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Equivariant Segre and Verlinde invariants for Quot schemes

Arkadij Bojko and Jiahui Huang

Abstract

Segre and Verlinde series have been studied in many cases, including virtual geometries of Quot schemes on surfaces and Calabi–Yau 4-folds. Our work is the first to address the equivariant setting for both $\mathbb{C}^2$ and $\mathbb{C}^4$ by examining higher degree contributions which have no compact analogue.

(i) For $\mathbb{C}^2$, we work mostly with virtual geometries of Quot schemes. After connecting the equivariant series in degree zero to the existing results of the first author for compact surfaces, we extend the Segre–Verlinde correspondence to all degrees and to the reduced virtual classes. Additionally, we conjecture that there is an equivariant symmetry of Segre series, which was also observed in the compact setting.

(ii) For $\mathbb{C}^4$, we give further motivation for the definition of the Verlinde series. Based on empirical data andtorsiopn additional structural results, we conjecture that there is an equivariant Segre–Verlinde correspondence and Segre symmetry analogous to the one for $\mathbb{C}^2$.

1. Introduction

1.1 Definitions of Segre and Verlinde invariants

Let Y be a smooth quasi-projective variety. For a torsion-free sheaf E, the Quot scheme $\mathrm{Quot}_Y(E,n)$ parameterizes quotients

$$E \twoheadrightarrow F$$

such that

$$\mathrm{rank}(F)=0, \quad c_1(F)=0, \quad \chi(F)=n.$$

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When $Y = \mathbb{C}^2$ is a toric surface or $Y = \mathbb{C}^4$ is a toric Calabi–Yau 4-fold, we shall define equivariant Segre and Verlinde invariants on these Quot schemes and find relations between them. In the non-equivariant case when Y is a smooth projective curve or surface, or a Calabi–Yau 4-fold, such relations are called Segre–Verlinde correspondences and are studied in [AJL+21, Boj21b, Boj21a, GM22, GK22, JOP21, MOP21].

For a vector bundle V on Y, the Segre and Verlinde invariants associated to it are constructed using the *tautological bundle*

$$V^{[n]} = p_*(\mathcal{F} \otimes q^*V)$$

on $\mathrm{Quot}_Y(E,n)$. Here

$$\mathrm{Quot}_Y(E,n) \xleftarrow{p} \mathrm{Quot}_Y(E,n) \times Y \xrightarrow{q} Y$$

are projections, and $\mathcal{F}$ is the universal quotient sheaf. This definition extends to the Grothendieck groups and associates to each $\alpha \in K^0(Y)$ the *tautological class* $\alpha^{[n]} \in K^0(\mathrm{Quot}_Y(E,n))$.

1.1.1 *Equivariant invariants on Hilbert schemes of surfaces.* Let S be a toric quasi-projective surface with an action of $\mathsf{T} = (\mathbb{C}^*)^2$. Before introducing the equivariant invariants, we give a brief summary of equivariant cohomology and equivariant integration with a further background in § 2.1. Given a T-representation V, we define its *equivariant characteristic classes* by considering the associated bundle

$$E\mathsf{T} \times_{\mathsf{T}} V \to E\mathsf{T} \times_{\mathsf{T}} \{\mathrm{pt}\} = B\mathsf{T}$$

and taking its characteristic classes in $H^*(B\mathsf{T}) = H^*_{\mathsf{T}}(\mathrm{pt})$. Denote by $c^{\mathsf{T}}, s^{\mathsf{T}}, e_{\mathsf{T}}, \mathrm{ch}_{\mathsf{T}}, \mathrm{td}_{\mathsf{T}}$ the equivariant Chern class, Segre class, Euler class, Chern character, and Todd class, respectively. From here forward, we will always use equivariant classes for toric varieties, and we will omit the torus T from the notations when it is clear from the context. In our equivariant setting, the integral $\int_Y$ refers to the equivariant push-forward defined in (4), and χ denotes the equivariant Euler characteristic of a K-theory class. By Euler characteristic, we always mean the holomorphic Euler characteristic.

Consider $E = \mathcal{O}_S$, so $\mathrm{Quot}_S(\mathcal{O}_S, n)$ is the smooth Hilbert scheme $\mathrm{Hilb}^n(S)$ parameterizing ideal sheaves of 0-dimensional subschemes of S of length n. The action of T on S lifts to $\mathrm{Hilb}^n(S)$, giving an equivariant structure to $\alpha^{[n]}$ for any equivariant K-theory class $\alpha \in K_{\mathsf{T}}(S)$.

In the context of Donaldson invariants, the *Segre and Chern series* first appeared in [Tyu94], and are, respectively, given by

$$I^{\mathcal{S}}(\alpha; q) := \sum_{n=0}^{\infty} q^n \int_{\mathrm{Hilb}^n(S)} s(\alpha^{[n]}),$$

$$I^{\mathcal{C}}(\alpha; q) := \sum_{n=0}^{\infty} q^n \int_{\mathrm{Hilb}^n(S)} c(\alpha^{[n]}). \tag{1}$$

We use notation which resembles that of [GM22] as we regularly rely on the ideas presented there. Note that $c(-E) = 1/c(E)$ for a vector bundle E which, together with [Ful13, § 3.2], implies that $c(\alpha^{[n]}) = s(-\alpha^{[n]})$.

The *Verlinde series* was originally defined for moduli spaces of bundles on curves in the context of conformal field theory [Ver88], and has since been extended to moduli spaces of torsion-free sheaves on surfaces in [GK22]. For Hilbert schemes on surfaces, the Verlinde

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series was considered in[EGL99, § 5]. To be consistent with the invariants above, we adopt the notation

$$I^{\mathcal{V}}(\alpha;q) := \sum_{n=0}^{\infty} q^n \chi(\mathrm{Hilb}^n(S), \det(\alpha^{[n]}))$$

from [JOP21].

1.1.2 *Virtual invariants on Quot schemes.* In general, Quot schemes are not smooth, in which case we do not have a deformation invariant fundamental class to integrate against. One way to resolve this issue is to work with a virtual fundamental class $[\mathrm{Quot}_Y(E,n)]^{\mathrm{vir}}$.

For a smooth projective surface S and a vector bundle E of rank N, a perfect obstruction theory for $\mathrm{Quot}_Y(E,n)$ of virtual dimension nN was constructed in [MOP15, Lemma 1] for trivial bundles and was generalized to arbitrary bundles in [Sta24, § 2.3]. We remark that the more general case of coherent sheaves is covered in [Lim20, § 1.3] but will not be pursued here. To this perfect obstruction theory, one may associate a virtual fundamental class $[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}$ (following [BF98, LT96]), and a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ (first proposed in [BF98, Remark 5.4]). The virtual invariants are defined in a similar way to what has gone before, with the usual fundamental class replaced by the virtual class $[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}$ and the Euler characteristic replaced by the virtual Euler characteristic $\chi^{\mathrm{vir}}(-) := \chi(- \otimes \mathcal{O}^{\mathrm{vir}})$, which are both invariant under deformations of the complex structure of S.

Unlike surfaces and Fano 3-folds, the usual obstruction theory for a Quot scheme of a Calabi–Yau 4-fold X is not perfect, so the previous method does not induce a virtual fundamental class. However, using the method of [Ric20] and tools from [OT20], a virtual class $[\mathrm{Quot}_X(E,n)]^{\mathrm{vir}}_{o(\mathcal{L})} \in H_{2nN}(\mathrm{Quot}_X(E,n),\mathbb{Z})$ was constructed in [Boj21a, § 2.1] when X is a *strict* Calabi–Yau 4-fold and E a simple rigid locally free sheaf. Here, $\mathcal{L}$ denotes the determinant line bundle $\det \mathbf{R}\mathscr{H}om_q(\mathcal{I},\mathcal{I})$ on $\mathrm{Quot}_X(E,n)$, where $\mathcal{I}$ is the universal subsheaf. As indicated by the subscript, this class is dependent on some choice of orientation $o(\mathcal{L})$, that is, a choice of a square root of the isomorphism

$$Q : \mathcal{L}\otimes\mathcal{L} \to \mathcal{O}_{\mathrm{Quot}_X(E,n)}$$

induced by Serre duality. There is also no canonical virtual structure sheaf, and instead we have the 'twisted' virtual structure sheaf $\hat{\mathcal{O}}^{\mathrm{vir}}$ defined in [OT20, (84), § 8]. The motivation for this notation comes from its relation to the twisted virtual structure sheaf $\hat{\mathcal{O}}^{\mathrm{vir}}_{\mathrm{NO}}$ of Nekrasov and Okounkov [NO14]. Without going into details, $\hat{\mathcal{O}}^{\mathrm{vir}}_{\mathrm{NO}}$ is defined from the usual virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ on a moduli space with obstruction theory by twisting with $\det^{\frac{1}{2}}(K^{\mathrm{vir}})$, where K^{vir} is the virtual cotangent bundle. Square roots of determinant line bundles appear also in the construction of $\hat{\mathcal{O}}^{\mathrm{vir}}$, but this time they are used to make the result independent of the choices made. Another distinguishing feature is that

$$\hat{\mathcal{O}}^{\mathrm{vir}} \in K_0(\mathrm{Quot}_Y(E,n),\mathbb{Z}[2^{-1}])$$

is not an integer class, precisely because of the square root determinant line bundles. To obtain integer invariants, the first author, in [Boj21b, § 5.3] and [Boj21a, § 1.4], introduced the *untwisted virtual structure sheaf*

$$\mathcal{O}^{\mathrm{vir}} = \hat{\mathcal{O}}^{\mathrm{vir}} \otimes \mathsf{E}^{\frac{1}{2}} \in K_0(\mathrm{Quot}_Y(E,n),\mathbb{Z}[2^{-1}]), \text{ where } \mathsf{E} = \det\left((E^*)^{[n]}\right).$$

The notation $(-)^{\frac{1}{2}}$ denotes the unique square root of a line bundle in $\mathbb{Z}[2^{-1}]$-valued K-theory (as established in [OT20, Remark 5.2]). In § 5.2, we motivate this definition further by showing,

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in the case $X = \mathbb{C}^4$, that $\mathsf{E}^{-\frac{1}{2}}$ naturally appears in the construction of $\hat{\mathcal{O}}^{\mathrm{vir}}$ as the only term that is not an integer class. As a consequence, we prove the following.

Proposition 1.1 (Proposition 5.3). *The untwisted virtual structure sheaf is integral:*

$$\mathcal{O}^{\mathrm{vir}} \in K_0(\mathrm{Quot}_X(E,n), \mathbb{Z})\,.$$

Before defining equivariant virtual invariants on $Y = \mathbb{C}^d$, $d = 2, 4$, we describe the torus action used on $\mathrm{Quot}_Y(E,n)$. Let

$$\mathsf{T}_0 = (\mathbb{C}^*)^d/(\sim) = \{(t_1, \dots, t_d) : t_1, \dots, t_d \neq 0\}/(\sim)$$

act on Y naturally, where we quotient by the subgroup $\langle\sim\rangle = \langle t_1t_2t_3t_4\rangle$ when $d = 4$. Let $\mathsf{T}_1 = (\mathbb{C}^*)^N = \{(y_1, \dots, y_N) : y_i \neq 0\}$ and $E = \oplus_{i=1}^N \mathcal{O}_Y\langle y_i\rangle$ be the T_1-equivariant bundle of rank N with weights $y_1, \dots, y_N$. This induces a $\mathsf{T}_0 \times \mathsf{T}_1$-action on $\mathrm{Quot}_Y(E,n)$ by acting on the middle term of the sequence

$$0 \to I \to E \to F \to 0.$$

Let $\alpha \in K_{\mathsf{T}_0}(Y)$, then we can write

$$\alpha = [\,\oplus_{i=1}^r \mathcal{O}_Y\langle v_i\rangle] - [\,\oplus_{i=r+1}^{r+s} \mathcal{O}_Y\langle v_i\rangle]$$

where $v_1, \dots, v_{r+s}$ are its T_0-weights. However, instead of thinking of the v_i as T_0-weights, we would like to view them as generic parameters. Therefore we introduce an additional torus $\mathsf{T}_2 = (\mathbb{C}^*)^{r+s}$ acting on $\mathbb{C}^r \times \mathbb{C}^s$. Set $\mathsf{T} := \mathsf{T}_0 \times \mathsf{T}_1 \times \mathsf{T}_2$ and write

$$\begin{aligned} K_{\mathsf{T}}(\mathrm{pt}) &= \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}; y_1^{\pm 1}, \dots, y_N^{\pm 1}; v_1^{\pm 1}, \dots, v_{r+s}^{\pm 1}]/(\sim),\\ H^*_{\mathsf{T}}(\mathrm{pt}) &= \mathbb{C}[\lambda_1, \dots, \lambda_d; m_1, \dots, m_N; w_1, \dots, w_{r+s}]/(\sim), \end{aligned}$$

where we quotient by the ideals $(\sim) = (t_1t_2t_3t_4 - 1)$ and $(\sim) = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$, respectively, when $d = 4$. Here λ_i, m_i, w_i are, respectively, the equivariant first Chern classes of the weights t_i, y_i, v_i.

In the surface case, the equivariant virtual invariants are defined using the virtual equivariant localization of [GP97, CFK09] in an analogous way to the non-virtual case. For toric Calabi–Yau 4-folds, the equivariant Donaldson invariants were first introduced by Cao and Leung in [CL14, § 8]; the K-theoretic equivariant invariants were predicted by Nekrasov [Nek20] and Cao, Kool and Monavari [CKM22]. They were formalized by Oh and Thomas [OT20, § 7], using the twisted virtual structure sheaf $\hat{\mathcal{O}}^{\mathrm{vir}}$ and their virtual equivariant localization. The *equivariant virtual Segre, Chern* and *Verlinde series* for α on $\mathrm{Quot}_Y(E,n)$ are, respectively,

$$\begin{aligned} \mathcal{S}_Y(E,\alpha;q) &:= \sum_{n=0}^{\infty} q^n \int_{[\mathrm{Quot}_Y(E,n)]^{\mathrm{vir}}} s(\alpha^{[n]}),\\ \mathcal{C}_Y(E,\alpha;q) &:= \sum_{n=0}^{\infty} q^n \int_{[\mathrm{Quot}_Y(E,n)]^{\mathrm{vir}}} c(\alpha^{[n]}),\\ \mathcal{V}_Y(E,\alpha;q) &:= \sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(\mathrm{Quot}_Y(E,n), \det(\alpha^{[n]})). \end{aligned}$$

Here the symbols $\int_{[\mathrm{Quot}_Y(E,n)]^{\mathrm{vir}}}, \chi^{\mathrm{vir}}$ are defined using virtual versions of (4). See Definition 3.2 for their precise definitions.

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Remark 1.2. It is important to note that the above definitions, when $Y = \mathbb{C}^4$, are dependent on a choice of orientation $o(\mathcal{L})$. This induces signs at each fixed point $Z \in \mathrm{Quot}_X(E,n)^{\mathsf{T}}$, which are suppressed in the notation. We denote the sign at Z to be $(-1)^{o(\mathcal{L})|_Z}$ and call $o(\mathcal{L})$ a choice of signs.

When $N = 1$, the weight on $E = \mathcal{O}_Y\langle y_1\rangle$ is not necessary as this extra action does not affect the fixed locus, so we sometimes ignore it by setting $y_1 = 1$. By definition, the coefficients of the Chern and Segre series are rational functions in the cohomological parameters $\lambda_1,\dots,\lambda_d, m_1,\dots,m_N, w_1,\dots,w_{r+s}$. For the Verlinde series, they are rational functions in the K-theoretic parameters $t_1,\dots,t_d, y_1\dots,y_N, v_1\dots,v_{r+s}$. Using the identification of Remark 2.4, they can be viewed as functions in the cohomological parameters as well. Define

$$*_{Y,i}(E,\alpha;q) := *_Y(E,\alpha;q)|_{\deg \vec{\lambda},\vec{m},\vec{w}=i}$$

to be the part with total degree i in those variables for $* \in \{\mathcal{S},\mathcal{C},\mathcal{V}\}$. More precisely, by restricting a multi-variable function to a certain degree, we mean the following.

