SCALES OF LOGARITHMIC METHODS OF SUMMABILITY

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1. Introduction. We suppose throughout that $p$ is a non-negative integer, and use the following notations:

$$
\pi_p(x) = \begin{cases} 
\frac{1}{\log_0 x \cdot \log_1 x \cdot \ldots \cdot \log_p x} & \text{for } x > e^p, \\
0 & \text{otherwise}, 
\end{cases}
$$

where $\log_0 x = x$ for $x > e_0 = 1$, and $\log_{n+1} x = \log(\log x)$ for $x > e_{n+1} = e^n$ $(n = 0, 1, 2, \ldots)$;

$$
\sigma_p(x) = \sum_{n=0}^{\infty} \pi_p(n) x^n \quad (-1 < x < 1);
$$

$$
S_n = \sum_{k=0}^{n} a_k 
$$

$$
(n = 0, 1, 2, \ldots);
$$

$$
t_p(n) = \frac{1}{\log_{p+1} n} \sum_{k=0}^{n} \pi_p(k) s_k \quad (n \geq e_{p+1}).
$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $L_p$ to $s$, and we write $\sum_{n=0}^{\infty} a_n = s(p)$ or $s \rightarrow s(p)$, if

$$
\lim_{x \rightarrow 1^{-}} \frac{1}{\sigma_p(x)} \sum_{n=0}^{\infty} \pi_p(n) s_n x^n = s.
$$


445
If \( t(n) \to s \) as \( n \to \infty \) the series \( \sum_{n=0}^{\infty} a_n \) is said to be summable \( \ell^p \) to \( s \), and we write \( \sum_{n=0}^{\infty} a_n = s(\ell^p) \) or \( s \to s(\ell^p) \) (see [5]).

Given two summability methods \( A, B \) we write \( A \supset B \) if any series summable \( B \) is summable \( A \) to the same sum; if in addition there is a series summable \( A \) but not summable \( B \) we write \( A \supset B \). If \( A \supset B \) and \( B \supset A \) the two methods are said to be equivalent and we write \( A \cong B \). It is known [5] that the \( \ell^p \) methods are regular and that \( L_0 \cong L, \ell_0 \cong \ell \) where \( L \) and \( \ell \) are standard logarithmic methods (for definitions see [3]). The aim of this paper is to establish various inclusion theorems for the two scales of methods.

2. Lemmas. We require four lemmas.

**Lemma 1.** If \( s \to s(\ell^p) \), then \( \frac{1}{\pi p+1(n)} \) and \( \frac{1}{\pi p+1(n+1)} \) and

\[
s_n = \frac{1}{\pi p+1(n)} \left[ t(n) \log_{p+1} n - t(n-1) \log_{p+1} (n-1) \right];
\]

hence

\[
\pi_{p+1}(n)s_n = t(n) - t(n-1) \frac{\log_{p+1}(n-1)}{\log_{p+1} n} \to 0,
\]

and so,

\[
\pi_{p+1}(n)a_n = \pi_{p+1}(n)s_n - \frac{\pi_{p+1}(n)}{\pi_{p+1}(n-1)} \pi_{p+1}(n-1)s_{n-1} \to 0.
\]
LEMMA 2. \( \ell_p \supseteq \ell_p \).

Proof. Since \( \ell_p \cong (N, q_n) \) with \( q_n = \pi_p(n) \), the lemma follows from a known result [4, Theorem 1].

LEMMA 3. If \( x \geq e_p \), \( y > 0 \), then

\[
(\log_p x)^{-y} = \int_0^\infty e^{-xt} \lambda_{p,y}(t) dt,
\]

where \( \lambda_{p,y}(t) \) is defined by the recursive formulae:

\[
\lambda_{0,y}(t) = \frac{t^{y-1}}{\Gamma(y)},
\]

\[
\lambda_{r+1,y}(t) = \frac{1}{\Gamma(y)} \int_0^\infty u^{y-1} \lambda_{r,u}(t) du \quad (r=0,1,2,\ldots).
\]

Proof. The lemma is true for \( p = 0 \), since, when \( x \geq e_0 = 1 \),

\[
(\log_0 x)^{-y} = x^{-y} = \frac{1}{\Gamma(y)} \int_0^\infty e^{-xt} t^{y-1} dt = \int_0^\infty e^{-xt} \lambda_{0,y}(t) dt.
\]

Assume the lemma is true for \( p = r \). Then, for \( x \geq e_{r+1} \) we have

\[
(\log_{r+1} x)^{-y} = \frac{1}{\Gamma(y)} \int_0^\infty e^{-u \log_{r+1} x} u^{y-1} du
\]

\[
= \frac{1}{\Gamma(y)} \int_0^\infty (\log_x)^{-u} u^{y-1} du
\]

447
\[
= \frac{1}{\Gamma(y)} \int_0^\infty u^{y-1}du \int_0^\infty e^{-xt} \lambda_{r,u} \, dt
\]
\[
= \int_0^\infty e^{-xt} \, dt \frac{1}{\Gamma(y)} \int_0^\infty u^{y-1} \lambda_{r,u} \, du
\]
\[
= \int_0^\infty e^{-xt} \lambda_{r+1,u} \, dt,
\]

the inversion in the order of integration being justified by Fubini's theorem since all the functions concerned are non-negative and Lebesgue measurable. The lemma is thus established by induction.

The case \( p = 1 \) of the next lemma is due to Hardy [2, page 268].

**Lemma 4.** If \( n > e_p \), \( y > 0 \), then

\[
(\log n)^{-y} = \int_0^1 t^n \phi(t) \, dt,
\]

where the function \( \phi \) is non-negative and independent of \( n \).

