Glasgow Math. J. **55** (2013) 369–380.
doi:10.1017/S0017089512000614.

$\aleph_n$-FREE MODULES OVER COMPLETE DISCRETE VALUATION DOMAINS WITH ALMOST TRIVIAL DUAL*

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(Received 17 February 2011; revised 14 September 2011; accepted 30 August 2012; first published online 25 February 2013)

Abstract. A module M over a commutative ring R has an almost trivial dual if there is no homomorphism from M onto a free R-module of countable infinite rank. Using a new combinatorial principle (the $\aleph_n$-Black Box), which is provable in ordinary set theory, we show that for every natural number n, there exist arbitrarily large $\aleph_n$-free R-modules with almost trivial duals, when R is a complete discrete valuation domain. A corresponding result for torsion modules is also obtained.

2010 *Mathematics Subject Classification.* 20A15, 20K10, 20K20, 20K21, 20K30, 13B10, 13L05.

1. Introduction. For a module M over a commutative ring R, let $M^* = \mathrm{Hom}_R(M, R)$ be the dual module over R. The problem of building uncountable R-modules M with trivial dual, i.e. $M^* = 0$, has attracted considerable attention in the research literature. It is a stronger form of the challenge of constructing κ-free non-free modules, since κ-free modules with trivial duals are clearly not free. (Recall that a module M is κ-free if all its submodules generated by $< \kappa$ elements are contained in a free R-submodule – see [**4**, **10**]). If R is a countable domain, but not a field, then it is clear how to construct proper classes of torsion-free R-modules with trivial dual, e.g. apply Corner [**1**] or Corner–Göbel [**2**]. In this case, in [**9**] the authors proved that there is even a proper class of $\aleph_n$-free R-modules M with trivial dual.

However, the result fails if R is uncountable, as can be seen from Kaplansky's [**11**] well-known splitting theorems for modules over the ring J_p of p-adic integers. Nevertheless, we want to extend the main result from [**9**] in the torsion-free case and also the torsion case to modules over complete discrete valuation domains (DVDs), in particular, to p-adic modules. Recall that an R-module M is κ-Σ-cyclic if every one

*Publication (GbShSm:981) in the second author's list of publications.

of its submodules generated by $< \kappa$ elements is contained in a direct sum of cyclic R-submodules. Our main theorem is as follows.

THEOREM 1.1. *If n is a natural number and R is a complete DVD, then there are proper classes of torsion-free modules M and torsion modules N with the following properties:*

(i) *M is an $\aleph_n$-free R-module such that $F = \bigoplus_{n\in\mathbb{N}} R$ is not an epimorphic image of M.*
(ii) *N is an $\aleph_n$-Σ-cyclic R-module such that $T = \bigoplus_{n\in\mathbb{N}} R/p^nR$ is not an epimorphic image of N.*

Note that a DVD R has a unique prime ideal pR and is complete in its p-adic topology, and that Teichmüller's Theorem (see [**11**, p. 43]) guarantees that there is a wide range of complete DVDs (see also [**12**, Section 3]).

The formulation of the above theorem is justified by the following observation. Let us assume that R is a complete DVD, e.g. the ring J_p of p-adic integers. In this case it was shown by Kaplansky [**11**, pp. 46, 52, 53] that any non-trivial, reduced R-module M has a non-trivial cyclic R-module which is a summand and hence an epimorphic image of M. (Recall that M is *reduced* if $\bigcap_{n<\omega} p^nR = 0$.) Thus, Kaplansky's decomposition theorem requires that we must adjust the trivial dual condition to extend the result from [**9**] to complete DVDs. Accordingly we define two kinds of test modules F and T. As in Theorem 1.1, let $F = \bigoplus_{n\in\mathbb{N}} R$ be a fixed free R-module of countable, infinite rank and let $T = \bigoplus_{n\in\mathbb{N}} R/p^nR$ be the canonical direct sum of torsion cyclic R-modules. We will distinguish the torsion-free case and the torsion case. In the torsion-free case we want to find κ-free modules M and in the torsion case we will construct κ-Σ-cyclic modules M for any $\kappa = \aleph_n$ ($n \in \mathbb{N}$). The notion in the torsion case also agrees with a parallel notion of κ-Σ-cyclic abelian p-groups investigated under $V = L$ (see e.g. [**3**]). In the torsion-free case we want to show that there are suitable torsion-free modules M such that F is not an epimorphic image of M, and in the torsion case we want to find torsion modules M with no epimorphic image T, which replaces the 'trivial dual' condition $M^* = 0$. We will say that M has *almost trivial dual* in either case. In order to deal with both cases simultaneously we will say that M is κ-free (κ-Σ-cyclic), where κ-Σ-cyclic refers to a torsion module which is κ-Σ-cyclic. In order to carry out our construction we will need that R is complete but $\bigoplus_\omega R$ is not. This will be ensured by the chosen topology which has a countable system of neighbourhoods of 0.

The paper is organized as follows. In Section 2 we consider a recent prediction principle from Shelah [**13**] adapted in [**9**] for algebraic applications. It has to be slightly strengthened to work in this situation (found in the next section). The crucial idea is that it will automatically provide the resources required to show that the constructed modules are $\aleph_n$-free ($\aleph_n$-Σ-cyclic). The duality test is an algebraic input which is 'classical'. The problem of realizing rings as endomorphism rings over modules with large free submodules in ordinary set theory, Zermelo–Fraenkel set theory with the axiom of choice (ZFC), is similar but much more complicated. This, however, is the work in progress [**6**].

In Section 3 we will carry out the main construction and prove the main theorem. We will also determine the cardinalities of the modules M and N. The minimal cardinal for these modules in Theorem 3.3 is $\beth_n(|R|)$ (which is the cardinality we get taking n-times the powerset of R). Again we would like to remind the reader that in the constructible universe L, $\beth_n = \beth_n(\aleph_0)$ equals the least possible value $\aleph_n$, since Generalised Continuum Hypothesis (GCH) holds in L. The additional work in ZFC results from the fact that a good combinatorial substitute, replacing the diamond prediction principle, is necessary to ensure the $\aleph_n$-freeness of the modules.