Definition 1.3. Let $f(z_1,\dots,z_k)$ be a function in the ring of fractions of $\mathbb{C}[\![z_1,\dots,z_k]\!]$. Consider the formal Laurent series expansion of $f(bz_1,\dots,bz_k)$ in the variable b:

$$f(bz_1,\dots,bz_k) = \sum_{i=-\infty}^{\infty} f_i(z_1,\dots z_k)b^i.$$

For $i \in \mathbb{Z}$, the part of f with *total degree* i is

$$f(z_1,\dots,z_k)|_{\deg \vec{z}=i} := f_i(z_1,\dots,z_k).$$

1.1.3 *Reduced invariants on Quot schemes of surfaces.* When S is a K-trivial surface, the obstruction on $\mathrm{Quot}_S(E,n)$ contains a trivial summand making $e(T^{\mathrm{vir}})$ vanish and, as a result, the invariants all vanish. We may instead consider the reduced classes and invariants from Gromov–Witten theory and stable pair theory [KT14], which have been employed to study the enumerative geometry of Hilbert schemes in, for instance, [GSY17]. In the case of $\mathrm{Quot}_S(E,n)$, a reduced perfect obstruction theory can be obtained by removing one copy of $\mathcal{O}_{\mathrm{Quot}_S(E,n)}$ from the usual obstruction for Quot schemes. The equivariant analogue of a K-trivial surface would be $S = \mathbb{C}^2$ with the action of the 1-dimensional torus $\mathsf{T}_0 = \{(t_1,t_2) : t_1t_2 = 1\}$. Let T^{red} be the virtual tangent bundle obtained from the reduced obstruction theory. For $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $\alpha = [\oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle] - [\oplus_{i=r+1}^s \mathcal{O}_S\langle v_i\rangle]$, we define the *reduced Segre, Chern,* and *Verlinde series* to be, respectively,

$$\mathcal{S}^{\mathrm{red}}(E,\alpha;q) := \sum_{n>0}^{\infty} q^n \int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{red}}} s(\alpha^{[n]}),$$

$$\mathcal{C}^{\mathrm{red}}(E,\alpha;q) := \sum_{n>0}^{\infty} q^n \int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{red}}} c(\alpha^{[n]}),$$

$$\mathcal{V}^{\mathrm{red}}(E,\alpha;q) := \sum_{n>0}^{\infty} q^n \int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{red}}} \mathrm{td}(T^{\mathrm{red}})\mathrm{ch}(\det(\alpha^{[n]})).$$

Again, the symbol $\int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{red}}}$ is notational, and the precise definition is given in § 3.4.

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1.2 Summary of results for surfaces

1.2.1 *Computation of Chern series.* Consider the case $S = \mathbb{C}^2$ with the $\mathsf{T} = (\mathbb{C}^*)^2$-action. Using the tools from [GM22], we are able to obtain the virtual Chern series of line bundles for Hilbert schemes, as we show in § 4.1. Note that in the non-equivariant setting, this can be retrieved from [OP22, Corollary 15].

Corollary 1.4. *Let $S = \mathbb{C}^2$, and let $L = \mathcal{O}_S\langle v_1\rangle$ be a T-equivariant line bundle over S. We have*

$$\mathcal{C}_S(\mathcal{O}_S, L; q) = \left(\frac{1}{1-q}\right)^{\int_S c(L)c_1(S)}.$$

By extracting the part with the lowest total degree in $\lambda_1, \lambda_2, w_1$, we obtain the following 2-dimensional analogue to the Donaldson–Thomas partition function for $\mathbb{C}^3$ [MNOP06b, Theorem 1], or Cao and Kool's formulation of Nekrasov's conjecture for $\mathbb{C}^4$ [CK17, Appendix B].

Corollary 1.5 (Corollary 4.2). *For $S = \mathbb{C}^2$, the following equality holds:*

$$\sum_{n=1}^{\infty} q^n \int_{[\mathrm{Hilb}^n(S)]^{\mathrm{vir}}} 1 = e^{\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}q}.$$

Remark 1.6. Suppose $F(q)$ is a power series. If Y is compact and $\sigma \in H^{2\dim Y}(Y;\mathbb{Q})$, then $\int_Y \sigma \in \mathbb{Q}$, and we make the identification $F(q)^\sigma := F(q)^{\int_Y \sigma}$. In the non-compact case $Y = \mathbb{C}^2$ or $Y = \mathbb{C}^4$, working with the equivariant push-forward $\int_Y : H^*_{\mathsf{T}}(Y) \to H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$, we will use a different notation. Suppose $\gamma \in H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$ and $F(0) = 1$. The expression $F(q)^\gamma$ is to be interpreted as

$$F(q)^\gamma := \exp(\gamma \log F(q)) = \sum_{n=0}^{\infty} \frac{(\gamma \log F(q))^n}{n!} \in H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}[\![q]\!].$$

Since $F(0) = 1$, we have $[q^0]\log F(q) = 0$, so the right-hand side does indeed belong to the ring $H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}[\![q]\!]$. We will be dealing exclusively with expressions of the form $F(q)^{\int_Y \sigma}$ for $\sigma \in H^*_{\mathsf{T}}(Y)$.

1.2.2 *Universal series expressions.* A common approach to find closed formulas for the Segre and Verlinde series is to compute their universal series. In the non-virtual case, for Hilbert schemes of projective surfaces, the main result of [EGL99] shows that the Segre and Verlinde invariants admit the following universal series expressions:

$$I^{\mathcal{C}}(\alpha; q) = A_0(q)^{c_2(\alpha)} A_1(q)^{\chi(\det(\alpha))} A_2(q)^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3(q)^{c_1(\alpha)K_S - \frac{1}{2}K_S^2} A_4(q)^{K_S^2},$$

$$I^{\mathcal{V}}(\alpha, q) = B_1(q)^{\chi(\det(\alpha))} B_2(q)^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3(q)^{c_1(\alpha)K_S - \frac{1}{2}K_S^2} B_4(q)^{K_S^2}, \tag{2}$$

see [EGL99, Theorem 5.3], and [Leh99, § 4.3]. Here, the products in the exponents refer to intersection products. The series $A_i(x), B_i(x)$ are universal, in the sense that they only depend on α through its rank and are independent of the surface. Explicit formulas for these series were conjectured and computed in [Leh99, MOP17, MOP21, EGL99, GM22]. The Segre–Verlinde correspondence in this case concerns the relations between A_i and B_i. It was first proposed by Johnson and Marian, Oprea and Pandharipande in relation to the study of Le Potier's strange duality [MOP17, Joh18], and was recently proved by Göttsche and Mellit [GM22].

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For virtual invariants on Quot schemes of a smooth projective surface S and torsion-free sheaf E, the universal series expressions are given by [Boj21a, Theorem 1.2]:

$$\mathcal{S}_S(E,\alpha;q) = A_1^{\mathrm{vir}}(q)^{c_1(S)c_1(\alpha)} A_2^{\mathrm{vir}}(q)^{c_1(S)^2} A_3^{\mathrm{vir}}(q)^{c_1(S)c_1(E)},$$
$$\mathcal{V}_S(E,\alpha;q) = B_1^{\mathrm{vir}}(q)^{c_1(S)c_1(\alpha)} B_2^{\mathrm{vir}}(q)^{c_1(S)^2} B_3^{\mathrm{vir}}(q)^{c_1(S)c_1(E)}.$$

The explicit formulas are computed in [AJL+21, Theorem 17] for A_1, A_2, B_1, B_2 and in [Boj21a, Theorem 1.2] for A_3, B_3.

Unlike the compact case where the invariants are simply numbers, the equivariant invariants can contain terms of various degrees in $H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$. This is reflected in the following theorem, where the virtual equivariant Segre and Verlinde invariants are written as infinite products of series labeled by partitions. The notation for partitions is given in § 2.2. For a partition μ and a K-theory class α, we write

$$c_\mu(\alpha) := \prod_{i=1}^{\ell(\mu)} c_{\mu_i}(\alpha).$$

Theorem 1.7 (Theorem 3.5). *Let $S = \mathbb{C}^2$. For any $r \in \mathbb{Z}$, $N > 0$, there exist universal power series $A_{\mu,\nu,\xi}(q), B_{\mu,\nu,\xi}(q)$, dependent on N and r, such that for $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $\alpha \in K_{\mathsf{T}}(S)$ of rank r, the equivariant virtual Segre and Verlinde series on $\mathrm{Quot}_S(E,n)$ can be written as the following infinite products:*

$$\mathcal{S}_S(E,\alpha;q) = \prod_{\mu,\nu,\xi \text{ partitions}} A_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)},$$
$$\mathcal{V}_S(E,\alpha;q) = \prod_{\mu,\nu,\xi \text{ partitions}} B_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)},$$
$$\mathcal{C}_S(E,\alpha;q) = \prod_{\mu,\nu,\xi \text{ partitions}} C_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)}.$$

The series in the above expressions are universal, in the sense that they depend on the input α only by its rank r. Sometimes, for clarity, we will add superscripts N, r to indicate the ranks of E and α. On the right-hand side of the above expressions, the exponents $\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)$ are computed by (4). They are thus homogeneous rational functions in

$$H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}} = \mathbb{C}(\lambda_1,\lambda_2; m_1,\dots,m_N; w_1,\dots,w_{r+s})$$

of degree $|\mu|+|\nu|+|\xi|-1$. Degree 0 terms occur when one of μ,ν,ξ is the partition (1) and the rest are the empty partition (0). The argument of § 3.2 shows that the series with degree 0 exponents are necessarily equal to the series from the projective case, that is

$$A_{(1),(0),(0)}(q) = A_1^{\mathrm{vir}}(q), \quad A_{(0),(1),(0)}(q) = A_2^{\mathrm{vir}}(q), \quad A_{(0),(0),(1)}(q) = A_3^{\mathrm{vir}}(q),$$
$$B_{(1),(0),(0)}(q) = B_1^{\mathrm{vir}}(q), \quad B_{(0),(1),(0)}(q) = B_2^{\mathrm{vir}}(q), \quad B_{(0),(0),(1)}(q) = B_3^{\mathrm{vir}}(q). \tag{3}$$

The universal series expressions of the reduced invariants take a much simpler form; as opposed to having series exponentiated to some powers of cohomology classes, we have the following additive expressions.

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Theorem 1.8 (Theorem 3.11). *When $S=\mathbb{C}^2$, the equivariant reduced Segre and Verlinde series for $E=\oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $\alpha\in K_{\mathsf{T}}(S)$ are*

$$\mathcal{S}^{\mathrm{red}}(E,\alpha;q)=\sum_{\mu,\nu,\xi}\log\left(A_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E),$$
$$\mathcal{V}^{\mathrm{red}}(E,\alpha;q)=\sum_{\mu,\nu,\xi}\log\left(B_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E),$$
$$\mathcal{C}^{\mathrm{red}}(E,\alpha;q)=\sum_{\mu,\nu,\xi}\log\left(C_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)$$

where $A_{\mu,\nu,\xi}$, $B_{\mu,\nu,\xi}$ and $C_{\mu,\nu,\xi}$ are the same series as in Theorem 1.7.

1.2.3 *Virtual Segre–Verlinde correspondence.* When Y is compact, the virtual Segre–Verlinde correspondence has been proved, for compact surfaces and Calabi–Yau 4-folds [Boj21a, Theorem 1.6] and for torsion-free sheaves E, to be

$$\mathcal{S}_Y(E,\alpha;q)=\mathcal{V}_Y(E,\alpha;(-1)^Nq).$$

As a corollary of this result, Theorem 1.7 and the relations (3), we first prove the following 'weak' equivariant Segre–Verlinde correspondence.

Corollary 1.9 (Corollary 3.7). *In the setting of Theorem 1.7, we have the following correspondence*

$$A_{\mu,\nu,\xi}(q)=B_{\mu,\nu,\xi}((-1)^Nq)$$

whenever one of μ,ν,ξ is (1) and the other two are (0). In particular, the degree 0 part satisfies

$$\mathcal{S}_{S,0}(E,\alpha;q)-\mathcal{V}_{S,0}(E,\alpha;(-1)^Nq)=\sum_{n=2}^{\infty}\frac{f_n}{(\lambda_1\lambda_2)^{n-2}}\cdot\left(\int_S c_1(S)\right)^2\cdot q^n$$

for some terms $f_n\in H^{2n-2}_{\mathsf{T}}(\mathrm{pt})$ dependent on α through its rank and Chern classes.

This is weak, in the sense that only the series whose powers are degree 0 satisfy the usual correspondence, while computations for small values of n show that the terms f_n can be non-zero, so the naive correspondence does not hold in all degrees. One might instead ask whether there are any other relations between the series in all cohomological degrees, generalizing the above corollary.

We are able to give a complete answer relating $B_{\mu,\nu,\xi}$ and $C_{\mu,\nu,\xi}$ once the sizes of the partitions $|\mu|,|\xi|$ are fixed and $\nu=(0)$. This is represented in Figure 1, but before stating our result, some further notation is needed. Given any partition μ, integer $a\in\mathbb{Z}$ and $n>0$, the binomial coefficients for μ are

$$\binom{a}{\mu}:=\prod_{i=1}^{\ell(\mu)}\binom{a}{\mu_i},$$

and the downward factorial of a by n is

$$(a)_{(n)}:=a\cdot(a-1)\cdots(a-n+1),$$

$$(a)_{(-1)}:=\frac{1}{a+1}.$$

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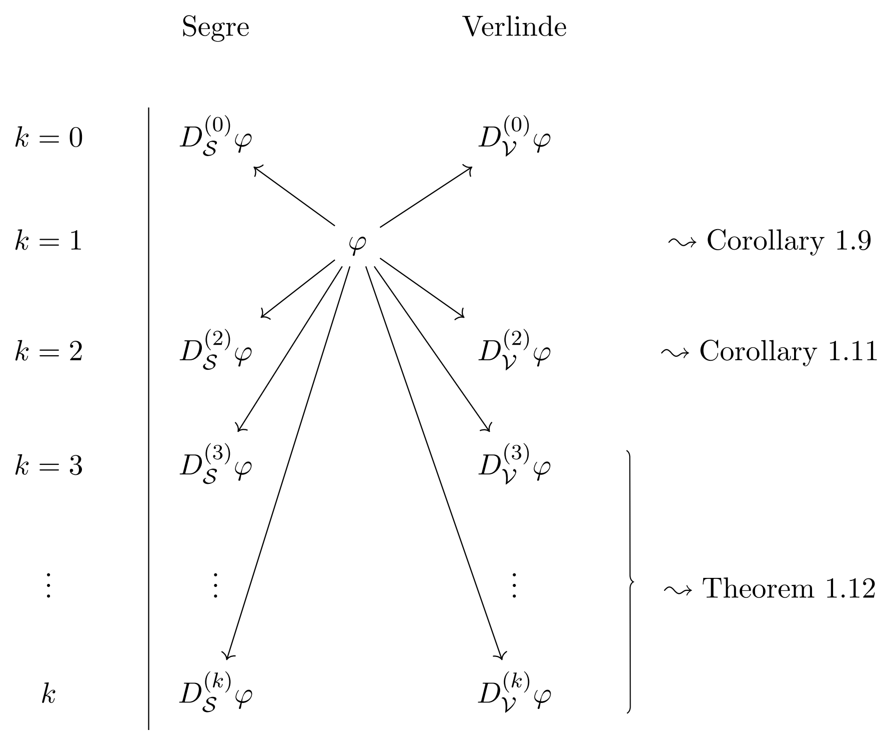

Figure 1. Relating Segre and Verlinde invariants by the common function φ.

