$$(\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))_{f,q} = \Lambda^p_{u,v}(\omega),$$

where $v = (\omega_0/\omega_1)^{p_0p_1/(p_1-p_0)}$, $u = (\omega_1^{p_1}/\omega_0^{p_0})^{1/(p_1-p_0)}$ and $\omega = t^{1/p_0}f(t^{1/p_0-1/p_1})$.

If $1/q = (1-\vartheta)/p_0 + \vartheta/p_1$ $(0 < \vartheta < 1)$, again

$$(\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))_{\vartheta,q} = \Lambda^q(\omega_0^{1-\vartheta}\omega_1^{\vartheta}).$$

(b) If $p_0 = p_1 = p$, then

$$(\Lambda^p(\omega_0), \Lambda^p(\omega_1))_{f,p} = \Lambda^p(\omega_0 f(\omega_1/\omega_0)),$$

Finally, recall the definition of the Lorentz-Sharpley space associated to a symmetric Banach space X whose fundamental function is the function parameter $\Phi_X(t) = \|\chi_{(0,t)}\|_X$ (cf. [20]):

$$\Lambda^p(X) = \{f \in L_0;\ f^* \in L_p^*(\Phi_X)\} = L_p^*(\Phi_X)^s$$

and, in the case $p = \infty$,

$$M(X) = \Lambda^\infty(X) = \{f \in L_0;\ \sup_{s>0} f^*(s)\Phi_X(s) < \infty\}.$$

By interpolation of couples of such spaces we obtain Lorentz spaces:

Theorem 22. *Let (X_0, X_1) be a couple of symmetric Banach spaces, f a function parameter and $0 < q \le \infty$. Then $(M(X_0)^d, M(X_1)^d)$ is a Marcinkiewicz pair and*

$$(M(X_0), M(X_1))_{f,q} = (L_\infty(\Phi_{X_0}), L_\infty(\Phi_{X_1}))^s_{f,q}.$$

If, additionally, the lower fundamental indices (defined as in [3]) satisfy $\alpha_{X_j} > 0$ and $1 \le p_0, p_1 \le \infty$, then

$$(\Lambda^{p_0}(X_0), \Lambda^{p_1}(X_1))_{f,q} = (L^*_{p_0}(\Phi_{X_0}), L^*_{p_1}(\Phi_{X_1}))^s_{f,q}.$$

Proof. The fundamental functions Φ_{X_j} are increasing and satisfy the Δ_2-condition. It follows from Theorem 11 that $(M(X_0), M(X_1))_{f,q} = (L_\infty(\Phi_{X_0}), L_\infty(\Phi_{X_1}))^s_{f,q}$.

Now we assume that X_j is a symmetric space such that $\alpha_{X_j} > 0$. In this case, Φ_{X_j} is an increasing function such that

$$\inf_{x>0} \frac{\Phi_{X_j}(2x)}{\Phi_{X_j}(x)} > 1,$$

it follows from Theorem 13 that $(L^*_{p_0}(\Phi_{X_0})^d, L^*_{p_1}(\Phi_{X_1})^d)$ is a Marcinkiewicz pair. □

Final remark. For simplicity, in the hypotheses we have considered Δ_2-weights, but we only apply D_2 as a bounded operator on the decreasing functions f^*_u and in fact what is needed is only that the weights

$$W(x) = \int_0^x \omega(t)\, dt$$

satisfy this Δ_2-condition. This fact follows from the identity

$$\int_0^\infty f^*_u(t)^p \omega(t)\, dt = p \int_0^\infty y^{p-1} \left(\int_0^{\lambda^u_f(y)} \omega(t)\, dt \right) dy$$

with $\lambda^u_f(y) = \int_{\{x;|f(x)|>y\}} u(x)\, dx$ (cf. [7, Theorem 2.1]).

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