Finally, in Section 4 we state some possible generalizations and their limitations.

2. The $\aleph_n$-free black box from [9]. Here we outline the new combinatorial principle from [13], which was simplified in [9]. It is designed to construct $\aleph_n$-free abelian groups in ZFC (without any additional axioms). Like Shelah's ordinary Black Box (see [10]) this prediction principle imitates the well-known diamond principle (which holds in the constructible universe L), and although it predicts much less, it is nevertheless strong enough to be very useful for applications in algebra. This new so-called $\aleph_n$-Black Box was first used to construct arbitrarily large $\aleph_n$-free abelian groups with trivial dual. Here we want to apply it for the construction of $\aleph_n$-free ($\aleph_n$-Σ-cyclic) R-modules over complete DVDs with almost trivial duals (see the notations in the Introduction). Thus, this set-theoretic machinery requires some modifications for complete DVDs in Section 3. We will first state the prediction principle and provide details of its proof useful for the construction of $\aleph_n$-free modules and $\aleph_n$-Σ-cyclic modules and possibly further applications.

The $\aleph_n$-Black Box is defined relative to a finite sequence $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{k_*} \rangle$, $k_* < \omega$ of cardinals such that for the cardinals

$$\chi_l := \lambda_l^{\aleph_0} \ (l \leq k_*) \tag{2.1}$$

the following ■-*conditions* (2.2) hold:

$$\chi_{l+1}^{\chi_l} = \chi_{l+1} \ (l < k_*). \tag{2.2}$$

The sequence $\overline{\lambda}$ will be called a *black box sequence* (■-sequence); clearly its members satisfy $\chi_l = \chi_l^{\aleph_0} < \chi_{l+1}$.

EXAMPLE 2.1. It is easy to find ■-sequences:

(a) Given any cardinal λ, then define inductively a $\overline{\lambda}$-sequence by $\chi_1 = \lambda^{\aleph_0}$ and if λ_l is defined for $l < k_*$, then choose a suitable $\lambda_{l+1} > \lambda_l$ with (2.2), e.g. put $\lambda_{l+1} = \chi_l^{\chi_l}$.

(b) $\langle \beth_1, \ldots, \beth_{k_*} \rangle$ is a ■-sequence.

If λ is a cardinal, then ${}^{\omega\uparrow}\lambda$ denotes all infinite branches, i.e. all *order-preserving* maps $\eta : \omega \to \lambda$ on λ, while ${}^{\omega\uparrow>}\lambda$ denotes the family of all order-preserving *finite branches* $\eta : n \to \lambda$ on λ, where the ordinals n, λ and ω are considered as sets, i.e. $n = \{0, \ldots, n-1\}$, thus the finite branch η has length n.

Now let $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{k_*} \rangle$ be a ■-sequence and put

$$\Lambda = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow}\lambda_{k_*}.$$

If $1 \leq m \leq k_*$, then we define

$$\Lambda_m = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow>}\lambda_m \times \cdots \times {}^{\omega\uparrow}\lambda_{k_*} \text{ and let } \Lambda_* = \dot{\bigcup_{m \leq k_*}} \Lambda_m. \tag{2.3}$$

The elements of Λ, Λ_* are written as sequences $\overline{\eta} = (\eta_1, \ldots, \eta_{k_*})$ with $\eta_l \in {}^{\omega\uparrow}\lambda$ or $\eta_l \in {}^{\omega\uparrow>}\lambda$ respectively. These $\overline{\eta}$s are used as supports of elements of the module to be constructed and hence will ensure $\aleph_n$-freeness.

DEFINITION 2.2. If $\overline{\eta} = (\eta_1, \ldots, \eta_{k_*}) \in \Lambda$ and $m \leq k_*$, $n < \omega$, then let $\overline{\eta}\restriction\langle m, n\rangle$ be the following element in Λ_m (thus in Λ_*)

$$(\overline{\eta}\restriction\langle m, n\rangle)_l = \begin{cases} \eta_l & \text{if } m \neq l \leq k_* \\ \eta_m \restriction n & \text{if } l = m \end{cases}.$$

With each $\overline{\eta}$ a *support* is associated in the following way: $[\overline{\eta}] = \{\overline{\eta}\restriction\langle m, n\rangle \mid m \leq k_*, n < \omega\}$, which is a countable subset of Λ_*. Similarly, for $m \leq k_*$ also let $[\overline{\eta}\restriction m] = \{\overline{\eta}\restriction\langle m, n\rangle \mid n < \omega\} \subseteq [\overline{\eta}]$.

DEFINITION 2.3. Let C be a set of size $\leq \chi_1$ and define a *set-trap (for* Λ, C*)* as a map $\varphi_{\overline{\eta}} : [\overline{\eta}] \to C$ with a label $\overline{\eta} \in \Lambda$.

The following lemma and theorem were proved in [**9**, Lemma 2.3, Theorem 2.4] and constitute the combinatorial version of the $\aleph_n$-Black Box.

LEMMA 2.4. *Let λ be an infinite cardinal, $\chi = \lambda^{\aleph_0}$ and $\mathfrak{P}$ a set of size $|\mathfrak{P}| = \chi$. Then there is a sequence $\langle \Phi_\eta \mid \eta \in {}^{\omega\uparrow}\lambda\rangle$ such that*

(a) $\Phi_\eta = \langle \Phi_{\eta n} \mid n < \omega\rangle$, *with* $\Phi_{\eta n} \in \mathfrak{P}$.
(b) *If $\overline{f} = \{f_\nu \mid f_\nu \in \mathfrak{P}, \nu \in {}^{\omega\uparrow>}\lambda\}$, $\alpha \in \lambda$ and $\rho \in {}^{\omega\uparrow>}\lambda$, then there is $\eta \in {}^{\omega\uparrow}\lambda$ such that $\rho \subset \eta$ and $\Phi_{\eta n} = f_{\eta\restriction n}$ for all $n < \omega$.*

We slightly modify [**9**] by predicting a finite sequence of values $0\eta_1, \ldots, 0\eta_{k_*}$ for some $\overline{\eta} \in \Lambda$ (not just $0\eta_{k_*}$ as in [**9**]). The changes of the proof of the next theorem are minor and thus follow easily from the proof of [**9**, Theorem 2.4].