Theorem 1.10 (Theorem 4.3). *For rank $r \geqslant 0$ and integers $k_1, k_2 \geqslant 0$ with $k := k_1 + k_2$, the universal series of Theorem 1.7 satisfy*

$$k_1!k_2! \sum_{|\mu|=k_1} \sum_{|\xi|=k_2} \binom{r}{\mu}\binom{N}{\xi} \log A_{\mu,(0),\xi}(q) = -r \sum_{n=1}^{\infty} \frac{(-n(r+N))_{(k-1)}}{n} \binom{-nr-1}{nN-1} q^n,$$

$$k_1!k_2!r^{k_1} \sum_{|\xi|=k_2} \binom{N}{\xi} \log B_{(1)_{k_1},(0),\xi}(q) = -r^k \sum_{n=1}^{\infty} n^{k-2} \binom{-nr-1}{nN-1} \left((-1)^N q\right)^n .$$

Furthermore, we have

$$k_1!k_2! \sum_{|\mu|=k_1} \sum_{|\xi|=k_2} \binom{r}{\mu}\binom{N}{\xi} \log C_{\mu,(0),\xi}(q) = r \sum_{n=1}^{\infty} \frac{(n(r-N))_{(k-1)}}{n} \binom{nr-1}{nN-1} q^n,$$

which can be compared to the identities above by replacing r with $-r$.

In particular, setting $k=1$, the first two equalities of this theorem give

$$A_{(1),(0),(0)}(q) = \left(B_{(1),(0),(0)}((-1)^N q)\right),$$

$$A_{(0),(0),(1)}(q) = \left(B_{(0),(0),(1)}((-1)^N q)\right).$$

This is consistent with the Segre–Verlinde correspondence in degree 0 from Corollary 1.9. Going one degree higher by setting $k=2$, we get the following correspondence.

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Corollary 1.11 (Corollary 4.4). *For rank $r \geqslant 0$, the universal series of Theorem 1.7 satisfy the following correspondences*

$$A_{(1,1),(0),(0)}(q)^{-r}A_{(2),(0),(0)}(q)^{\frac{-(r-1)}{2}} = B_{(1,1),(0),(0)}\left((-1)^N q\right)^{r+N},$$

$$A_{(1),(0),(1)}(q)^{-r} = B_{(1),(0),(1)}\left((-1)^N q\right)^{r+N}, \text{ and}$$

$$A_{(0),(0),(1,1)}(q)^{-rN}A_{(0),(0),(2)}(q)^{\frac{-r(N-1)}{2}}$$
$$= B_{(0),(0),(1,1)}\left((-1)^N q\right)^{N(r+N)} B_{(0),(0),(2)}\left((-1)^N q\right)^{\frac{(N-1)(r+N)}{2}}.$$

It is natural to expect that applying Lagrange inversion to the expressions in Theorem 1.10 will simplify the formulae. In § 4.3, we use this to phrase the higher degree Segre and Verlinde series in terms of applying differential operators of comparable form to a single function $\varphi(t) = \log(1+t)$ with common variable changes except for the usual sign $q \rightsquigarrow (-1)^N q$.

Theorem 1.12 (Theorem 4.6). *Let $\varphi(t) = \log(1+t)$ and $\psi(t) = Nt^{-1} + r(1+t)^{-1}$. Define the differential operator*

$$D_\psi = \frac{1}{\psi}\cdot\frac{d}{dt}.$$

Furthermore, use the notation

$$D_{\mathcal{S}}^{(k)} = (-(r+N)D_\psi)_{(k-1)}, \quad D_{\mathcal{V}}^{(k)} = r^{k-1}D_\psi^{k-1}$$

for $k \geqslant 0$ where $D_\psi^{-1}(-)$ denotes integrating $\psi\cdot(-)$ assuming a constant term 0. In the setting of Theorem 1.10, we have the following relations

$$k_1!k_2!\sum_{|\mu|=k_1}\sum_{|\xi|=k_2}\binom{r}{\mu}\binom{N}{\xi}\log A_{\mu,(0),\xi}(q) = -r\sum_{i=1}^N\left(D_{\mathcal{S}}^{(k)}\varphi\right)(H_i),$$

$$k_1!k_2!r^{k_1}\sum_{|\xi|=k_2}\binom{N}{\xi}\log B_{(1)_{k_1},(0),\xi}\left((-1)^N q\right) = -r\sum_{i=1}^N\left(D_{\mathcal{V}}^{(k)}\varphi\right)(H_i),$$

where H_i are the Newton–Puiseux solutions to $H_i^N = q(1+H_i)^{-r}$.

If the rank r of α is negative, then the above identities no longer hold. For such α, we may instead consider the Chern series and $C^{N,r}_{\mu,(0),\xi}$. Because of the previously discussed identity $c(\alpha) = s(-\alpha)$, this is equivalent to working with $A^{N,-r}_{\mu,(0),\xi}(q)$, which satisfies the above theorem for $-r > 0$.

Setting $k = 1$ allows us to retrieve the degree 0 parts for both the Segre and Verlinde invariants using the function φ, and hence the Segre–Verlinde correspondence of Corollary 1.9. When $k = 2$, we get a relation in the degree 1 part which is equivalent to Corollary 1.11. In general, the Segre and Verlinde relations are obtained by applying $D_{\mathcal{S}}^{(k)}$ and $D_{\mathcal{V}}^{(k)}$ to φ. Figure 1 summarizes these observations.

Remark 1.13. The previous results only deal with the $\nu = (0)$ case. When $\nu = (1)$, techniques similar to those of § 4.2 can be applied to the term [Boj21a, (4.24)], if an explicit expression for it is found. For $|\nu| > 1$, the same techniques no longer work because of the limitations of Lemma 4.1.

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A symmetry for virtual Segre series is also proved in [Boj21a, Theorem 1.7] for the compact case, and states

$$\mathcal{S}_Y(E,V;(-1)^N q) = \mathcal{S}_Y(V,E;(-1)^r q)$$

for torsion-free sheaves E and V of rank N and r respectively. As in Corollary 1.9, we have the following weak version of this symmetry in the equivariant case.

COROLLARY 1.14. *In the setting of Theorem 1.7, for $\alpha = V = \oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle$, we have the following symmetry*

$$A^{N,r}_{\mu,\nu,\xi}((-1)^N q) = A^{r,N}_{\xi,\nu,\mu}((-1)^r q)$$

whenever one of μ,ν,ξ is (1) and the other two are (0). In degree 0, we have

$$\mathcal{S}_{S,0}(E,V;(-1)^N q) - \mathcal{S}_{S,0}(V,E;(-1)^r q) = \sum_{n=1}^{\infty} \frac{g_n}{(\lambda_1\lambda_2)^{n-2}} \cdot \left(\int_S c_1(S)\right)^2 \cdot q^n$$

for some terms $g_n \in H^{2n-2}_{\mathsf{T}}(\mathrm{pt})$.

Motivated by our study of the higher degree Segre series in § 4.4 which, as we see after using Theorem 1.10, are symmetric under interchanging μ and ξ, the partitions keeping track of Chern classes of α and E, respectively, we conjecture the following *strong Segre symmetry*.

CONJECTURE 1.15. Let $r,N>0$, $S=\mathbb{C}^2$. For $E=\oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$, $V=\oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle$, we have the following symmetry

$$\mathcal{S}_S(E,V;(-1)^N q) = \mathcal{S}_S(V,E;(-1)^r q).$$

Using a program, we checked that this condition holds in the cases summarized in (28).

1.2.4 *Reduced Segre–Verlinde correspondence.* For the reduced invariants on $S=\mathbb{C}^2$, let us denote by $\mathcal{S}_i^{\mathrm{red}}$ and $\mathcal{V}_i^{\mathrm{red}}$ the degree i parts of the reduced Segre and Verlinde series, respectively, in the sense of Definition 1.3. In this setting, the fact that the series from Theorem 1.8 are the same as those from Theorem 1.7 allows us to obtain relations from the non-reduced case. For example, the Segre–Verlinde correspondence of Corollary 1.9 and symmetry of Corollary 1.14 give the following reduced Segre–Verlinde correspondence and symmetry in degree -1.

COROLLARY 1.16. *We have the following correspondence in degree -1:*

$$\mathcal{S}^{\mathrm{red}}_{-1}(E,V;q) = \mathcal{V}^{\mathrm{red}}_{-1}(E,\alpha;(-1)^N q).$$

When $\alpha = V$ is an equivariant vector bundle, we have the following symmetry:

$$\mathcal{S}^{\mathrm{red}}_{-1}(E,V;(-1)^N q) = \mathcal{S}^{\mathrm{red}}_{-1}(V,E;(-1)^r q).$$

Let $\alpha \in K_{\mathsf{T}}(S)$ with rank $r>0$. Write $c_1\lambda = c_1(\alpha)$ and $c_2\lambda^2 = c_2(\alpha)$ for some $c_1, c_2 \in \mathbb{Q}$. Then we can use the Theorem 1.8 to show that, for some series $A_2(q), A_1(q), A_0(q), B_1(q), B_0(q)$, dependent on r and N,

$$\begin{aligned}
\mathcal{S}^{\mathrm{red}}_0(E,\alpha;q)|_{m_1=\ldots=m_N=0} &= -\log\left(A_{(2),(0)}(q)\right)\cdot c_2 - \log\left(A_{(1,1),(0)}(q)\right)\cdot c_1^2 + \log\left(A_{(0),(2)}(q)\right)\\
&=: A_2(q)c_2 + A_1(q)c_1^2 + A_0(q),\\
\mathcal{V}^{\mathrm{red}}_0(E,\alpha;q)|_{m_1=\ldots=m_N=0} &= -\log\left(B_{(1,1),(0)}(q)\right)\cdot c_1^2 + \log\left(B_{(0),(2)}(q)\right)\\
&=: B_1(q)c_1^2 + B_0(q).
\end{aligned}$$

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These degree 0 terms should correspond to the ones for compact K3-surfaces and their reduced invariants. In particular, one can use the results obtained in Theorems 1.10 and 1.12 and the ones below to predict the Segre and Verlinde series in the compact case and describe their correspondences.

Corollary 1.11 provides relations between A_1, A_2 and B_1, and we have the following formula for $B_1(q)$ obtained from Theorem 1.10 by setting $k_1 = 2, k_2 = 0$.

Corollary 1.17. *The series $B_1(q)$ is explicitly given by*

$$[q^n]B_1(q) = -\frac{1}{2}\binom{n(N+r)-1}{nr}$$

for $n, r > 0$.

We give some conjectural formulas for A_0, B_0 in terms of A_1, B_1 when $N = 1$, which we have checked for $n \leqslant 20, r < 5$.

Conjecture 1.18. When $N = 1$ and $r \in \mathbb{Z}$, we have

$$[q^n]B_0(q) = \frac{1}{6}\left(\binom{r+1}{2}(n-1) - 1\right)[q^n]B_1(q).$$

When $N = 1$, $r < -1$ and $n > 1$, we have

$$[q^n]A_0(q) = \frac{1}{12}r(nr+n+2)[q^n]A_1(q).$$

1.3 Correspondence for 4-folds and other observations

1.3.1 *Segre–Verlinde correspondence.* Since all toric Calabi–Yau 4-folds are non-compact, we do not know how the invariants in the non-compact case relate to the ones in the compact case. We have seen for the surface case that the exponents on the universal series have a factor of $c_1(S)$. Considering the series given in [Boj21b, Proposition 4.13] and [Boj21a, (3.38)], together with § 5.4, we see that this term should be replaced by $c_3(X)$ in the 4-fold case. Motivated by Corollaries 1.9 and 1.14, we conjecture a weak Segre–Verlinde correspondence and symmetry for $X = \mathbb{C}^4$.

Conjecture 1.19. Let $X = \mathbb{C}^4$, $E = \oplus_{i=1}^N \mathcal{O}_X\langle y_i\rangle$, $V = \oplus_{i=1}^r \mathcal{O}_X\langle v_i\rangle$, and $\alpha \in K_{\mathsf{T}}(X)$, then for some choice of signs $o(\mathcal{L})$, we have the following symmetry and correspondence

$$\mathcal{S}_X(E, V; (-1)^N q) = \mathcal{S}_X(V, E; (-1)^r q),$$

$$\mathcal{S}_{X,0}(E,\alpha;q) - \mathcal{V}_{X,0}(E,\alpha;(-1)^N q) = \sum_{n=1}^{\infty} \frac{F_n}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{n-2}} \cdot \left(\int_X c_3(X)\right)^2 \cdot q^n$$

for some terms $F_n \in H_{\mathsf{T}}^{4n-6}(\mathrm{pt})$ dependent on α through its rank and Chern classes.

We checked both the correspondence and the symmetry using a computer program, as summarized in (40) and (41).

1.3.2 *Nekrasov's conjectures.* In [CK17, Appendix B], Cao and Kool gave a mathematical formulation of Nekrasov's conjecture [Nek20, § 5]. We generalize this to Quot schemes of $\mathbb{C}^2$ and $\mathbb{C}^4$ as follows.

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CONJECTURE 1.20. Let $Y = \mathbb{C}^d$, $E = \oplus_{i=1}^{r+1} \mathcal{O}_Y\langle y_i\rangle$, and $V = \oplus_{i=1}^{r} \mathcal{O}_Y\langle v_i\rangle$. When $d = 2$, or $d = 4$ with some choice of signs $o(\mathcal{L})$, we have

$$\mathcal{C}_Y(E, V; q) = \exp\left(q \int_Y c_{d-1}(Y)\right).$$

Note that when $N = 1$ and $Y = \mathbb{C}^2$, this is exactly Corollary 1.5. When $Y = \mathbb{C}^4$, we shall show that this conjecture is a consequence of Nekrasov and Piazzalunga's conjecture [NP19, § 2.5] using a Quot scheme version of Cao, Kool and Monavari's cohomological limit [CKM22, Appendix A] in Proposition 5.9. For $Y = \mathbb{C}^2$, we check it for $N = 2, 3, 4, 5$ up to and including $n = 7, 4, 2, 2$, respectively.

Since $[\mathrm{Quot}_Y(E, n)]^{\mathrm{vir}}$ has virtual dimension nN, we have $C_Y^N(V; q) = 1$ when $N > r$ in the compact case, simply for degree reason, which means the corresponding Verlinde series is also trivial because of the Segre–Verlinde correspondence. In the non-compact case, the above conjectures suggest that they may contain negative-degree terms. However, with a computer calculation for $Y = \mathbb{C}^2$, $N = 2, 3, 4, 5$ and all possible r, up to and including $n = 8, 5, 2, 2$, we see a complete vanishing when $N > r + 1$ for Chern numbers, and when $r < N$ for rank $-r$ Verlinde numbers.

CONJECTURE 1.21. Let $Y = \mathbb{C}^d$, $N > 1$, and $E = \oplus_{i=1}^{N} \mathcal{O}_Y\langle y_i\rangle$. When $d = 2$, or $d = 4$ with some choice of signs, we have, for $r = 0, 1, \ldots, N-2$ and $V = \oplus_{i=1}^{r} \mathcal{O}_Y\langle v_i\rangle$,

$$\mathcal{C}_Y(E, V; q) = 1.$$

Furthermore, for $r = 1, \ldots, N-1$, we have

$$\mathcal{V}_Y(E, -[V]; q) = 1.$$

In Proposition 5.9, we also show that the Chern series part of this conjecture for $d = 4$ is a consequence of Nekrasov and Piazzalunga's Conjecture 5.7.