**Proof.** By Lemma 3,

\[
(\log n)^{-y} = \int_0^\infty e^{-nx} \lambda_{p,y} \, (x) \, dx = \int_0^1 t^n \phi(t) \, dt,
\]

where \( \phi(t) = \frac{1}{t} \lambda_{p,y} (\log \frac{1}{t}) \).

3. **Inclusion Theorems.**

**Theorem 1.** There is a series summable \( \ell_{p+1} \) but not summable \( L_p \) i.e. \( L_p \not\subset \ell_{p+1} \).

448
Proof. Let $N$ be the integer such that $N - 1 < e_{p+1} < N$, and, with $i = \sqrt{-1}$, let

$$a_n = \begin{cases} \pi(n)(\log_{p+1} n)^{-1-i} & \text{for } n \geq e_{p+1}, \\ 0 & \text{for } n < e_{p+1}. \end{cases}$$

Then

$$s_{n-1} - \left( (\log_{p+1} n)^{-1} - (\log_{p+1} N)^{-1} \right)$$

$$= \sum_{k=N}^{n-1} \pi(k)(\log_{p+1} k)^{-1-i} - \int_{N}^{n} (\log_{p+1} t)^{-1-i} \pi(t) dt$$

$$= \sum_{k=N}^{n-1} \xi_k,$$

where

$$\xi_k = \int_{k}^{k+1} \left( \int_{k}^{t} \left( - \frac{d}{dx} \pi(x)(\log_{p+1} x)^{-1-i} \right) dx \right) dt$$

$$= \int_{k}^{k+1} \left( \int_{k}^{t} (\pi'(x))(\log_{p+1} x)^{-1-i} \left( \sum_{r=0}^{p} \frac{\pi(x)}{r} + (1+i)(\log_{p+1} x)^{-1} \right) dx \right) dt$$

$$= \int_{k}^{k+1} \left( \int_{k}^{t} \frac{x^{1/2}}{x} dx \right) dt$$

$$= O\left( \frac{1}{k^2} \right).$$
Hence \( \sum_{k=N}^{\infty} \phi_k \) converges, and so \( s_{n-1}^{-1} (\log_{p+1} n)^{-i} \) tends to a finite limit as \( n \to \infty \). Since \( s_n = s_{n-1} + \pi_p(n) (\log_{p+1} n)^{-i} \), we have that \( s_n = i(\log_{p+1} n)^{-i} + k_n \) where \( k_n \) tends to a finite limit as \( n \to \infty \).

Consequently \( \{s_n\} \) is bounded but does not converge, and as \( a_n = O(\pi_p(n)) \), it follows from a known tauberian theorem [5, Corollary] that \( \sum_{n=0}^{\infty} a_n \) is not \( L_p \) summable.

We now show that \( \sum_{n=0}^{\infty} a_n \) is \( L_{p+1} \) summable. For \( m \geq N \), we have that

\[
t_{p+1}(m) = \frac{1}{\log p+2} \sum_{n=N}^{m} \pi_{p+1}(n) \left( \frac{(\log_{p+1} n)^{-i}}{i} + k_n \right)
\]

\[
= \frac{1}{i \log p+2} \sum_{n=N}^{m} \pi_p(n)(\log_{p+1} n)^{-i} - i
\]

\[+ \frac{1}{\log p+2} \sum_{n=0}^{m} \pi_{p+1}(n)k_n \]

and hence \( t_{p+1}(m) \) tends to a finite limit as \( m \to \infty \).

**Theorem 2.** \( L_{p+1} \supset L_p \).
Proof. By Lemma 4, for \( n \geq e_{p+1}' \)

\[
\frac{\pi_{p+1}(n)}{\pi(n)} = (\log_{p+1} n)^{-1} = \int_{0}^{1} t^n \phi(t) dt,
\]

where \( \phi(t) \) is non-negative and independent of \( n \), and hence, by a result due to Borwein [1, Theorem A], \( L_{p+1} \supseteq L_p \). The stronger inclusion follows immediately from Theorem 1 and Lemma 2.

**THEOREM 3.** \( L_p \supset L_p \).

**Proof.** We consider a series used to show the existence of a series summable by the Abel method \( A \), but not summable by any Cesàro method [2, Theorem 56].

\[
e^{1/(1+x)} = \sum_{n=0}^{\infty} a_n x^n.
\]

It is known that \( a_n \) is not \( O(n^r) \) for any \( r \), and hence, by Lemma 1, \( \sum_{n=0}^{\infty} a_n \) is not summable \( L_p \). Since the series is summable \( A \), and [2, page 81] \( A \subseteq L \preceq L_0 \subseteq L_p \), the theorem can now be deduced from Lemma 2.

**THEOREM 4.** \( f_{p+1} \supset f_p \).

**Proof.** The inclusion \( f_{p+1} \supseteq f_p \) follows immediately from a known theorem for \( \bar{N} \) methods [2, Theorem 14]. The stronger inclusion may be deduced from Theorem 1. However a direct proof is easy.

Consider

\[
s_n = (-1)^n \frac{1}{\pi_{p+1}(n)} \quad (n \geq e_{p+1}).
\]
Then $s_n \to 0 \left( \ell_{p+1} \right)$, i.e. $\sum_{n=0}^{\infty} a_n$ is summable $\ell_{p+1}$, but $s_n \neq o\left( \frac{1}{\pi_{p+1}(n)} \right)$; hence, by Lemma 1, $\sum_{n=0}^{\infty} a_n$ is not $\ell_p$ summable.

REFERENCES


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