THE $\overline{\chi}$-BLACK BOX 2.5 ([**9**, Theorem 2.4]). *Let $\langle \lambda_1, \ldots, \lambda_{k_*}\rangle$ be a ■-sequence satisfying (2.2), where Λ and Λ_* are as in (2.3), and let C be a set of size $\leq \chi_1$. Then there is a family of set-traps $\langle \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda\rangle$ satisfying the following:*

PREDICTION PRINCIPLE: *If $\varphi : \Lambda_* \to C$ is any map and $\alpha_1 \in \lambda_1, \ldots, \alpha_{k_*} \in \lambda_{k_*}$, then for some $\overline{\eta} \in \Lambda$ there is a set-trap $\varphi_{\overline{\eta}}$ with $\varphi_{\overline{\eta}} \subseteq \varphi$ and $0\eta_1 = \alpha_1, \ldots, 0\eta_{k_*} = \alpha_{k_*}$.*

In order to construct $\aleph_n$-free (torsion) modules, the following freeness condition and thus the proof of the Freeness Proposition 2.7 need some further changes (in comparison with [**9**, Proposition 2.6]). Thus, we will be more explicit, also use [**13**], and carry out the arguments in detail. Recall that $\Lambda_*^{\leq\omega}$ means the set of all subsets of Λ_* that are at most countable.

We say that a function $F : \Lambda \to \Lambda_*^{\leq\omega}$ is *regressive* if for every $\overline{\eta} \in \Lambda$ and every $l \leq k_*$ we have

$$\sup(\mathrm{Im}(\eta_l)) > \sup\left(\bigcup_{\overline{\nu}\in F(\overline{\eta})} \mathrm{Im}(\nu_l)\right).$$

DEFINITION 2.6. Let $F : \Lambda \to \Lambda_*^{\leq\omega}$ be a regressive map. A subset $\Omega \subseteq \Lambda$ is *free* (with respect to F) if there is an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \alpha_*\rangle$ of Ω (we write $\Omega_\alpha = \{\overline{\eta}^\beta \mid \beta < \alpha\}$) and there are $\ell_\alpha \leq k_*$, $n_\alpha < \omega$ $(\alpha < \alpha_*)$ such that for $\alpha < \alpha_*$ and $n_\alpha \leq n$

$$\overline{\eta}^\alpha\restriction\langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta\restriction\langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F.$$

https://doi.org/10.1017/S0017089512000614 Published online by Cambridge University Press

Moreover, Ω is *κ-free* (with respect to F) for a cardinal κ if the above holds for all subsets of Ω of cardinality $< \kappa$.

That is to say, every newly chosen element $\overline{\eta}^{\alpha}$ picks up some unused element from Λ_* in its support that is not touched by the function F. Note that the enumeration of Ω in Definition 2.6 does not permit repetitions.

FREENESS PROPOSITION 2.7. *With the notation from Theorem 2.5 and Definition 2.6, the set Λ is $\aleph_{k_*}$-free with respect to any regressive function $F : \Lambda \to \Lambda_*^{\leq\omega}$. For any $k < k_*$, $\Omega \subseteq \Lambda$ of cardinality $|\,\Omega\,| \leq \aleph_k$ and $\langle u_{\overline{\eta}} \subseteq \{1, \ldots, k_*\} \mid |\,u_{\overline{\eta}}\,| > k, \overline{\eta} \in \Omega\rangle$ we can find an enumeration $\langle \overline{\eta}^{\alpha} \mid \alpha < \aleph_k\rangle$ of Ω, $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $n_\alpha < \omega$ $(\alpha < \aleph_k)$ such that*

$$\overline{\eta}^{\alpha} \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^{\beta} \upharpoonright \langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F \textit{ for all } n \geq n_\alpha.$$

Proof. The proof follows by induction on k. We begin with $k = 0$, hence we may assume that $|\,\Omega\,| = \aleph_0$. Let $\Omega = \{\overline{\eta}^{\alpha} \mid \alpha < \omega\}$ be an enumeration without repetitions. From $0 = k < |\,\overline{u}_{\overline{\eta}}\,|$ it follows $\overline{u}_{\overline{\eta}} \neq \emptyset$ and we can choose any $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ for all $\alpha < \omega$. To be definite, we may choose $\ell_\alpha = \min u_{\overline{\eta}^\alpha}$. If $\alpha \neq \beta < \omega$, then $\overline{\eta}^{\alpha} \neq \overline{\eta}^{\beta}$ and there is $n_{\alpha,\beta} \in \omega$ such that $\overline{\eta}^{\alpha} \upharpoonright \langle \ell_\alpha, n\rangle \neq \overline{\eta}^{\beta} \upharpoonright \langle \ell_\alpha, n\rangle$ for all $n \geq n_{\alpha\beta}$. Since Ω_α is finite, we may write it as $\Omega_\alpha = \{\overline{\eta}^{\gamma_1}, \cdots, \overline{\eta}^{\gamma_s}\}$. Moreover, for any $\overline{\eta}^{\gamma_i} \in \Omega_\alpha$ there is n_i such that $\overline{\eta}^{\gamma_i} \upharpoonright \langle \ell_\alpha, n\rangle \notin \overline{\eta}^{\gamma_i} F$ for all $n > n_i$ by the regressiveness of F. Choosing $n_{\alpha,\beta}$ large enough, i.e. larger than all n_i and all n_{γ_i,γ_j}, we may even get that $\overline{\eta}^{\gamma_i} \upharpoonright \langle \ell_\alpha, n\rangle \notin \Omega_\alpha F$ and also $\overline{\eta}^{\alpha} \upharpoonright \langle \ell_\alpha, n\rangle \notin \Omega_\alpha F$ for all $n \geq n_{\alpha,\beta}$. If $n_\alpha = \max_{\beta<\alpha} n_{\alpha,\beta}$, then $\overline{\eta}^{\alpha} \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^{\beta} \upharpoonright \langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \Omega_\alpha F$ for all $n \geq n_\alpha$. Hence, case $k = 0$ is settled and we let $k' = k + 1$ and assume that the proposition holds for k.