2. Preliminaries

2.1 Equivariant cohomology and K-theory

Given a topological group G acting on a topological space M, the *equivariant cohomology* $H_G^*(M)$ is defined to be $H^*(EG \times M/G)$, where $EG \to BG$ is the universal principle G-bundle on the classifying space BG. The map $M \to \mathrm{pt}$ induces a ring homomorphism $H_G^*(\mathrm{pt}) \to H_G^*(M)$, making $H_G^*(M)$ a module over $H_G^*(\mathrm{pt})$ for any M, and we can view $H_G^*(\mathrm{pt})$ as a 'coefficient ring'.

DEFINITION 2.1. Given a G-representation V, viewed as a vector bundle $V \to \{\mathrm{pt}\}$, we define its equivariant characteristic classes by taking the associated bundle

$$EG \times_G V \to EG \times_G \{\mathrm{pt}\} = BG$$

and taking its characteristic classes in $H^*(BG) = H_G^*(\mathrm{pt})$. We denote by $c_i^G, e_G, \mathrm{ch}_G, \mathrm{td}_G$ the equivariant versions of the i-th Chern class, the Euler class, the Chern character, and the Todd class, respectively.

Example 2.2. For the action of a d-dimensional torus $\mathsf{T} = (\mathbb{C}^*)^d = \{(t_1, \ldots, t_d) : t_i \neq 0\}$, the coefficient ring is

$$H_{\mathsf{T}}^*(\mathrm{pt}) = H^*(B\mathsf{T}) = H^*((\mathbb{C}P^\infty)^d) = \mathbb{C}[\lambda_1, \ldots, \lambda_d]$$

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and $\lambda_1, \dots, \lambda_d$ are exactly the equivariant first Chern classes of 1-dimensional T-representations with weights $t_1, \dots, t_d$, respectively. In general, from [Edi97, § 3.2],

$$c_1^{\mathsf{T}}(\mathbb{C}\langle t_1^{w_1} \dots t_d^{w_d}\rangle) = w_1\lambda_1 + \dots + w_d\lambda_d.$$

For the $(d-1)$-dimensional subtorus $\mathsf{T}' = \{(t_1, \dots, t_d) \in \mathbb{C}^d : t_1 \dots t_d = 1\} \subseteq \mathsf{T}$, the inclusion induces the following isomorphism to the quotient ring:

$$H^*_{\mathsf{T}'}(\mathrm{pt}) \cong \mathbb{C}[\lambda_1, \dots, \lambda_d]/(\lambda_1 + \dots + \lambda_d).$$

We can construct the *equivariant K-group* $K_G(M)$ from the G-equivariant vector bundles. When $M = \mathbb{C}^d$, the vector bundles over M are trivial, but they may carry non-trivial G-actions. Therefore, the equivariant bundles on $\mathbb{C}^d$ correspond to finite-dimensional G-representations. The equivariant characteristic classes of vector bundles can then be extended to the K-theory classes. For example, the Euler class of $\alpha = [V] - [W] \in K_G(\mathrm{pt})$ is $e^G(\alpha) = e^G(V)/e^G(W)$, which lives in the ring of fractions $H^*_G(\mathrm{pt})_{\mathrm{loc}}$; the Chern character is $\mathrm{ch}_G(\alpha) = \mathrm{ch}_G(V) - \mathrm{ch}_G(W)$, which lives in $\prod_{i=0}^{\infty} H^i_G(\mathrm{pt})$.

Example 2.3. When $Y = \mathbb{C}^d$ with the natural action by $\mathsf{T} = (\mathbb{C}^*)^d$ or $\mathsf{T}' \cong (\mathbb{C}^*)^{d-1}$, we have the following character rings

$$K_{\mathsf{T}}(Y) \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \cong K_{\mathsf{T}}(\mathrm{pt}),$$

$$K_{\mathsf{T}'}(Y) \cong \frac{\mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]}{(t_1 \cdots t_d - 1)} \cong K_{\mathsf{T}'}(\mathrm{pt}),$$

where, for any weight $w = (w_1, \dots, w_d)$, the line bundle $\mathcal{O}_Y\langle t^w\rangle := \mathcal{O}_Y \otimes t^w$ simply corresponds to its character $t^w = t_1^{w_1} \cdots t_d^{w_d}$.

Remark 2.4. We will occasionally identify Chern characters, which are power series in cohomology, with elements in K-theory by

$$t_1^{w_1} \cdots t_d^{w_d} \leftrightarrow \mathrm{ch}_{\mathsf{T}}(\mathcal{O}_Y\langle t_1^{w_1} \cdots t_d^{w_d}\rangle) = e^{w_1\lambda_1 + \dots + w_d\lambda_d}.$$

This allows us to consider certain classes in cohomology as elements of $K_{\mathsf{T}}(\mathrm{pt})$. For example, for the line bundle $L = \mathcal{O}_Y\langle t_1^{w_1} \cdots t_d^{w_d}\rangle$, we write

$$\mathrm{ch}_{\mathsf{T}}(\Lambda_{-1}L^\vee) = 1 - e^{-c_1^{\mathsf{T}}(L)} = 1 - t_1^{-w_1} \cdots t_d^{-w_d} \in K_{\mathsf{T}}(Y).$$

The reason we consider equivariant cohomology is for equivariant integration. The integration formula of [EG95, Corollary 1] via equivariant localization states that, on a smooth complete variety Y with the action of a torus T, for λ an equivariant cohomological class, we have

$$\pi_{Y*}(\lambda) = \sum_F \pi_{F*}\left(\frac{i_F^*\lambda}{e_{\mathsf{T}}(N_F Y)}\right),$$

where the sum goes through the components F of the fixed locus, $N_F Y$ denotes the normal bundle, π denotes projection to a point, and i denotes the inclusion map. The right-hand side of this formula can be used to define equivariant integration in general; for Y an arbitrary smooth variety with finitely many fixed (reduced) points, the *equivariant push-forward* of π_Y is

$$\begin{aligned} \int_Y : H^*_{\mathsf{T}}(Y) &\to H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}, \\ \alpha &\mapsto \sum_{x \in Y^{\mathsf{T}}} \frac{i_x^*\alpha}{e_{\mathsf{T}}(T_x Y)}. \end{aligned} \tag{4}$$

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Example 2.5. Again let $Y=\mathbb{C}^d$ with the natural $\mathsf{T}=(\mathbb{C}^*)^d$-action. The only T-fixed point of Y is the origin. At the origin, the character for the tangent space is $T_0Y=t_1+t_2+\cdots+t_d\in K_T(\mathrm{pt})$, so $e_{\mathsf{T}}(T_0Y)=\lambda_1\cdots\lambda_d$. Substituting into (4), we have

$$\int_Y \alpha=\frac{\alpha}{\lambda_1\cdots\lambda_d}.$$

2.2 Partitions and solid partitions

A *partition* μ is a finite sequence $(\mu_1,\mu_2,\ldots,\mu_\ell)$ of non-increasing positive integers. The size $|\mu|$ is the sum of the μ_i's and we call $\ell=\ell(\mu)$ the length of the partition μ . The empty sequence (0) is the *empty partition* with size $|(0)|=0$. Each partition μ corresponds to a Young diagram which consists of pairs of non-negative integers $(i,j)\in\mathbb{Z}^2_{\geqslant 0}$ as follows:

$$\mu\leftrightarrow\{(i,j):j<\mu_{i+1}\}.$$

A pair $\square=(i,j)$ in the above set is called a box in μ, which we denote $\square\in\mu$. The *conjugate partition* μ^t is defined to be the partition whose boxes are $\{(j,i):(i,j)\in\mu\}$. Denote by $c(\square),r(\square),a(\square),l(\square)$ the *column index*, *row index*, *arm length* and *leg length* of $\square=(i,j)\in\mu$, defined explicitly as follows:

$$\begin{aligned}&c(\square)=j,\quad r(\square)=i,\\&a(\square)=\mu_{i+1}-j-1,\quad l(\square)=\mu^t_{j+1}-i-1.\end{aligned}$$

When $i,j>0$, a necessary condition for box (i,j) to be in μ is that both $(i-1,j)$ and $(i,j-1)$ are in μ. When $i=0$ (respectively $j=0$), we only need $(i,j-1)\in\mu$ (respectively $(i-1,j)\in\mu$).

A *solid partition* π is a finite sequence $(\pi_{ijk})_{i,j,k\geqslant 1}$ of positive integers such that

$$\pi_{ijk}\geqslant\pi_{i+1,j,k},\quad \pi_{ijk}\geqslant\pi_{i,j+1,k},\quad \pi_{ijk}\geqslant\pi_{i,j,k+1}.$$

The size of $|\pi|$ is the sum of the integers π_{ijk}. As a 4-dimensional analogue to a partition, a solid partition can also be viewed as a collection of boxes

$$\pi\leftrightarrow\{(i,j,k,l):l<\pi_{i,j,k}\}\subseteq\mathbb{Z}^4_{\geqslant 0}.$$

As for partitions, we have

$$(i,j,k,l)\in\pi\text{ implies}\begin{cases}(i-1,j,k,l)\in\pi\text{ unless }i=0,\\(i,j-1,k,l)\in\pi\text{ unless }j=0,\\(i,j,k-1,l)\in\pi\text{ unless }k=0,\\(i,j,k,l-1)\in\pi\text{ unless }l=0.\end{cases}\tag{5}$$

For a positive integer N, an N-*colored partition* of size n is an N-tuple of partitions $\mu=(\mu^{(1)},\ldots,\mu^{(N)})$ such that $|\mu|:=\sum|\mu^{(i)}|=n$. Figure 2 illustrates how the partitions are colored based on their index. Similarly, an N-*colored solid partition* is an N-tuple of solid partitions.

2.3 Admissible functions and universal series

We consider the notion of admissibility in the sense of [Mel18], which will be an important condition in finding universal series for equivariant invariants.

DEFINITION 2.6. Let $F(Q_1,Q_2\ldots;q_1,\ldots,q_d)\in\mathbb{Q}(q_1,\ldots,q_d)[\![Q_1,Q_2,\ldots]\!]$ be a series in finitely many variables $Q_1,Q_2,\ldots$ with constant term equal to 1. Then, using the plethystic exponential

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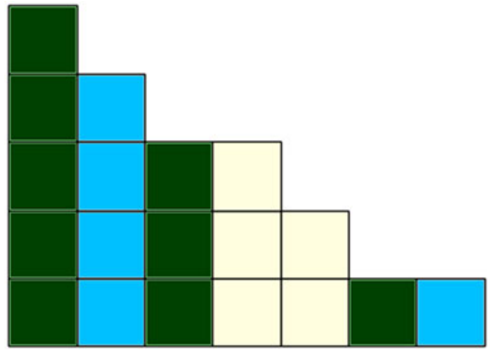

Figure 2. A 3-colored partition $\mu = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)})$ of size $|\mu| = 19$ where $\mu^{(1)} = (5,3,1)$, $\mu^{(2)} = (4,1)$, $\mu^{(3)} = (3,2)$ correspond to different colors.

Exp, we can write

$$F = \mathrm{Exp}\left(\frac{L}{(1-q_1)\cdots(1-q_d)}\right)$$

such that L is a power series in the variables $Q_1, Q_2, \dots$ whose coefficients are rational functions in $q_1, \dots, q_d$. The series F is called admissible with respect to the variables $q_1, \dots, q_d$ if the coefficients of L are polynomials in $q_1, \dots, q_d$.

Suppose $F(Q; m_1, \dots, m_N; w_1, \dots, w_r; q_1, \dots, q_d) \in \mathbb{Q}(q_1, \dots, q_d)[\![Q; m_1, \dots, m_N; w_1, \dots, w_r]\!]$ is admissible with respect to $q_1, \dots, q_d$ with constant term 1. We have the following Laurent expansion:

$$\log F(Q; \vec{m}; \vec{w};; e^{\lambda_1}, \dots, e^{\lambda_d}) = \sum_{k_1,\dots,k_d=-\infty}^{\infty} H_{k_1,\dots,k_d}(Q; \vec{m}; \vec{w}) \lambda_1^{k_1} \dots \lambda_d^{k_d}.$$

Since F is admissible, by the definition of plethystic exponential,

$$(1-q_1)\cdots(1-q_d)\log F(Q; \vec{m}; \vec{w}; \vec{q})$$

is regular in a neighbourhood of $q_1 = \ldots = q_d = 0$ as a power series in $q_1, \dots, q_d$, meaning we have a lower bound $k_1, \dots, k_d \geqslant -1$ for the above summation.

Furthermore, suppose F is symmetric in $w_1, \dots, w_r$ and symmetric in $m_1, \dots, m_N$, then we can expand in the following elementary symmetric polynomial basis:

$$\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \dots, e^{\lambda_d}) = \sum_{\substack{\mu,\xi \text{ partitions} \\ k_1,\dots,k_d \geqslant -1}} H_{\mu,\xi,\vec{k}}(Q) \prod_{i=1}^{\ell(\mu)} e_{\mu_i}(\vec{w}) \prod_{i=1}^{\ell(\xi)} e_{\xi_i}(\vec{m}) \lambda_1^{k_1} \dots \lambda_d^{k_d}$$

for some series $H_{\mu,\xi,\vec{k}}$.

Let $Y = \mathbb{C}^d$ and

$$\mathsf{T}_0 = (\mathbb{C}^*)^d, \mathsf{T}_1 = (\mathbb{C}^*)^N, \mathsf{T}_2 = (\mathbb{C}^*)^r$$

with the natural actions on Y, $E = \mathbb{C}^N \otimes \mathcal{O}_Y$, $V = \mathbb{C}^r \otimes \mathcal{O}_Y$, respectively. We write $\mathsf{T} = \mathsf{T}_0 \times \mathsf{T}_1 \times \mathsf{T}_2$. Say the equivariant cohomology ring of T is $H^*_{\mathsf{T}}(\mathrm{pt}) = \mathbb{C}[\lambda_1, \dots, \lambda_d; m_1, \dots, m_N; w_1, \dots, w_r]$. Then V as a T-equivariant bundle has equivariant Chern roots $w_1, \dots, w_r$, and E has Chern

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roots $m_1, \ldots, m_N$, so $e_i(m_1, \ldots, m_N) = c_i^{\mathsf{T}}(E), e_i(w_1, \ldots, w_r) = c_i^{\mathsf{T}}(V)$. Therefore

$$\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_d}) = \sum_{\substack{\mu,\xi \text{ partitions} \\ k_1,\ldots,k_d \geqslant -1}} H_{\mu,\xi,\vec{k}}(Q) c_\mu(V) c_\xi(E) \lambda_1^{k_1} \ldots \lambda_d^{k_d}.$$

For $\vec{k} = (k_1, \ldots, k_d)$ where $k_1, \ldots, k_d \geqslant -1$, there exist polynomials $E_{\vec{k}}$ such that

$$\frac{1}{d!} \sum_{\tau \text{ permutation}} \lambda_1^{k_{\tau(1)}} \cdots \lambda_d^{k_{\tau(d)}} = \frac{E_{\vec{k}}(e_1(\lambda_1, \ldots, \lambda_d), \ldots, e_d(\lambda_1, \ldots, \lambda_d))}{\lambda_1 \cdots \lambda_d}.$$

Now suppose F is symmetric in the variables $q_1, \ldots, q_d$, so $H_{\mu,k} = H_{\mu,\tau(k)}$ for any permutation τ. Hence

$$\begin{aligned} &\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_d}) \\ &= \sum_{\substack{\mu \text{ partition} \\ k_1,\ldots,k_d \geqslant -1}} H_{\mu,\xi,\vec{k}}(Q) E_{\vec{k}}(e_1(\lambda_1, \ldots, \lambda_d), \ldots, e_d(\lambda_1, \ldots, \lambda_d)) c_\mu(V) c_\xi(E). \end{aligned}$$

Note the equivariant weights of the tangent space $T_0 Y$ are exactly $\lambda_1, \ldots, \lambda_d$, so $e_i(\lambda_1, \ldots, \lambda_d) = c_i^{\mathsf{T}}(Y)$. By Example 2.5, we have

$$\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_d}) = \sum_{\substack{\mu \text{ partition} \\ k_1,\ldots,k_d \geqslant -1}} H_{\mu,\xi,\vec{k}}(Q) \int_Y E_{\vec{k}}(c_1(Y), \ldots, c_d(Y)) c_\mu(V) c_\xi(E). \tag{6}$$

Redistributing the terms, we get

$$\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_d}) = \sum_{\mu,\nu,\xi \text{ partitions}} H_{\mu,\nu,\xi}(Q) \int_Y c_\mu(V) c_\nu(Y) c_\xi(E)$$

for some series $H_{\mu,\nu,\xi}$. We exponentiate both sides, and we obtain the following *universal series expression* for F.