Let $|\,\Omega\,| = \aleph_{k'}$ and choose an $\aleph_{k'}$-filtration $\Omega = \bigcup_{\delta<\aleph_{k'}} \Omega_\delta$ with $\Omega_0 = \emptyset$ and $|\,\Omega_1\,| = \aleph_k$. The crucial idea comes from [**13**]: We can also assume that this chain is closed, meaning that for any $\delta < \aleph_{k'}$, $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$ and $\overline{\eta} \in \Omega$ with

$$\{\eta_m \mid m \leq k_*\} \subseteq \{\nu_m, \nu'_m, (\overline{\nu}F)_m, (\overline{\nu}'F)_m \mid m \leq k_*\} \text{ it follows } \overline{\eta} \in \Omega_\delta. \tag{2.4}$$

The proof of (2.4) is a simple, slightly extended purity-argument (as known from module theory). We may assume that $\Omega' \subseteq \Omega$ is given and of cardinality $|\,\Omega'\,| = \aleph_k < \aleph_{k'}$. First we enlarge Ω' to get Ω'_1 by adding $\overline{\eta} \in \Omega$ whenever there are $\overline{\nu}, \overline{\nu}' \in \Omega'$ satisfying the hypothesis of (2.4). Clearly $|\,\Omega'_1\,| = \aleph_k$ and we repeat this process ω many times running through all pairs $\overline{\nu}, \overline{\nu}' \in \Omega'_n$ obtained so far. Let $\Omega_c = \bigcup_{n<\omega} \Omega'_n$. We claim that Ω_c is closed as Ω_δ in (2.4). If $\overline{\eta} \in \Omega$ satisfies the hypothesis of (2.4) for Ω_c, then there is some $n < \omega$ and (2.4) holds for $\overline{\eta} \in \Omega$ at stage Ω'_n. Now $\overline{\eta} \in \Omega'_{n+1} \subseteq \Omega_c$ is as required, and a closed chain as in (2.4) can be constructed.

Thus, if $\overline{\eta} \in \Omega_{\delta+1} \setminus \Omega_\delta$, then the set

$$u^*_{\overline{\eta}} = \{\ell \leq k_* \mid \exists n < \omega, \overline{\nu} \in \Omega_\delta \text{ such that } \overline{\eta} \upharpoonright \langle \ell, n\rangle = \overline{\nu} \upharpoonright \langle \ell, n\rangle \text{ or } \overline{\eta} \upharpoonright \langle \ell, n\rangle \in \overline{\nu}F\}$$

is empty or a singleton. Otherwise there are $n, n' < \omega$ and distinct $\ell, \ell' \leq k_*$ with $\overline{\eta} \upharpoonright \langle \ell, n\rangle \in \{\overline{\nu} \upharpoonright \langle \ell, n\rangle\} \cup \bigcup \overline{\nu}F$ and $\overline{\eta} \upharpoonright \langle \ell', n'\rangle \in \{\overline{\nu}' \upharpoonright \langle \ell', n'\rangle\} \cup \bigcup \overline{\nu}'F$ for certain $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$. Hence, $\{\eta_m \mid m \leq k_*\} \subseteq \{\nu_m, \nu'_m, (\overline{\nu}F)_m, (\overline{\nu}'F)_m \mid m \leq k_*\}$, and the closure property implies the contradiction $\overline{\eta} \in \Omega_\delta$.

If $\delta < \aleph_{k'}$, then let $D_\delta = \Omega_{\delta+1} \setminus \Omega_\delta$ and $u'_{\overline{\eta}} := u_{\overline{\eta}} \setminus u^*_{\overline{\eta}}$ must have size $> k' - 1 = k$. Thus, the induction hypothesis applies and we find an enumeration $\overline{\eta}^{\delta\alpha}$ $(\alpha < \aleph_k)$ of D_δ as in the proposition. Finally we put these chains for each $\delta < \aleph_{k'}$ together

https://doi.org/10.1017/S0017089512000614 Published online by Cambridge University Press

with the induced ordering to get an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_{k'} \rangle$ of Ω satisfying the proposition. □

We are now ready to prepare the algebraic setting for our main theorem. As we are working over a complete DVD R, there is a prime p which represents the unique maximal ideal pR such that completion in R comes from the p-adic topology. As mentioned earlier, we will consider only reduced R-modules M. Hence, $p^n M$ $(n \in \omega)$ generates a Hausdorff p-adic topology on M. We intend to construct modules M in the torsion and the torsion-free case and choose the basic submodules for the Ms first.

In the torsion-free and torsion cases we write

$$B = \bigoplus_{\overline{\eta} \in \Lambda} Be_{\overline{\eta}} \text{ with } B_{\overline{\eta}} = \bigoplus_{m \le k_*} \bigoplus_{n < \omega} Re_{\overline{\eta} \upharpoonright \langle m,n \rangle}. \tag{2.5}$$

However, we distinguish the annihilators of the generators $e_{\overline{\eta} \upharpoonright \langle m,n \rangle}$ in (2.5). If (2.5) represents a basic module in the torsion-free case, then let

$$\mathrm{Ann}_R(e_{\overline{\eta} \upharpoonright \langle m,n \rangle}) = 0 \text{ for all } \overline{\eta} \in \Lambda, n < \omega, m \le k_*. \tag{2.6}$$

In the torsion case let

$$\mathrm{Ann}_R(e_{\overline{\eta} \upharpoonright \langle m,n \rangle}) = p^n R \text{ for all } \overline{\eta} \in \Lambda, n < \omega, m \le k_*. \tag{2.7}$$

Next we consider two p-adic completions of B. In the torsion-free case, let $\widehat{B}$ be the usual p-adic completion of the free R-module B, and in the torsion case let $\widehat{B}$ be the torsion completion of the torsion module B (which is a direct sum of various copies of cyclic R-modules $R/p^n R$); see [**10**] for elementary facts on the two distinct, reduced R-modules $\widehat{B}$. Any $b \in \widehat{B}$ (in both cases) can be expressed as a countable sum $b = \sum_{\overline{\eta} \in \Lambda_*} b_{\overline{\eta}} e_{\overline{\eta}}$ with coefficients $0 \neq b_{\overline{\eta}} \in R = \widehat{R}$. Let

$$[b]_{\Lambda_*} = \{\overline{\eta} \in \Lambda_* \mid b_{\overline{\eta}} \neq 0\} \text{ be the support of } b.$$

We adopt the notion of a trap from Corner–Göbel [**2**, Definition 3.2]. Recall that $A \subseteq_* B$ means that A is a pure submodule of B.

Definition 2.8. Let G be any R-module such that $B \subseteq G \subseteq_* \widehat{B}$. A *trap* is a partial R-homomorphism $\varphi_{\overline{\eta}} : B \longrightarrow G$ with label $\overline{\eta} \in \Lambda$ such that $B_{\overline{\eta}} \subseteq \mathrm{Dom}(\varphi_{\overline{\eta}}) \subseteq B$.

From the set-theoretic version of the Black Box 2.5 follows as in [**9**, Theorem 3.3] (by easy modification) its algebraic counterpart, which we want to apply in Section 3.

The $\overline{\chi}$-Black Box 2.9. *Given a* ■*-sequence* $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{k_*} \rangle$ *with (2.2) and an* R*-module* G *of size* $|G| \le \chi_1$*, let* Λ, Λ_* *be as above in (2.3). Then there is a family of traps* $\varphi_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda)$ *with the following property:*

The prediction: *If* $\varphi : B \to G$ *is an* R*-homomorphism and* $\alpha_1 \in \lambda_1, \ldots, \alpha_{k_*} \in \lambda_{k_*}$*, then there is* $\overline{\eta} \in \Lambda$ *with* $0\eta_1 = \alpha_1, \ldots, 0\eta_{k_*} = \alpha_{k_*}$ *such that* $\varphi_{\overline{\eta}} \subseteq \varphi$.

3. The main theorem. We recall some elementary facts and notations for R-modules over complete DVDs from [**11**]. Let tM be the torsion submodule of the R-module M. Its *torsion-free rank* $\mathrm{rk}_0 M$ is the dimension $\dim_Q(Q \otimes M/tM)$ of the vector-space $Q \otimes M/tM$ over the quotient field Q of R. And if tM is bounded, then the module decomposes into $M = tM \oplus M/tM$. We will say that M is *special* if tM

is bounded *and* M/tM has finite rank. Otherwise, either tM contains as a submodule an unbounded infinite direct sum of cyclic (torsion) R-modules or M contains a free R-module of infinite rank. We get an immediate observation. If M is special, then M is a direct sum of a free R-module of finite rank and a bounded R-module with a possibly infinite number of cyclic R-modules which are bounded, and by a theorem of Baer and Prüfer (see Fuchs [**5**, Theorem 17.2, Vol. 1]) the same holds for epimorphic images of M. Thus, epimorphic images of M are well known and it remains to consider R-modules which are not special, and this is the case we want to deal with in our main theorem.

We choose the basic modules B from (2.5), (2.6) and (2.7). In the torsion-free case as well as in the torsion case we want to find M as a pure submodule $B \subseteq M \subseteq_* \widehat{B}$ satisfying the demands of Theorem 3.3. Let $D \in \{T, F\}$ be the test module depending on both torsion and torsion-free cases. By density, any homomorphism $G \longrightarrow D$ is uniquely determined on B. Thus (by Theorem 2.9), any surjection $\varphi : M \to D$ had been predicted by one of the traps $\varphi_{\overline{\eta}} \subseteq \varphi$ on its domain $\mathrm{Dom}(\varphi_{\overline{\eta}})$. Moreover, we can predict the initial values $0\eta_1, \ldots, 0\eta_{k_*}$. This will be used in two steps in the construction to ensure that no such φ exists for M.

Firstly, we will show that for φ there is some element $m \in \widehat{M}$ (the completion of M in both torsion and the torsion-free cases) such that $m\varphi \notin D$, using that D is not complete in the appropriate p-adic (torsion) completion of D. We follow the convention that we do not rename homomorphisms on B and their unique extensions on G. Secondly, we will prove a *killing lemma* that provides elements y, which will be added to M, hence $y\varphi \in D$ which will also imply $m\varphi \in D$. But including y will not destroy $\aleph_n$-freeness or Σ-cyclicness of the extended M. This contradiction will ensure that the final module M does not have an epimorphic image D.

To start with we prove a very simple fact.

PROPOSITION 3.1. *Let R be a complete DVD.*

(i) *Let M be any reduced, torsion-free R-module and $\varphi : M \to D$ be an epimorphism, where $D = \bigoplus_{n\in\omega} Rd_n$ is free. Then there is some element $m \in \widehat{M}$ such that $m\varphi \in \widehat{D} \setminus D$.*

(ii) *Let M be a reduced, torsion R-module and $\varphi : M \to D$ be an epimorphism, where $D = \bigoplus_{n\in\omega} Rd_n$ with $\mathrm{Ann}\, d_n = p^nR$. Then there is some $m \in \widehat{M}$ such that $m\varphi \in \widehat{D} \setminus D$.*