Proposition 2.7. *Let $F(Q; \vec{m}; \vec{w}; \vec{q}) \in \mathbb{Q}(q_1, \ldots, q_d)[\![Q; m_1, \ldots, m_N; w_1, \ldots, w_r]\!]$ be admissible with respect to the variables $q_1, \ldots, q_d$. Suppose F is symmetric in $w_1, \ldots, w_r$, in $m_1, \ldots, m_N$, and in $q_1, \ldots, q_d$. Then there exist power series $G_{\mu,\nu,\xi}(Q)$ labeled by partitions μ, ν, ξ, such that*

$$F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_n}) = \prod_{\mu,\nu,\xi \text{ partitions}} G_{\mu,\nu,\xi}(Q)^{\int_Y c_\mu(V) c_\nu(Y)} c_\xi(E).$$

3. Segre and Verlinde invariants on $\mathbb{C}^2$

3.1 Virtual equivariant invariants on Hilbert schemes

Before defining virtual invariants, we recall the notion of a perfect obstruction theory in the sense of [BF98, Definition 4.4]. For our purposes, we use the following simplified version.

Definition 3.1. Let X be a scheme over $\mathbb{C}$. An *obstruction theory* is a complex of vector bundles

$$E^\bullet = [\ldots \to E^{-2} \to E^{-1} \to E^0]$$

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for some $a\in\mathbb{Z}$, together with a morphism in the derived category $D(\mathrm{QCoh}(X))$ to the cotangent complex

$$\varphi: E^\bullet \to L_X^\bullet$$

such that $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is surjective. It is a *(2-term) perfect obstruction theory* if $E^i=0$ for $i\neq 0,-1$. The *virtual tangent bundle* $T^{\mathrm{vir}}=E_\bullet=(E^\bullet)^*$ is the class of the dual complex of a given obstruction theory.

Let S be a surface and E a vector bundle. It is known that $\mathrm{Quot}_S(E,n)$ admits an obstruction theory given by the dual complex of $\mathbf{R}\mathscr{H}om_p(\mathcal{I},\mathcal{F})$, where $\mathcal{I},\mathcal{F}$ are, respectively, the universal subsheaf and the quotient sheaf. When S is a projective surface, this obstruction theory is perfect and of virtual dimension nN. An explicit construction of this perfect obstruction theory can be found in [Sta24, § 2.3]. Using this, we can define a virtual fundamental class $[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}$ via the methods from [BF98, LT96], as well as a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ using [CFK09]. Applying the same argument for $S=\mathbb{C}^2$ gives us an equivariant perfect obstruction theory. We note that, since the quotients $E\twoheadrightarrow F$ are of rank 0 and Euler characteristic n, F is compactly supported, and the Ext-groups $\mathrm{Ext}^i(E,F)$ are finite-dimensional vector spaces. Thus the steps involving Serre duality still work.

Let $S=\mathbb{C}^2$ and $E=\oplus_{i=1}^N\mathcal{O}_S\langle y_i\rangle$. Recall from the introduction the following tori:

$$\mathsf{T}_0=(\mathbb{C}^*)^2,\quad \mathsf{T}_1=(\mathbb{C}^*)^N,\quad \mathsf{T}_2=(\mathbb{C}^*)^{r+s}.$$

Set $\mathsf{T}=\mathsf{T}_0\times\mathsf{T}_1\times\mathsf{T}_2$, with

$$K_{\mathsf{T}}(\mathrm{pt})=\mathbb{Z}[t_1^{\pm1},t_2^{\pm1};y_1^{\pm1},\dots,y_N^{\pm1};v_1^{\pm1},\dots,v_{r+s}^{\pm1}],$$
$$H^*_{\mathsf{T}}(\mathrm{pt})=\mathbb{C}[\lambda_1,\lambda_2;m_1,\dots,m_N;w_1,\dots,w_{r+s}],$$

where λ_i,m_i,w_i are, respectively, the equivariant first Chern classes of the 1-dimensional T-representations with weights t_i,y_i,v_i (see Example 2.2). This gives us a T-action on $\mathrm{Quot}_S(E,n)$. For a T-equivariant bundle V, the tautological bundle $V^{[n]}$ is also T-equivariant. The T_1-fixed quotients of $\mathrm{Quot}_S(E,n)$ decompose into the form

$$0\to\oplus_{i=1}^N I_i\langle y_i\rangle\to\oplus_{i=1}^N\mathcal{O}_S\langle y_i\rangle\to\oplus_{i=1}^N F_i\langle y_i\rangle\to 0.$$

Thus the T_1-fixed locus can be identified as

$$\bigsqcup_{n_1+\ldots+n_N=n}\mathrm{Hilb}^{n_1}(S)\times\cdots\times\mathrm{Hilb}^{n_N}(S).$$

The T_0-fixed locus of these Hilbert schemes consists of collections of finitely many reduced points, labeled by partitions. Consequently, the fixed locus $\mathrm{Quot}_S(E,n)^{\mathsf{T}}$ consists of finitely many reduced points of the form

$$Z_\mu=([Z_1],[Z_2],\dots,[Z_N])\in\mathrm{Hilb}^{n_1}(S)\times\cdots\times\mathrm{Hilb}^{n_N}(S),$$

labeled by N-colored partitions $\mu=\big(\mu^{(1)},\dots,\mu^{(N)}\big)$.

For a vector bundle V over Y, define

$$\Lambda_z(V):=\sum_{i\geqslant0}[\Lambda^iV]z^i\in K^0(Y)[z],\quad \Lambda_z(-V):=\sum_{i\geqslant0}[\mathrm{Sym}^iV](-z)^i\in K^0(Y)[\![z]\!],$$

which extends to a homomorphism $\Lambda_z:(K^0(Y),+)\to(K^0(Y)[\![z]\!],\cdot)$. We define the following equivariant invariants using virtual equivariant localization [GP97] and K-theoretic virtual equivariant localization [CFK09, Theorem 5.3.1]. Since we are interested in comparing the Segre and

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Verlinde series, we convert the Verlinde invariants into cohomological invariants using the virtual Hirzebruch–Riemann–Roch formula [RS21, Corollary 1.2].

DEFINITION 3.2. Let $S = \mathbb{C}^2$ and

$$\alpha = [\oplus_{i=1}^{r} \mathcal{O}_Y \langle v_i \rangle] - [\oplus_{i=r+1}^{r+s} \mathcal{O}_Y \langle v_i \rangle] \in K_{\mathsf{T}}(S).$$

The *equivariant virtual Segre, Chern, and Verlinde series* on Quot schemes are, respectively,

$$\mathcal{S}_S(E, \alpha; q) := \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_S(E,n)^{\mathsf{T}}} \frac{s(\alpha^{[n]}|_Z)}{e(T_Z^{\mathrm{vir}})},$$

$$\mathcal{C}_S(E, \alpha; q) := \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_S(E,n)^{\mathsf{T}}} \frac{c(\alpha^{[n]}|_Z)}{e(T_Z^{\mathrm{vir}})},$$

$$\mathcal{V}_S(E, \alpha; q) := \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_S(E,n)^{\mathsf{T}}} \frac{\mathrm{ch}(\det(\alpha^{[n]}|_Z))}{\mathrm{ch}(\Lambda_{-1}(T_Z^{\mathrm{vir}})^{\vee})}.$$

We shall describe how to calculate these invariants, and refer to [FMR21, § 5.1] and [Lim21, § 3.3] for the following argument. On each T_1-fixed locus

$$D = \mathrm{Hilb}^{n_1}(S) \times \cdots \times \mathrm{Hilb}^{n_N}(S),$$

the universal subsheaf and universal quotient sheaf of D are $\bigoplus_{i=1}^{N} I_{\mathcal{Z}_i}\langle y_i \rangle$ and $\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{Z}_j}\langle y_i \rangle$ where $\mathcal{Z}_i$ is the universal subscheme of $\mathrm{Hilb}^{n_i}(S)$. The virtual tangent bundle over D is then

$$T_D^{\mathrm{vir}} = \bigoplus_{i,j=1}^{N} \mathbf{R}\mathscr{H}om_p(I_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_j})\langle y_i^{-1} y_j \rangle$$

where $p : D \times X \to D$ is the projection. Further restricting to each $Z_\mu = ([Z_1], [Z_2], \ldots, [Z_N]) \in \mathrm{Quot}_S(E, n)^{\mathsf{T}}$ gives the virtual tangent bundle at Z_μ as follows:

$$T_{Z_\mu}^{\mathrm{vir}} = \bigoplus_{i,j=1}^{N} \mathrm{Ext}(I_{Z_i}, \mathcal{O}_{Z_j})\langle y_i^{-1} y_j \rangle \in K_{\mathsf{T}}(S). \tag{7}$$

To give an explicit formula for T^{vir}, we consider a T_0-equivariant free resolution of I_{Z_i}. We refer to [Eis95, p. 439] for the following *Taylor resolution*. Say I_{Z_i} is generated by monomials $m_1, \ldots, m_s$. For each $k = 0, \ldots, s$, let F_k be the free $\mathbb{C}[x_1, \ldots, x_n]$-module with basis $\{e_I\}$, indexed by subsets $I \subseteq \{1, \ldots, s\}$ of size k. Set

$$m_I = \text{least common multiple of} \{m_i : i \in I\}.$$

For $k = 1, \ldots, s$, define the differential $d_k : F_k \to F_{k-1}$ by

$$d_k(e_I) = \sum_{j=1}^{k} (-1)^j \frac{m_I}{m_{I-\{i_j\}}} e_{I-\{i_j\}}$$

for each subset $I = \{i_1, \ldots, i_k\}$ such that $i_1 < \cdots < i_k$. Giving each e_I the weight of m_I, we obtain the T_0-equivariant free resolution

$$0 \to F_s \to \ldots \to F_1 \to I_{Z_i} \to 0$$

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where

$$F_k = \bigoplus_{I\subseteq\{1,\ldots,s\},|I|=k} \mathcal{O}_S\langle m_I(t)\rangle$$

for some $d_{kI}\in\mathbb{Z}^2$. Define

$$P(I_{Z_i}) = \sum_{k,|I|=k} (-1)^k m_I(t). \tag{8}$$

Note that the character of $\mathcal{O}_S=\mathbb{C}[x_1,x_2]$ is $\sum_{i,j\geqslant 0} t_1^{-i}t_2^{-j} = 1/(1-t_1^{-1})(1-t_2^{-1})$, so the character of $\mathcal{O}_{Z_i}=\mathcal{O}_S/I_{Z_i}$ is

$$Q_i := \frac{1-P(I_{Z_i})}{(1-t_1^{-1})(1-t_2^{-1})}.$$

Therefore, the character of $T^{\mathrm{vir}}_{Z_\mu}$ in $K_{\mathsf{T}}(\mathrm{pt})$ can be expressed as

$$\begin{aligned}
\bigoplus_{i,j=1}^N \mathrm{Ext}(I_{Z_i},\mathcal{O}_{Z_j})\langle y_i^{-1}y_j\rangle &= \sum_{i,j=1}^N \sum_{k,|I|=k} (-1)^k \mathrm{Hom}(\mathcal{O}_S\langle m_I(t)\rangle,\mathcal{O}_{Z_j})y_i^{-1}y_j \\
&= \sum_{i,j=1}^N \sum_{k,I}(-1)^k\mathcal{O}_{Z_j}\langle m_I(t)^{-1}\rangle y_i^{-1}y_j \\
&= \sum_{i,j=1}^N \overline{P(I_{Z_i})}Q_j y_i^{-1}y_j \\
&= \sum_{i,j=1}^N (Q_j-(1-t_1)(1-t_2)\overline{Q_i}Q_j)\cdot y_i^{-1}y_j
\end{aligned} \tag{9}$$

where $\overline{(\cdot)}$ denotes the involution $t_i\mapsto t_i^{-1}$.

For the T-equivariant bundle $V=\oplus_{i=1}^r\mathcal{O}_S\langle v_i\rangle$, the fiber of $V^{[n]}$ over $Z_\mu=(Z_1,\ldots Z_N)$ is the rn-dimensional representation

$$\bigoplus_{i=1}^r\bigoplus_{j=1}^N \mathcal{O}_{Z_j}\langle v_iy_j\rangle = \sum_{i=1}^r\sum_{j=1}^N\sum_{\square\in\mu^{(j)}} v_iy_jt_1^{-c(\square)}t_2^{-r(\square)}. \tag{10}$$

We could also replace the terms $\sum_{\square\in\mu^{(j)}} t_1^{-c(\square)}t_2^{-r(\square)}$ by Q_j, but we elect to use the above expansion to match with one of our main references [GM22].

Substituting the above calculations into the definition, we obtain the following expressions for the Chern and Verlinde series of vector bundles

$$\begin{aligned}
\mathcal{C}_S(E,V;q) &:= \sum_\mu q^{|\mu|}\frac{\prod_{j=1}^N\prod_{\square\in\mu^{(j)}}\prod_{i=1}^r(1+w_i+m_j-c(\square)\lambda_1-r(\square)\lambda_2)}{e(T^{\mathrm{vir}}|_{Z_\mu})}, \\
\mathcal{V}_S(E,V;q) &:= \sum_\mu q^{|\mu|}\frac{\prod_{j=1}^N\prod_{\square\in\mu^{(j)}}\prod_{i=1}^r v_iy_jt_1^{-c(\square)}t_2^{-r(\square)}}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)}.
\end{aligned} \tag{11}$$

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3.2 Relation to projective toric surfaces

We consider what the equivariant invariants will be for a smooth projective toric surface S', and compare them with the case $S=\mathbb{C}^2$. More details on this reduction can be found in [GM22, § 3.2]; see also [Arb21, § 6.2] and [LYZ02, § 3.2].

Suppose we are interested in the integral

$$\int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}} f(V^{[n]})$$

for a T-equivariant bundle V and some multiplicative genus f. On S where V has Chern roots $w_1,\dots,w_r$ and E has Chern roots $m_1,\dots,m_N$, this can be computed as

$$\begin{aligned} F(\vec{\lambda},\vec{m},\vec{w}) &:= \int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}} f(V^{[n]}) \\ &= \sum_{|\mu|=n} \frac{\prod_{j=1}^N \prod_{\square\in\mu^{(j)}} \prod_{i=1}^r f\left(w_i+m_j-c(\square)\lambda_1-r(\square)\lambda_2\right)}{e(T^{\mathrm{vir}}|_{Z_\mu})}. \end{aligned}$$

We write $T'=(\mathbb{C}^*)^2$ with cohomological variables λ_1',λ_2' so that $H^*_{\mathsf{T}'}(\mathrm{pt})=\mathbb{C}[\lambda_1',\lambda_2']$. Let S' be a toric projective surface with a natural action by $\mathsf{T}'=(\mathbb{C}^*)^2$. Say the fixed points are $p_1,\dots,p_M$ and the Chern roots of the tangent space of S' at p_i are $\lambda_1^{(i)},\lambda_2^{(i)}$, which live in $H^*_{\mathsf{T}'}(\mathrm{pt})=\mathbb{C}[\lambda_1',\lambda_2']$. Suppose that E' is a T'-equivariant bundle with Chern roots $m_1^{(i)},\dots,m_N^{(i)}\in H^*_{\mathsf{T}'}(\mathrm{pt})$ at each p_i such that the fixed locus of $\mathrm{Quot}_{S'}(E',n)$ is a finite collection of reduced points for all n. Let V' be a T'-equivariant bundle on S' with Chern roots $w_1^{(i)},\dots,w_r^{(i)}\in H^*_{\mathsf{T}'}(\mathrm{pt})$. By virtual equivariant localization, we have

$$\int_{[\mathrm{Quot}_{S'}(E',n)]^{\mathrm{vir}}} f(V') = \left(\sum_{i=1}^M F\left(\vec{\lambda}^{(i)},\vec{m}^{(i)},\vec{w}^{(i)}\right)\right)\Big|_{\lambda_1'=\lambda_2'=0}.$$

Since S' is compact, the sum on the right-hand side lives in $H^*_{\mathsf{T}}(\mathrm{pt})=\mathbb{C}[\lambda_1',\lambda_2']$. Therefore, it is indeed valid to set $\lambda_1'=\lambda_2'=0$, and the equality follows from the virtual Bott residue formula.

If we define $I_S(E,V;q):=\sum_{n=0}^\infty q^n\int_{[\mathrm{Quot}_S(E,n)]} f(V)$, then from the above expansions, we obtain

$$I_{S'}(E',V';q)=\left(\sum_{i=1}^M I_S(E,V;q)\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)},\vec{m}=\vec{m}^{(i)},\vec{w}=\vec{w}^{(i)}}\right)\Big|_{\vec{\lambda}'=0}. \tag{12}$$

Suppose furthermore that the series in the above expression have the following universal structure

$$\begin{aligned} I_{S'}(E',V';q) &= \prod_{\substack{\mu,\nu,\xi \text{ partitions}\\ |\mu|+|\nu|+|\xi|=2}} U'_{\mu,\nu,\xi}(q)^{c_\mu(V')c_\nu(S')c_\xi(E')}, \\ I_S(E,V;q) &= \prod_{\mu,\nu,\xi \text{ partitions}} U_{\mu,\nu,\xi}(q)^{\int_S c_\mu(V)c_\nu(S)c_\xi(E)} \end{aligned} \tag{13}$$

where $\int_S$ is given by (4). The exponents $c_\mu(V')c_\nu(S')c_\xi(E')$ are regarded as numbers in $H^2(S')$. Exponentiating a series to a power of an element of $H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$ follows the recipe in Remark 1.6.

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Then (12) implies

$$I_{S'}(E',V';q) = \left(\prod_{\mu,\nu,\xi \text{ partitions}} U_{\mu,\nu,\xi}(q)^{\int_S c_\mu(V)c_\nu(S)c_\xi(E)}\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)},\vec{m}=\vec{m}^{(i)},\vec{w}=\vec{w}^{(i)}} \right)\Big|_{\vec{\lambda}'=0}$$

$$= \prod_{\substack{\mu,\nu,\xi \text{ partitions} \\ |\mu|+|\nu|+|\xi|=2}} U_{\mu,\nu,\xi}(q)^{c_\mu(V')c_\nu(S')c_\xi(E')}.$$

The final sum is only over $|\mu|+|\nu|+|\xi|=2$ because if $|\mu|+|\nu|+|\xi|\neq 2$, then the term

$$\left(\sum_{i=1}^M \int_S c_\mu(V)c_\nu(S)c_\xi(E)\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)},\vec{m}=\vec{m}^{(i)},\vec{w}=\vec{w}^{(i)}} \right)\Big|_{\vec{\lambda}'=0} = \int_S c_\mu(V')c_\nu(S')c_\xi(E')$$

vanishes for degree reasons. By universality, we conclude that if the series structure (13) exists, then the series $U^{\mathrm{vir}}_{\mu,\nu,\xi}(q)$ of $I_{S'}(E',V';q)$ for the projective case coincide with the series $U_{\mu,\nu,\xi}(q)$ of $I_S(E,V;q)$ whose exponents lie in $H^0_{\mathsf{T}}(\mathrm{pt})$.

3.3 Universal series expansion

A general tactic for studying Segre and Verlinde series is by using a more general genus. In the non-virtual setting for Hilbert schemes of surfaces, this could be [GM22, (1.1)]. In the virtual case, we use the invariant (14) defined below. For Calabi–Yau 4-folds, we consider the Nekrasov genus (35) introduced by [NP19].

For projective surfaces, we define an auxiliary virtual invariant as follows

$$\mathcal{N}_S(E,\alpha;q,z) := \sum_{n=0}^\infty q^n \chi^{\mathrm{vir}}\left(\mathrm{Quot}_S(E,n), \Lambda_{-z}\alpha^{[n]}\right). \tag{14}$$

The variable z is considered as the weight of an extra $\mathbb{C}^*$-action that is trivial on S and $\mathrm{Quot}_S(E,n)$. We shall refer to this as the *Nekrasov genus* for Quot schemes of surfaces (cf. (35)).

We generalize this to the equivariant setting using virtual equivariant localization. On $S=\mathbb{C}^2$ for vector bundles, this is given by:

$$\mathcal{N}_S(E,V;q,z) := \sum_{\mu} q^{|\mu|} \frac{\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - t_1^{-c(\Box)} t_2^{-r(\Box)} v_i y_j z)}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)}$$

$$\in \mathbb{Q}(t_1,t_2;y_1,\dots,y_N)[\![q,z]\!]. \tag{15}$$

The following Chern and Verlinde limits are satisfied, analogous to [GM22, Proposition 3.5].

Lemma 3.3. *For $S=\mathbb{C}^2$, the Chern series and the Verlinde series can be retrieved from $\mathcal{N}_S$ by taking limits. We have*

$$\mathcal{C}_S(E,V;q) = \lim_{\varepsilon\to 0} \mathcal{N}_S\left(E,V;(-1)^N q\varepsilon^{N-r}(1+\varepsilon)^r, (1+\varepsilon)^{-1}\right)|_{\vec{\lambda}\rightsquigarrow -\varepsilon\vec{\lambda},\vec{w}\rightsquigarrow-\varepsilon\vec{w},\vec{m}\rightsquigarrow -\varepsilon\vec{m}},$$

$$\mathcal{V}_S(E,V;q) = \lim_{\varepsilon\to 0} \mathcal{N}_S\left(E,V;(-1)^r q\varepsilon^r, \varepsilon^{-1}\right).$$

Here we use the identification of Remark 2.4 with $c_1(t_i)=\lambda_i$, $c_1(y_i)=m_i$, $c_1(v_i)=w_i$.

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Proof. For the Chern limit, first consider the substitutions $\lambda_i \rightsquigarrow -\varepsilon\lambda_i, w_i \rightsquigarrow -\varepsilon w_i, m_i \rightsquigarrow -\varepsilon m_i$. This turns the term $\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - t_1^{-c(\Box)} t_2^{-r(\Box)} v_i y_j (1+\varepsilon)^{-1})$ into

$$\begin{aligned}
&\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r 1 - \frac{e^{-\varepsilon(w_i+m_j-c(\Box)\lambda_1-r(\Box)\lambda_2)}}{1+\varepsilon}\\
=&\frac{1}{(1+\varepsilon)^{r|\mu|}} \prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1+\varepsilon - e^{-\varepsilon(w_i+m_j-c(\Box)\lambda_1-r(\Box)\lambda_2)})\\
=&\left(\frac{\varepsilon}{1+\varepsilon}\right)^{r|\mu|} \prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - c(\Box)\lambda_1 - r(\Box)\lambda_2 + w_i + m_j + O(\varepsilon)).
\end{aligned}$$

For the denominator in the sum (15), we note that for a Chern root x, substituting it by $-\varepsilon x$ turns $1-e^{-x} = x - \frac{x^2}{2} + \ldots$ into $1 - e^{\varepsilon x} = -\varepsilon(x + O(\varepsilon))$. Therefore, after the substitution, the denominator $\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)$ becomes

$$(-1)^{N|\mu|}\varepsilon^{N|\mu|}(e(T^{\mathrm{vir}}_{Z_\mu}) + O(\varepsilon)).$$

Substituting back into (15), the Chern limit becomes the limit of

$$\begin{aligned}
&\sum_\mu (-1)^{N|\mu|} q^{|\mu|} \varepsilon^{(N-r)|\mu|}(1+\varepsilon)^{r|\mu|} \cdot \frac{\varepsilon^{r|\mu|}}{(-1)^{N|\mu|}\varepsilon^{N|\mu|}(1+\varepsilon)^{r|\mu|}}\cdot\\
&\frac{\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - c(\Box)\lambda_1 - r(\Box)\lambda_2 + w_i + m_j + O(\varepsilon))}{(e(T^{\mathrm{vir}}_{Z_\mu}) + O(\varepsilon))}\\
=&\sum_\mu q^{|\mu|} \frac{\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - c(\Box)\lambda_1 - r(\Box)\lambda_2 + w_i + m_j + O(\varepsilon))}{(e(T^{\mathrm{vir}}_{Z_\mu}) + O(\varepsilon))}
\end{aligned}$$

which converges to $\mathcal{C}_S(E,V;q)$ by (11).

For the Verlinde series, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \mathcal{N}_S(E,V;(-1)^r q\varepsilon^r, \varepsilon^{-1})\\
=&\lim_{\varepsilon\to 0} \sum_\mu (-1)^{r|\mu|} q^{|\mu|} \varepsilon^{r|\mu|} \cdot \frac{\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} t_1^{c(\Box)} t_2^{r(\Box)} \prod_{i=1}^r (1 - t_1^{-c(\Box)} t_2^{-r(\Box)} v_i y_j \varepsilon^{-1})}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)}\\
=&\lim_{\varepsilon\to 0} \sum_\mu q^{|\mu|} \frac{\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (t_1^{-c(\Box)} t_2^{-r(\Box)} v_i y_j - \varepsilon)}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)}\\
=&\mathcal{V}_S(E,V;q).
\end{aligned}$$

□

Before starting the proof for universal series expressions, let us discuss what the expansion of $\mathcal{N}_S(E,V;q,z)$ as a formal Laurent series in the variables $\vec{\lambda}, \vec{m}, \vec{w}, q, z$ would look like. In [Arb21, Proposition 3.2], Arbesfeld shows that invariants such as $[q^n]\mathcal{N}_S(E,V;q,z)$ can be written as a quotient whose numerator is a Laurent polynomial in $\vec{t}, \vec{y}, \vec{v}, z$ and whose denominator is of the form $\prod_{\mathsf{w}}(1-\mathsf{w})$ for some *non-compact weights* w in the sense of the following definition.

Definition 3.4 [Arb21, Definition 3.1]. Let M be a quasi-projective scheme with an action by some torus T. For a weight $\mathsf{w} \in \mathsf{T}^\vee$, denote by T_{w} the maximal subtorus of T containing $\ker \mathsf{w}$. If the fixed locus $M^{\mathsf{T}_{\mathsf{w}}}$ is proper, then w is a compact weight; otherwise, it is a non-compact weight.

Fasola, Monavari and Ricolfi used this to prove that the K-theoretic Donaldson–Thomas partition functions on $\mathbb{C}^3$ are Laurent polynomials with respect to the variables $y_1, \ldots, y_N$ [FMR21, Theorem 6.5]. We give an outline of their argument, applied to the invariant $\mathcal{N}_S$ for $S = \mathbb{C}^2$. First note that, by (9), for any N-colored partition μ, we have

$$\frac{1}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)} = A(\vec{t}) \prod_{1\leqslant i,j\leqslant N, i\neq j} \frac{\prod_{a\in A_{ij}}(1 - y_i^{-1}y_j t^a)}{\prod_{b\in B_{ij}}(1 - y_i^{-1}y_j t^b)}$$

for some series $A(\vec{t}) \in \mathbb{Q}[\![t_1, t_2]\!]_{\mathrm{loc}}$ and some sets of weights A_{ij}, B_{ij}. We shall show that the denominator of $\mathcal{N}_S$ does not have factors of the form $(1 - y_i^{-1}y_j t^b)$ for any $i \neq j$ and $b \in \mathbb{Z}^2$. By [Arb21, Proposition 3.2], we need to prove that $\mathsf{w} = y_i^{-1}y_j t^b$ is a compact weight. Since

$$\ker \mathsf{w} = \{(\vec{t}, \vec{y}, \vec{v}) : y_i = y_j t^b\}$$

is itself a torus, we have $\mathsf{T}_\mathsf{w} = \ker \mathsf{w}$. By definition, it suffices to show that $\mathrm{Quot}_S(E,n)^{\mathsf{T}_\mathsf{w}}$ is proper. With the automorphism $\mathsf{T} \to \mathsf{T}$ defined by

$$(\vec{t}, y_1, \ldots, y_j, \ldots, y_N, \vec{v}) \mapsto (\vec{t}, y_1, \ldots, y_j t^b, \ldots, y_N, \vec{v}),$$

we identify the subgroup T_w as $\mathsf{T}_0 \times \{(w_1, \ldots, w_N) : w_i = w_j\} \times \mathsf{T}_2$, which contains the subgroup $\mathsf{T}_0 = \mathsf{T}_0 \times \{(1, \ldots, 1)\}$. This gives us an inclusion

$$\mathrm{Quot}_S(E,n)^{\mathsf{T}_\mathsf{w}} \hookrightarrow \mathrm{Quot}_S(E,n)^{\mathsf{T}_0}.$$

The quotients in the fixed locus $\mathrm{Quot}_S(E,n)^{\mathsf{T}_0}$ are all supported at the origin $0 \in \mathbb{C}^2$, so the fixed locus lies inside the punctual Quot scheme $\mathrm{Quot}_S(E,n)_0$. The punctual Quot scheme is proper since it is a fiber of the Quot-to-Chow map $\mathrm{Quot}_S(E,n) \to \mathrm{Sym}^n S$, which is a proper morphism [FMR21, Remark 3.4]. In conclusion, $[q^n]\mathcal{N}_S(E, V; q, z)$ is a Laurent polynomial with respect to the variables $y_1, \ldots, y_N$, so it can be expanded into a power series with respect to the cohomological parameters $m_1, \ldots, m_N$.

Furthermore, if w is a weight that contains both t_1 and t_2, then we have $\mathsf{T}_\mathsf{w} \cong \{(t_1, t_2) : t_1 t_2 = 1\} \times \mathsf{T}_1 \times \mathsf{T}_2$. The fixed locus of this subgroup remains the same as that of T, as explained in the next section for reduced invariants. Therefore, w is a compact weight, and the denominator of $\mathcal{N}_S$ will not contain factors of the form $(1 - t_1^a t_2^b)$ for any $a \neq 0$, $b \neq 0$. This means that in cohomology $[q^n]\mathcal{N}_S(E, V; q, z)$ can be expanded into a Laurent series in λ_1, λ_2 whose coefficients are power series in $\vec{m}, \vec{w}, z$, where the degrees on λ_1, λ_2 are bounded below individually. We shall see the importance of this lower bound in the proof of the following theorem.