Proof. The claims (i) and (ii) will be shown simultaneously. Let $\varphi : M \to D$ be an epimorphism. Since D is not complete in either case (in the p-adic topology if M is torsion-free, and with respect to torsion completion in the torsion case), there is an element $d \in \widehat{D} \setminus D$. By density we may write $d = \sum_{n\in\omega} d_n \in \widehat{D}$ with $d_n \in p^nD$ and in the torsion case there is a $k < \omega$ such that $\mathrm{Ann}_R(d_n) \subseteq p^kR$ for all $n < \omega$. Each d_n can be expressed as $d_n = m_n\varphi$ with $m_n \in p^nR$ for all $n < \omega$; put $m = \sum_{n\in\omega} m_n$. Thus, m goes to

$$m\varphi = \sum_{n\in\omega}(m_n\varphi) = \sum_{n\in\omega} d_n = d \in \widehat{D} \setminus D.$$

□

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We now show our *killing lemma*. Assume that the cardinal sequence $\overline{\lambda}$ is chosen such that $\lambda_{k_*} = \lambda_{k_*}^{\aleph_0}$, hence $|\widehat{B}| \le \lambda_{k_*}$ and there is a surjection $\delta : \lambda_{k_*} \to \widehat{B}$ with $\alpha \mapsto x_\alpha$ such that $\alpha > \sup\{\mathrm{Im}(\overline{\nu}) : \overline{\nu} \in [x_\alpha]\}$. Moreover, we want to define a regressive function $F : \Lambda \to \Lambda_*^{\le\omega}$ such that the following holds:

If the mapping $\overline{\eta} \mapsto [x_{0\eta_{k_*}}]$ is regressive, then let $F(\overline{\eta}) = [x_{0\eta_{k_*}}]$, otherwise put $F(\overline{\eta}) = \emptyset$.

PROPOSITION 3.2. *For any $\overline{\eta} \in \Lambda$ there are $\epsilon \in \{0, 1\}$, $k \in \mathbb{N}$ and*

$$y_{\overline{\eta}} = \sum_{n<\omega} p^n \left(\sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,n\rangle} \right) + \epsilon x_{0\eta_{k_*}} \text{ in the torsion-free case, and} \tag{3.1}$$

$$y_{\overline{\eta}} = \sum_{n<\omega} p^{n-k} \left(\sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,n\rangle} \right) + \epsilon x_{0\eta_{k_*}} \text{ in the torsion case} \tag{3.2}$$

such that there is no homomorphism $\varphi : \langle B, y_{\overline{\eta}} \rangle_* \longrightarrow D$ with $\varphi \restriction B_{[\overline{\eta}]} = \varphi_{\overline{\eta}}$ and $x_{0\eta_{k_*}}\varphi \in \widehat{D} \setminus D$.

Proof. The proof in the torsion case is very similar and thus left to the reader. Assume that the claim does not hold for $\epsilon = 0$ and $\epsilon = 1$. Hence, there are elements

$$y_{\overline{\eta}}^1 = \sum_{n<\omega} p^n \left(\sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,n\rangle} \right) + x_{0\eta_{k_*}} \quad \text{and} \quad y_{\overline{\eta}}^0 = \sum_{n<\omega} p^n \left(\sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,n\rangle} \right)$$

and homomorphisms

$$\varphi^1 : \langle B, y_{\overline{\eta}}^1 \rangle_* \longrightarrow D \quad \text{and} \quad \varphi^0 : \langle B, y_{\overline{\eta}}^0 \rangle_* \longrightarrow D$$

such that

$$\varphi^1 \restriction B_{[\overline{\eta}]} = \varphi_{\overline{\eta}} \quad \text{and} \quad \varphi^0 \restriction B_{[\overline{\eta}]} = \varphi_{\overline{\eta}}$$

and

$$x_{0\eta_{k_*}}\varphi^1 \in \widehat{D}\setminus D \quad \text{and} \quad x_{0\eta_{k_*}}\varphi^0 \in \hat{D}\setminus D.$$

We calculate the difference $y_{\overline{\eta}}^1\varphi^1 - y_{\overline{\eta}}^0\varphi^0$. Since the homomorphisms φ^1 and φ^0 coincide with $\varphi_{\overline{\eta}}$ on $B_{[\overline{\eta}]}$, we obtain $y_{\overline{\eta}}^1\varphi^1 - y_{\overline{\eta}}^0\varphi^0 = x_{0\eta_{k_*}}\varphi^1 \in D$, the desired contradiction. □

If κ is an infinite cardinal, then we iterate powersets beginning at κ and define $\beth_0(\kappa) = \kappa$, and if $\beth_\ell(\kappa)$ is defined for $\ell < \omega$, then let $\beth_{\ell+1}(\kappa) = 2^{\beth_\ell(\kappa)}$.

We are now ready for our main theorem.

THEOREM 3.3. *Let R be a complete discrete valuation domain of cardinality $|R| \le \kappa$. Given $k_* \in \mathbb{N}$, there is an $\aleph_{k_*}$-free, respectively an $\aleph_{k_*}$-Σ-cyclic torsion, R-module M of cardinality $\beth_{k_*}(\kappa)$ such that the following holds:*

(i) *In the torsion-free case $F = \bigoplus_{\aleph_0} R$ is not an epimorphic image of the $\aleph_{k_*}$-free R-module M.*

(ii) *In the torsion case $T = \bigoplus_{n\in\mathbb{N}} R/p^nR$ is not an epimorphic image of the $\aleph_{k_*}$-Σ-cyclic torsion R-module M.*

Proof. As before, the torsion-free and torsion cases are proved similarly, so we will concentrate on the torsion-free case. For any $\overline{\eta} \in \Lambda$ we will apply Proposition 3.2 and obtain elements $y_{\overline{\eta}} \in \widehat{B}$. Let

$$M = \left\langle B, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \right\rangle_* \subseteq \widehat{B}.$$

Obviously, M has size $\beth_{k_*}(\kappa)$. We must show that M is $\aleph_{k_*}$-free and that F is not an epimorphic image of M. We begin with the second claim and show the following:

LEMMA 3.4. *There is no epimorphism from M onto F.*

Proof. Assume that $\varphi : M \to F$ is an epimorphism. By Proposition 3.1 there is some element $b \in \widehat{B}$ such that φ maps $b\varphi \in \widehat{F} \setminus F$. Let $\delta(\alpha_{k_*}) = b$ with $\alpha_{k_*} > \sup\{\mathrm{Im}(\overline{\nu}) : \overline{\nu} \in [b_\alpha]\}$ and choose $\alpha_1, \ldots, \alpha_{k_*-1}$ greater than α_{k_*}. By the Black Box 2.9 there is $\overline{\eta} \in \Lambda$ with $0\eta_1 = \alpha_1, \ldots, 0\eta_{k_*} = \alpha_{k_*}$ and $\varphi_{\overline{\eta}} \subseteq \varphi$. Let $y_{\overline{\eta}}$ be the element from Proposition 3.2 for $\overline{\eta}$. Then $y_{\overline{\eta}} \in M$ and $\varphi \restriction \langle B, y_{\overline{\eta}} \rangle$ extends $\varphi_{\overline{\eta}}$ and satisfies $b\varphi \in \widehat{F} \setminus F$. This contradicts the choice of $y_{\overline{\eta}}$ in Proposition 3.2. Hence, there is no epimorphism $\varphi : M \to F$. □

It remains to show that M is $\aleph_{k_*}$-free, thus we will apply Proposition 2.7, which also permits that the function F of Proposition 3.5 is regressive. We first need some often used arithmetic of P-divisor chains. If $\overline{\eta} \in \Lambda$, then write $b_{0\eta_{k_*}} = \sum_{n<\omega} p^n b_{\overline{\eta}n}$ where $b_{\overline{\eta}n} \in B$ and let $y_{\overline{\eta}k} = \sum_{n \geq k} p^{n-k} (\sum_{m=1}^{k_*} e_{\overline{\eta} \upharpoonleft \langle m,n \rangle} + b_{\overline{\eta}n})$. Moreover, let $y_{\overline{\eta}} = y_{\overline{\eta}0}$. From $py_{\overline{\eta}k+1} = \sum_{n \geq k+1} p^{n-k} (\sum_{m=1}^{k_*} e_{\overline{\eta} \upharpoonleft \langle m,n \rangle} + b_{\overline{\eta}n})$ and $y_{\overline{\eta}k} - py_{\overline{\eta}k+1} = \sum_{m=1}^{k_*} (e_{\overline{\eta} \upharpoonleft \langle m,k \rangle} + b_{\overline{\eta}k})$, it follows that

$$py_{\overline{\eta}k+1} = y_{\overline{\eta}k} - \left(\sum_{m=1}^{k_*} e_{\overline{\eta} \upharpoonleft \langle m,k \rangle} - b_{\overline{\eta}k} \right). \tag{3.3}$$

PROPOSITION 3.5. *The module M is an $\aleph_{k_*}$-free R-module.*

Proof. Besides the Λ_*-support $[g]_{\Lambda_*}$ any element g of the module $M = \langle B, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle_*$ has a refined natural finite support $[g]$. It consists of all those elements of Λ and Λ_* contributing to g. We observe that g is generated by elements $y_{\overline{\eta}}$ and $e_{\overline{\eta} \upharpoonleft \langle m,n \rangle}$ and simply collect the $\overline{\eta}$s and $\overline{\eta} \upharpoonleft \langle m, n \rangle$ needed. Clearly $[g]$ is a finite subset of $\Lambda \cup \Lambda_*$. Hence, any subset H' of M has a natural support $[H']$ taking the union of supports of its elements. If H is a submodule of M that is generated by $< \kappa$ elements, say by the set H', for some infinite cardinal κ, then there is a subset $\Omega \subseteq \Lambda$ of size $|\Omega| < \kappa$ such that H is a submodule of the pure R-submodule

$$M_\Omega = \langle e_{\overline{\eta} \upharpoonleft \langle m,n \rangle}, b_{\overline{\eta}n}, y_{\overline{\eta}} \mid \overline{\eta} \in \Omega, m \leq k_*, n < \omega \rangle_* \subseteq \widehat{B},$$

which also has size $< \kappa$. Thus, in order to show $\aleph_{k_*}$-freeness of M, it suffices to consider any $\Omega \subseteq \Lambda$ of size $|\Omega| < \aleph_{k_*}$ and show the freeness of the module M_Ω. We may assume that $|\Omega| = \aleph_{k_*-1}$. Let $F : \Lambda \to \Lambda_*^{\leq \omega}$ be the regressive map defined before Proposition 3.2.

By Proposition 2.7 we can express the generators of M_Ω in the form

$$M_\Omega = \langle e_{\overline{\eta}^\alpha \upharpoonleft \langle m,n \rangle}, e_{\overline{\nu}}, y_{\overline{\eta}^\alpha n} \mid \alpha < \aleph_{k_*-1}, m \leq k_*, n < \omega, \overline{\nu} \in \overline{\eta}^\alpha F \rangle$$

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and find a sequence of pairs $(\ell_\alpha, n_\alpha) \in (k_* + 1) \times \omega$ such that for $n \geq n_\alpha$

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup_{\beta<\alpha} \overline{\eta}^\beta F. \tag{3.4}$$

Let $M_\alpha = \langle e_{\overline{\eta}^\gamma \upharpoonright \langle m,n\rangle}, e_{\overline{\eta}^\gamma F}, y_{\overline{\eta}^\gamma n} \mid \gamma < \alpha, m \leq k_*, n < \omega\rangle$ for any $\alpha < \aleph_{k_*-1}$; thus

$$\begin{aligned} M_{\alpha+1} &= M_\alpha + \langle e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y_{\overline{\eta}^\alpha n} \mid m \leq k_*, n < \omega, \overline{\nu} \in \overline{\eta}^\alpha F\rangle \\ &= M_\alpha + \langle e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle' *} \mid n < n_\alpha\rangle + \langle y_{\overline{\eta}^\alpha n} \mid n \geq n_\alpha\rangle \\ &\quad + \langle e_{\overline{\nu}}, e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle} \mid \ell_\alpha \neq m \leq k_*, n < \omega, \overline{\nu} \in \overline{\eta}^\alpha F\rangle. \end{aligned}$$