Theorem 3.5. *Let $S = \mathbb{C}^2$. For any $r \in \mathbb{Z}$, $N > 0$, there exist universal power series $A_{\mu,\nu,\xi}(q), B_{\mu,\nu,\xi}(q), C_{\mu,\nu,\xi}(q)$, dependent on N and r, such that for $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i \rangle$ and $\alpha \in K_\mathsf{T}(S)$ of rank r, the equivariant virtual Segre, Verlinde and Chern series on $\mathrm{Quot}_S(E,n)$ can be written as the following infinite products*

$$\mathcal{S}_S(E, \alpha; q) = \prod_{\mu,\nu,\xi \text{ partitions}} A_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)},$$

$$\mathcal{V}_S(E, \alpha; q) = \prod_{\mu,\nu,\xi \text{ partitions}} B_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)},$$

$$\mathcal{C}_S(E, \alpha; q) = \prod_{\mu,\nu,\xi \text{ partitions}} C_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)}.$$

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Proof. We begin with the case where α is a vector bundle V. Assume $V = \oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle$. At the end of the proof, we can generalize this to arbitrary T-equivariant bundles by substituting T-weights into the variables $v_1, \dots, v_r$.

Begin by expanding $\log \mathcal{N}_S(E, V; q, z)$ as a Laurent series in λ_1, λ_2 as follows:

$$\log \mathcal{N}_S(E, V; q, z) = \sum_{(j,k)\in\mathbb{Z}^2} H_{j,k}(q, z; \vec{m}; \vec{w})\lambda_1^j\lambda_2^k$$

for some series $H_{j,k} \in \mathbb{Q}[\![q, z; m_1, \dots, m_N; w_1, \dots, w_r]\!]$. By the symmetry in $w_1, \dots, w_r$ and the symmetry in $m_1, \dots, m_N$, this expands to

$$\begin{aligned}\log \mathcal{N}_S(E, V; q, z) = &\sum_{\substack{\mu,\xi \text{ partitions}\\ j,k\geqslant -1}} G_{\mu,\xi,j,k}(q, z)\cdot \lambda_1^j\lambda_2^k c_\mu(V)c_\xi(E)\\ &+ \sum_{\substack{\mu,\xi \text{ partitions}\\ \min\{j,k\}\leqslant -2}} G_{\mu,\xi,j,k}(q, z)\cdot \lambda_1^j\lambda_2^k c_\mu(V)c_\xi(E)\end{aligned}$$

for some series $G_{\mu,\xi,j,k} \in \mathbb{Q}[\![q, z]\!]$.

Our goal is to get a universal series expression by exponentiating the above equality. To do so, we show that the terms in the second summation vanish, using the relations from § 3.2. This proves that $\mathcal{N}_S(E, V; q, z)$ is admissible, from which we deduce the desired expressions by taking the limits of Lemma 3.3.

Let S' be a toric projective surface with $\mathsf{T}' = (\mathbb{C}^*)^2$-action, fixed points $p_1, \dots, p_M$ and vector bundles E', V' as in § 3.2. Then (12) applied to $\mathcal{N}$ becomes

$$\mathcal{N}_{S'}(E', V'; q, z) = \left(\prod_{i=1}^M \mathcal{N}_S(E, V; q, z)\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)}, \vec{m}=\vec{m}^{(i)}, \vec{w}=\vec{w}^{(i)}}\right)\Bigg|_{\vec{\lambda}'=0}.$$

Instead of taking $\vec{m}^{(i)}, \vec{w}^{(i)}$ as elements in $H^*_{\mathsf{T}'}(\mathrm{pt}) = \mathbb{C}[\vec{\lambda}']$, we could regard them as formal variables $\vec{m}, \vec{w}$, independent of i. This can be done by adding trivial actions by $\mathsf{T}_1, \mathsf{T}_2$ to S', and replacing E', V' by

$$E' = \oplus_{j=1}^N \mathcal{O}_{S'}\langle y_j\rangle, \quad V' = \oplus_{j=1}^r \mathcal{O}_{S'}\langle v_j\rangle,$$

which are trivial bundles whose equivariant Chern roots at each fixed point p_i are, respectively, $\vec{m}$ and $\vec{w}$, independent of i. In this setting, the substitutions $\vec{m} = \vec{m}^{(i)}, \vec{w} = \vec{w}^{(i)}$ are no longer necessary in the above equation for $\mathcal{N}_{S'}$, which gives

$$\mathcal{N}_{S'}(E', V'; q, z) = \left(\prod_{i=1}^M \mathcal{N}_S(E, V; q, z)\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)}}\right)\Bigg|_{\vec{\lambda}'=\vec{m}=\vec{w}=0}.$$

Substituting the previous expansion of $\log \mathcal{N}_S(E, V; q, z)$, we see that

$$\log \mathcal{N}_{S'}(E', V'; q, z) = \left(\sum_{i=1}^M \sum_{\substack{\mu,\xi \text{ partitions}\\ j,k\in\mathbb{Z}}} G_{\mu,\xi,j,k}(q, z)\cdot (\lambda_1^{(i)})^j(\lambda_2^{(i)})^k c_\mu(V)c_\xi(E)\right)\Bigg|_{\vec{\lambda}'=\vec{w}=\vec{m}=0}.$$

Again, the reason that we can evaluate $\vec{\lambda}' = \vec{w} = \vec{m} = 0$ on the right-hand side is that, by compactness of S', the terms in the brackets are power series in $\vec{\lambda}', \vec{w}, \vec{m}$. Since the terms

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$c_\mu(V), c_\xi(E)$ form a basis for symmetric polynomials in the variables $\vec{w}, \vec{m}$, we conclude that each summand

$$\sum_{i=1}^{M}\sum_{j,k\in\mathbb{Z}} G_{\mu,\xi,j,k}(q,z)\cdot(\lambda_1^{(i)})^j(\lambda_2^{(i)})^k \tag{16}$$

is a power series in λ_1', λ_2'.

Let $S' = \mathbb{P}^1\times\mathbb{P}^1$, with T_0-action

$$(t_1,t_2)\cdot([x_0:x_1],[y_0:y_1]) = ([x_0:t_1x_1],[y_0,t_2y_1]).$$

We refer to [LYZ02, § 3.7] for the following computations of equivariant weights. The fixed points are

$$p_1 = p_{00} = ([1:0],[1:0]),\quad p_2 = p_{01} = ([1:0],[0:1]),$$

$$p_3 = p_{10} = ([0:1],[1:0]),\quad p_4 = p_{11} = ([0:1],[0:1]).$$

The corresponding weights are $\vec{\lambda}^{(1)} = \vec{\lambda}^{(00)}, \vec{\lambda}^{(2)} = \vec{\lambda}^{(01)}, \vec{\lambda}^{(3)} = \vec{\lambda}^{(10)}, \vec{\lambda}^{(4)} = \vec{\lambda}^{(11)}$ where

$$\lambda_1^{(ij)} = (-1)^i\lambda_1',\quad \lambda_2^{(ij)} = (-1)^j\lambda_2'$$

for $i,j\in\{0,1\}$ for each μ,ξ. Substituting into (16), we see that the summands with odd j or k would cancel each other out, leaving us with

$$\sum_{j,k \text{ even}} G_{\mu,\xi,j,k}(q,z)\cdot 4(\lambda_1')^j(\lambda_2')^k.$$

By our previous observation, this expression is a power series in λ_1', λ_2' for each μ,ξ and does not contain negative-degree terms. Therefore, if $\min\{j,k\}\leqslant -2$, we have $G_{\mu,\xi,j,k}(q) = 0$.

Having dealt with the case where j,k are both even, we would like to apply the same argument to the other cases. To do so, we need to solve the problem that the summands vanish whenever one of j,k is odd. We claim that if each w_i is replaced by $w_i(l+\lambda_1+\lambda_2)^2$ in $\mathcal{N}_S$ for $i=1,\dots,r$, where l is a fixed number, then the resulting coefficients in front of q,z are of the form

$$\tilde{F}(\vec{\lambda},\vec{m},\vec{w}) := \int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}} \tilde{f}(c_i(K_S^{[n]}), c_j(\mathcal{O}_S^{[n]}), e_k(\vec{w}))_{i,j,k>0} \in H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$$

for some series $\tilde{f}$. The number l here will be substituted by integers later to get more specific conditions. Then we may repeat the above argument and get that $\sum_{i=1}^M \tilde{F}(\vec{\lambda}^{(i)},\vec{m}^{(i)},\vec{w})\in H^*_{\mathsf{T}}(\mathrm{pt})$ is a power series in the variables $\vec{\lambda}',\vec{m},\vec{w}$. Separating out the coefficients of $c_\mu(V), c_\xi(E)$ as we did in (16), this gives that the coefficients of q,z in

$$\sum_{i=1}^{M}\sum_{j,k} G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot(l+\lambda_1+\lambda_2)^{2|\mu|}\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)}}$$

are power series in λ_1', λ_2'.

Let us prove the claim above. Note that the coefficients of q,z in $\mathcal{N}_S(E,V;q,z)$ are of the form

$$\sum_{\mu}\frac{f(c_j(\mathcal{O}_S^{[n]}), e_k(\vec{w}))_{j,k>0}}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})}$$

for some series f. Set $\vec{Y} = (m_j - c(\square)\lambda_1 - r(\square)\lambda_2)_{j=1,\dots,N}^{\square\in\mu^{(j)}}$ the collection of Chern roots of $\mathcal{O}_S^{[n]}$ and $\vec{X} = (\lambda_1+\lambda_2+m_j - c(\square)\lambda_1 - r(\square)\lambda_2)_{j=1,\dots,N}^{\square\in\mu^{(j)}}$ the collection of Chern roots of $K_S^{[n]}$. Let us

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also denote $x=\lambda_1+\lambda_2$. To prove the claim, it suffices to show that, for any series f, there exists some series $\tilde f$ such that

$$f(c_j(\mathcal{O}_S^{[n]}), e_k((l+\lambda_1+\lambda_2)^2\vec{w})_{j,k>0} = \tilde f(c_i(K_S^{[n]}), c_j(\mathcal{O}_S^{[n]}), e_k(\vec{w}))_{i,j,k\geqslant 0}.$$

We may rewrite the left-hand side and see that this is equivalent to showing

$$g(x, e_j(\vec{Y}), e_k(\vec{w}))_{j,k>0} = \tilde f(e_i(\vec{X}), e_j(\vec{Y}), e_k(\vec{w}))_{i,j,k>0}$$

for any series g. This follows from Lemma 3.6 by taking $\vec{x}=x, \vec{y}=\vec{Y}$.

As a result of the previous paragraph, the coefficients of

$$\sum_{i=1}^{M}\sum_{j,k} G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot(l+\lambda_1+\lambda_2)^{2|\mu|}\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)}}$$
$$=\sum_{s=0}^{2|\mu|}\sum_{i=1}^{M}\sum_{j,k}\binom{2|\mu|}{s}G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot l^{2|\mu|-s}(\lambda_1+\lambda_2)^s\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)}}$$

are power series in λ_1', λ_2' for any integer $l\geqslant 0$. When $\mu\neq(0)$, the matrix formed by the vectors

$$\left(\binom{2|\mu|}{0}l^{2|\mu|}, \binom{2|\mu|}{1}l^{2|\mu|-1}, \binom{2|\mu|}{2}l^{2|\mu|-2}, \dots, \binom{2|\mu|}{2|\mu|}l^0\right)$$

for $l=1,2,3,\dots$ has maximal rank. We may take a linear combination of the above expression evaluated at different integer values of l, and get that

$$\sum_{i=1}^{M}\sum_{j,k\in\mathbb{Z}} G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot(\lambda_1+\lambda_2)^s\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)}}$$

is a power series in λ_1', λ_2' for each $s=0,1,\dots,2|\mu|$.

Taking $s=2$, we get that the following series is a power series in λ_1', λ_2'

$$\sum_{i=1}^{M}\sum_{j,k} G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot(\lambda_1+\lambda_2)^2\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)}}$$
$$=\sum_{j,k\ \mathrm{odd}} G_{\mu,\xi,j,k}(q,z)\cdot 8(\lambda_1')^{j+1}(\lambda_2')^{k+1}.$$

This means the coefficients $G_{\mu,\xi,j,k}(q,z)$ vanish for terms $(\lambda_1')^{j+1}(\lambda_2')^{k+1}$ whenever the degree $j+1$ or $k+1$ is negative, which happens exactly when $\min\{j,k\}\leqslant -2$. Hence, we conclude $G_{\mu,\xi,j,k}=0$ whenever j,k are odd, $\min\{j,k\}\leqslant -2$, and $\mu\neq(0)$.

If one of j,k is odd and the other is even, continuing to assume $\mu\neq(0)$, we take $s=1$ and get

$$\sum_{i=1}^{M} G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot(\lambda_1+\lambda_2)\Big|_{\vec{\lambda}=\vec{\lambda}^{(i)}}$$
$$=\begin{cases} G_{\mu,\xi,j,k}(q,z)\cdot 8(\lambda_1')^{j+1}(\lambda_2')^{k}, \text{if } j \text{ odd}, k \text{ even},\\ G_{\mu,\xi,j,k}(q,z)\cdot 8(\lambda_1')^{j}(\lambda_2')^{k+1}, \text{if } k \text{ odd}, j \text{ even}.\end{cases}$$

Although these are not polynomials when $\min\{j,k\}\leqslant -2$, we see there might be some linear dependence because we may only conclude that

$$G_{\mu,\xi,j,k} = -G_{\mu,\xi,j+1,k-1}$$

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for j odd and k even. To solve this issue, we further apply the argument to the $s=3$ case and obtain the following dependencies

$$G_{\mu,\xi,j,k} = -G_{\mu,\xi,j+3,k-3}$$

for all j odd, k even and $\min\{j,k\} \leqslant -4$. Consequently, $G_{\mu,\xi,j,k} = G_{\mu,\xi,j+2,k-2}$ for any j even and k odd with $\min\{j,k\} \leqslant -3$. Combining these relations we see that, for $\min\{j,k\} \leqslant -2$, there exist some constants $C^{\pm}_{\mu,\xi,a,b,l}$, labeled by the partitions μ,ξ, integers a,b,l and a sign $\pm$, such that

$$\begin{aligned} G_{\mu,\xi,j,k}(q,z) &= \sum_{a,b}((-1)^j-(-1)^k)C^{\pm}_{\mu,\xi,a,b,j+k}q^az^b \\ &= \begin{cases} \sum_{a,b} 2C^{+}_{\mu,\xi,a,b,j+k}q^az^b & \text{if } j \text{ even}, \ k \text{ odd}, \ j \geqslant 0, \\ \sum_{a,b} -2C^{+}_{\mu,\xi,a,b,j+k}q^az^b & \text{if } j \text{ odd}, \ k \text{ even}, \ j > 0, \\ \sum_{a,b} 2C^{-}_{\mu,\xi,a,b,j+k}q^az^b & \text{if } j \text{ even}, \ k \text{ odd}, \ j < 0, \\ \sum_{a,b} -2C^{-}_{\mu,\xi,a,b,j+k}q^az^b & \text{if } j \text{ odd}, \ k \text{ even}, \ j < 0. \end{cases} \end{aligned}$$

The reason for the superscript $\pm$ is due to cases such as $j=-1, k=0$, where we would have $\min\{j,k\} > -2$, so the dependence does not necessarily hold. Because of this gap, we cannot always relate the coefficient when $j \geqslant 0$ to $j<0$, resulting in separated cases. By the paragraphs preceding this theorem, for a fixed a, the degrees j,k on λ_1', λ_2' of the $[q^a]$ coefficient are bounded below. However the above indicates that the constants $C^{\pm}_{\mu,\xi,a,b,l}$ only depend on the value $l = j+k$, and we can make j or k arbitrarily small. Hence $C^{\pm}_{\mu,\xi,a,b,l} = 0$ whenever $\mu \neq (0)$.