Hence, any element in $M_{\alpha+1}/M_\alpha$ can be represented in $M_{\alpha+1}$ modulo M_α in the form

$$\sum_{n\geq n_\alpha} r_n y_{\overline{\eta}^\alpha n} + \sum_{n<n_\alpha} r'_n e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle} + \sum_{\overline{\nu}\in\overline{\eta}^\alpha F} r_{\overline{\nu}} e_{\overline{\nu}} + \sum_{n<\omega} \sum_{\ell_\alpha\neq m\leq k_*} r''_{mn} e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}.$$

Moreover, the summands involving the $e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}$s have disjoint supports. Now condition (3.4) applies recursively. And by the disjointness (identifying elements of $\{e_{\overline{\nu}} \mid \overline{\nu} \in \overline{\eta}^\alpha F\}$ with one of the $e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}$s if possible) it also follows that all coefficients r, r'_n, r''_{mn} must be zero, showing that the set

$$\{e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, k\rangle}, e_{\overline{\nu}}, e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle} \mid k < n_\alpha, \ell_\alpha \neq m \leq k_*, n < \omega, \overline{\nu} \in \overline{\eta}^\alpha F\} \setminus M_\alpha$$

freely generates $M_{\alpha+1}/M_\alpha$. Thus, M_Ω has an ascending chain with only free factors; it follows that M_Ω is free. □

The theorem has a simple corollary. If M in Theorem 3.3(i) has a free direct summand P, then P must have finite rank. Moreover, if M decomposes into a non-trivial direct sum $M = \bigoplus_{i\in E} M_i$, then each summand M_i (by the above discussion of modules over complete DVDs) splits off a copy of R and by the first remark it follows that E is finite, thus (by a classical definition) M is almost indecomposable. Similarly, by Theorem 3.3(ii) any summand S of a torsion module N of countable rank must be bounded. Thus, if N decomposes into an infinite direct sum $N = \bigoplus_{i\in E} N_i$ of submodules N_i which are unbounded, then again by the basic properties of modules over complete DVDs we will find an unbounded summand S of countable rank. Then it is easy to find an epimorphism of S onto N, a contradiction, which shows that E is finite in this case as well. Thus (by a classical definition), the torsion module N is essentially indecomposable. We can summarize these observations.

Corollary 3.6. *Let M and N be as in Theorem 3.3. If M is $\aleph_n$-free, then M is almost indecomposable. If N is $\aleph_n$-Σ-cyclic, then N is essentially indecomposable.*

This corollary illustrates an interesting connection with the main result in Göbel and Paras [**8**], where it was shown that this kind of indecomposability cannot appear if the infinite rank of M is $< 2^{\aleph_0}$: In this case it was proved that torsion-free modules M always decompose into $M = C \oplus \bigoplus_{\aleph_0} R$ for some R-module C. Note, by way of explanation of the apparent dissonance, the modules M as in Theorem 1.1 have size $\geq 2^{\aleph_0}$.

4. Some generalizations. In order to prove in the torsion-free case that the module from Theorem 3.3 has no epimorphisms onto $\bigoplus_\omega R$, we applied the simple Proposition 3.1. This was based on the incompleteness of the module $F = \bigoplus_\omega R$. Thus, the Main Theorem 3.3 still holds if we replace R by a different ring R such that $\bigoplus_\omega R$ is not complete in some $\mathbb{S}$-topology. We state the theorem without proof.

THEOREM 4.1. *Let R be a ring equipped with an $\mathbb{S}$-topology such that $\bigoplus_\omega R$ is not complete with respect to $\mathbb{S}$. Suppose $|R| \leq \kappa$. Given $k_* \in \mathbb{N}$, there is an $\aleph_{k_*}$-free, respectively an $\aleph_{k_*}$-Σ-cyclic torsion, R-module M of cardinality $\beth_{k_*}(\kappa)$ such that the following holds:*

(i) *In the torsion-free case $F = \bigoplus_{\aleph_0} R$ is not an epimorphic image of the $\aleph_{k_*}$-free R-module M.*
(ii) *In the torsion case $T = \bigoplus_{n\in\mathbb{N}} R/p^n R$ is not an epimorphic image of the $\aleph_{k_*}$-Σ-cyclic torsion R-module M.*

The following example shows that the hypothesis of Theorem 4.1 need not hold for general rings and topologies.

EXAMPLE 4.2. Given an uncountable cardinal α, there is a ring T equipped with a topology S' such that $\bigoplus_\beta T$ is complete in the S'-topology if and only if $\beta < \alpha$.

Proof. The claim follows from **[7]** but we include the construction briefly. Let $T = \mathbb{Z}[X]$, where X is a set of variables of size $|X| = \alpha$. Choose the S'-topology that consists of all monomials in X. Then T is S'-complete and moreover $\bigoplus_\beta T$ is S'-complete whenever $\beta < \alpha$. However, if we consider $M = \bigoplus_{S'} T$, then $(r_s : s \in S')$ is a Cauchy net with $r_s = (s\delta_l)_{l\in S'}$ since $r_s - r_{sl} \in sM$ for all $s, l \in S'$ but $\lim(r_s : s \in S') \notin M$. Hence, M is not S'-complete and so nor is any module $\bigoplus_\beta T$ for $\beta \geq \alpha$. □

We would like to remark that even for rings T and cardinals α as in Example 4.2 one can similarly construct $\aleph_n$-free modules M (in the appropriate sense) such that M does not have any epimorphism onto $\bigoplus_\alpha T$. The main point is that one has to consider an appropriate completion of a base module B and then add new elements to B as in the *killing lemma* Proposition 3.2. In order to have these elements at hand one has to know that in the end there is always a *witness* for the unwanted homomorphisms, i.e. for such homomorphisms there is an element in the completion that is mapped outside F (see Proposition 3.1).

ACKNOWLEDGEMENTS. The collaboration was supported by the project No. I-963-98.6/2007 of the German-Israeli Foundation for Scientific Research & Development and by the project STR 627/1-6 of the German Research Foundation (DFG).

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