With all the vanishings of $G_{\mu,\xi,j,k}$, we write

$$\begin{aligned} \log \mathcal{N}_S(E,V;q,z) = &\sum_{\substack{\mu,\xi \text{ partitions} \\ j,k \geqslant -1}} G_{\mu,\xi,j,k}(q,z)\cdot \lambda_1^j\lambda_2^k \cdot c_\mu(V) \\ &+ \sum_{\substack{\xi \text{ partition} \\ \min\{j,k\}\leqslant -2}} G_{(0),\xi,j,k}(q,z)\cdot \lambda_1^j\lambda_2^k. \end{aligned}$$

To deal with the terms $G_{(0),\xi,j,k}(q,z)$ for $\min\{j,k\} \leqslant -2$, we apply Lemma 4.1 and find

$$D_z G_{(0),\xi,j,k}(q,z) = rG_{(1),\xi,j,k}(q,z) = 0,$$

so $G_{(0),j,k}$ is constant with respect to the variable z. Let us attempt to extract the $[q^n\lambda_1^j\lambda_2^k c_{(0)}(V)c_\xi(E)]$ coefficient of the Chern series from $G_{(0),\xi,j,k}$ using the Chern limit of Lemma 3.3. This results in a limit $\varepsilon \to 0$ of the term $\varepsilon^{n(N-r)}\varepsilon^{|\xi|}\varepsilon^{j+k}$, which does not make sense when the rank r is sufficiently large, so we must have $G_{(0),\xi,j,k} = 0$ for such r. To generalize this to arbitrary ranks, we apply [GM22, Lemma 3.3] to $\mathcal{N}_S$, which says that the coefficients of $\mathcal{N}_S$ are polynomials in r when $r \geqslant 0$. Now we can write

$$\log \mathcal{N}_S(E,V;q,z) = \sum_{\substack{\mu,\xi \text{ partitions} \\ j,k \geqslant -1}} G_{\mu,\xi,j,k}(q,z)\cdot \lambda_1^j\lambda_2^k \cdot c_\mu(V)c_\xi(E).$$

As noted in [OP22, (31)], the obstruction on $\mathrm{Hilb}^n(S)$ at a fixed point $[Z_\mu]$ is $(K_S^{[n]})^\vee|_{Z_\mu}$. From (10), we see a copy of $K_S^\vee = t_1t_2$ is in $K_S^{[n]}|_{Z_\mu}$. By (9), the obstruction bundle on $\mathrm{Quot}_S(E,n)$ at

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any fixed point has at least one copy of $K_S^\vee$ as a direct summand. For a line bundle L, we have

$$\mathrm{ch}(\Lambda_{-1}L^\vee) = 1 - e^{-c_1(L)} = e(L)\cdot(1+\dots)$$

where $\dots$ are some omitted terms in $H^{\geqslant 0}_{\mathsf{T}}(\mathrm{pt})$. Therefore, $1/\mathrm{ch}(\Lambda_{-1}(T_Z^{\mathrm{vir}})^\vee)$ has a factor of $e(K_S^\vee) = c_1(S) = \lambda_1+\lambda_2$ in its numerator. We also note that this factor does not appear in the denominator, because if we pass to the subtorus $\{(t_1,t_2) : t_1t_2 = 1\}$, the Zariski tangent space has no T-fixed parts: by (7), the fixed part can only come from the direct summands with $i=j$, which correspond to the Hilbert scheme case, but, by [Arb21, (1.6)], these summands have no fixed parts because $a(\square), l(\square) \geqslant 0$ for any box $\square$. Therefore, we may extract this factor of $c_1(S)$ and obtain

$$\log \mathcal{N}_S(E,V;q,z) = \sum_{\substack{\mu,\xi \text{ partitions}\\ j,k\geqslant -1}} H_{\mu,\xi,j,k}(q,z)\cdot \lambda_1^j\lambda_2^k\cdot c_\mu(V)c_\xi(E)c_1(S).$$

for some series $H_{\mu,\xi,j,k}\in\mathbb{Q}[\![q,z]\!]$. Furthermore, since j,k are now bounded below by -1, multiplying by $\lambda_1\lambda_2$ would give us a power series expansion in λ_1,λ_2, allowing us to use the symmetry in λ_1,λ_2 and write

$$\log \mathcal{N}_S(E,V;q,z) = \sum_{\mu,\nu,\xi \text{ partitions}} H_{\mu,\nu,\xi}(q,z)\cdot\int_S c_\mu(V)c_\nu(S)c_\xi(E)c_1(S) \tag{17}$$

for some series $H_{\mu,\nu,\xi}\in\mathbb{Q}[\![q,z]\!]$.

Finally, taking Chern and Verlinde limits of $H_{\mu,\nu,\xi}$ as in Lemma 3.3 then exponentiating gives us series $C_{\mu,\nu,\xi}, B_{\mu,\nu,\xi}$ such that

$$\mathcal{C}_S(E,V;q) = \prod_{\mu,\nu,\xi \text{ partitions}} C_{\mu,\nu,\xi}(q)^{\int_S c_\mu(V)c_\nu(S)c_\xi(E)c_1(S)},$$
$$\mathcal{V}_S(E,V;q) = \prod_{\mu,\nu,\xi \text{ partitions}} B_{\mu,\nu,\xi}(q)^{\int_S c_\mu(V)c_\nu(S)c_\xi(E)c_1(S)}.$$

and the fact that $\mathcal{S}_S(E,V;q) = \mathcal{C}_S(E,-V;q)$ implies that there exists series $A_{\mu,\nu,\xi}$ such that

$$\mathcal{S}_S(E,V;q) = \prod_{\mu,\nu,\xi \text{ partitions}} A_{\mu,\nu,\xi}(q)^{\int_S c_\mu(V)c_\nu(S)c_\xi(E)c_1(S)}.$$

To generalize this to arbitrary K-theory classes $\alpha = [V'] - [V''] \in K_{\mathsf{T}}(S)$ for equivariant bundles V', V'' of rank m, l, respectively, we apply [GM22, Lemma 3.3] once more; it states that the invariants for α are obtained by substituting

$$r \rightsquigarrow m-l, \quad p_n(v_1,v_2,\dots,v_r) \rightsquigarrow p_n(v_1',v_2',\dots,v_m') - p_n(v_1'',v_2'',\dots,v_l''),$$

where the p_n are the power-sum symmetric polynomials. Hence, the above universal series expressions hold for all $\alpha\in K_{\mathsf{T}}(S)$. $\square$

Lemma 3.6. *Let $F(\vec{x},\vec{y})$ be a polynomial symmetric in $\vec{x} = (x_1,\dots,x_n)$ and symmetric in $\vec{y} = (y_1,\dots,y_m)$. Then F can be written as a polynomial expression of symmetric functions in $\{y_1,\dots,y_m\}$ and symmetric functions in $\{x_i+y_j\}_{i=1,\dots,n}^{j=1,\dots,m}$.*

Proof. Note that this lemma is additive and multiplicative on F, meaning that, if the result holds for F_1 and F_2, then it holds for F_1+F_2 and $F_1\cdot F_2$. By the additive property, we assume that F is homogeneous of degree d. We proceed by induction on d, where the base case of $d=0$ is trivial.

We expand F in the elementary symmetric polynomial basis with respect to the variables $y_1, \dots, y_m$:

$$F(\vec{x}, \vec{y}) = \sum_{\mu \text{ partition}} f_\mu(\vec{x}) e_\mu(\vec{y}).$$

Note that each $f_\mu(\vec{x})$ is symmetric in $\vec{x}$ and trivially symmetric in $\vec{y}$, and has degree strictly less than F if $\mu \neq (0)$. By the induction hypothesis, we can assume that $f_\mu(\vec{x})$ satisfies the lemma when $\mu \neq (0)$. Then, by the additive and multiplicative properties, it remains to prove $f_{(0)}(\vec{x})$ satisfies the lemma.

Since $f_{(0)}(\vec{x})$ is a symmetric polynomial, which can be written as a sum of products of the elementary symmetric polynomials $e_k(\vec{x})$, we may apply the additive and multiplicative properties again and conclude that the result follows once we prove it for $e_k(\vec{x})$. Since we are performing an induction on the degree d of F, we may assume that the result holds for $k < d$ and proceed to prove it for $k = d$.

We have

$$e_d(\{x_i + y_j\}_{i=1,\dots,n}^{j=1,\dots,m}) = K \cdot e_d(\vec{x}) + G(\vec{x}, \vec{y})$$

for some constant $K \in \mathbb{Z}$, and we have that G is a polynomial symmetric in $\vec{x}$ and in $\vec{y}$ and that every monomial term in G contains some y_j. Moreover, G is homogeneous of degree d, and we can repeat the previous argument and write

$$G(\vec{x}, \vec{y}) = \sum_{\mu \neq (0)} g_\mu(\vec{x}) e_\mu(\vec{y}).$$

Now each $g_\mu(\vec{x})$ has degree strictly less than d, and induction shows the result holds for each $g_\mu(\vec{x})$, and consequently for $G(\vec{x}, \vec{y})$ and $e_d(\vec{x})$. □

By the non-equivariant Segre–Verlinde correspondence [Boj21a, Theorem 1.7] and the relations between the non-equivariant series and equivariant series illustrated in § 3.2, we have a weak Segre–Verlinde correspondence as the following corollary. The same argument as in the following proof also gives us a weak symmetry in the form of Corollary 1.14.

Corollary 3.7. *In the setting of Theorem 3.5, we have the following correspondence*

$$A_{\mu,\nu,\xi}(q) = B_{\mu,\nu,\xi}((-1)^N q)$$

whenever one of μ, ν, ξ is (1) and the other two are (0). In particular, the degree 0 part satisfies

$$\mathcal{S}_{S,0}(E, \alpha; q) - \mathcal{V}_{S,0}(E, \alpha; (-1)^N q) = \sum_{n=2}^{\infty} \frac{f_n}{(\lambda_1 \lambda_2)^{n-2}} \cdot \left(\int_S c_1(S) \right)^2 \cdot q^n$$

for some terms $f_n \in H^{2n-2}_{\mathsf{T}}(\mathrm{pt})$ dependent on α through its rank and Chern classes.

Proof. By the last paragraph of § 3.2, the universal series in Theorem 3.5, when passed to a toric projective surface, must give the Segre–Verlinde correspondence of [Boj21a, Theorem 1.7] in degree 0. Since the degree 0 terms occur only when one of μ, ν, ξ is (1) and the other two are (0), we have

$$A_{\mu,\nu,\xi}(q) = B_{\mu,\nu,\xi}((-1)^N q)$$

in those cases.

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Note that when we take exp of (17), the total degree 0 part might come from the product of a negative-degree term and a positive-degree term, but since each term in the integrand is accompanied by a copy $c_1(S)$, we know this difference must be a multiple of $c_1(S)^2$. We also see that the $[q^n]$ coefficients are sums of products of at most n such integrals, giving a denominator of $\lambda_1^n\lambda_2^n$, so we are done.

For illustration, we shall extract this difference, and express it explicitly. This is just a standard computation. For a partition $\mu=(\mu_1,\mu_2,\ldots,\mu_L)$ of size n with length L, and a sequence of positive integers $\kappa=(k_1,k_2,\ldots,k_L)$, write $\kappa|\mu$ if each $k_i|\mu_i$. We also associate a set of tuples of partitions to κ by

$$M_\kappa := \left\{ \left((\mu^{(i)})_{i=1}^L, (\nu^{(i)})_{i=1}^L, (\xi^{(i)})_{i=1}^L\right) \;\middle|\; \begin{array}{l} \mu^{(i)}, \nu^{(i)}, \xi^{(i)} \text{ are partitions for each } i, \text{ s.t.} \\ \quad \sum_{i=1}^L k_i(|\mu^{(i)}|+|\nu^{(i)}|+|\xi^{(i)}|-1)=0 \text{ and} \\ \quad \mu_i=\nu_i=\xi_i=0 \text{ for some } i. \end{array} \right\}.$$

For each $n>0$, we would like to find the degree 0 part of the $[q^n]$ coefficient of exp of (17). By expanding the exponential using definition, we observe that these terms come from products of integrals labeled by $\mu^{(i)},\nu^{(i)},\xi^{(i)}$ in M_κ for some tuples $\kappa|\pi$ for some partition π of size n. A more precise description is given by the following equation. Suppose $\log(A_{\mu,\nu,\xi}(q))=\sum_{i=1}^\infty a_{\mu,\nu,\xi,i}q^i$ and $\log(B_{\mu,\nu,\xi}(q))=\sum_{i=1}^\infty b_{\mu,\nu,\xi,i}q^i$. Then we have

$$\begin{aligned}
&[q^n](\mathcal{S}_{S,0}(E,\alpha;q)-\mathcal{V}_{S,0}(E,\alpha;(-1)^N q)) \\
&= \sum_{\substack{|\pi|=n\\ \ell(\pi)>1}} \sum_{\kappa|\pi} \prod_{(\vec\mu,\vec\nu,\vec\xi)\in M_\kappa} \prod_{i=1}^{\ell(\pi)} \frac{1}{k_i!}\left(a_{\mu^{(i)},\nu^{(i)},\xi^{(i)},\pi_i/k_i}\int_S c_{\mu^{(i)}}(\alpha)c_{\nu^{(i)}}(S)c_{\xi^{(i)}}(E)c_1(S)\right)^{k_i} \\
&\quad -(-1)^{Nn}\sum_{\substack{|\pi|=n\\ \ell(\pi)>1}} \sum_{\kappa|\pi} \prod_{(\vec\mu,\vec\nu,\vec\xi)\in M_\kappa} \prod_{i=1}^{\ell(\pi)} \frac{1}{k_i!}\left(b_{\mu^{(i)},\nu^{(i)},\xi^{(i)},\pi_i/k_i}\int_S c_{\mu^{(i)}}(\alpha)c_{\nu^{(i)}}(S)c_{\xi^{(i)}}(E)c_1(S)\right)^{k_i} \\
&= \sum_{\substack{|\pi|=n\\ \ell(\pi)>1}} \sum_{\kappa|\pi} \prod_{(\vec\mu,\vec\nu,\vec\xi)\in M_\kappa} \left(\prod_{i=1}^{\ell(\pi)} a^{k_i|M_k|}_{\mu^{(i)},\nu^{(i)},\xi^{(i)},\pi_i/k_i} - (-1)^{Nn}\prod_{i=1}^{\ell(\pi)} b^{k_i|M_k|}_{\mu^{(i)},\nu^{(i)},\xi^{(i)},\pi_i/k_i}\right) \\
&\quad \cdot \prod_{i=1}^{\ell(\pi)} \frac{c_1(S)^{k_i}}{k_i!\lambda_1^{k_i}\lambda_2^{k_i}}\left(c_{\mu^{(i)}}(\alpha)c_{\nu^{(i)}}(S)c_{\xi^{(i)}}(E)\right)^{k_i}.
\end{aligned}$$

In the summation we have $\ell(\pi)>1$ because M_κ is empty for any $\kappa|\pi$ if $\pi=(n)$, by definition. Therefore $2\leqslant\sum_{i=1}^{\ell(\pi)}k_i\leqslant n$. By multiplying the denominator and numerator of the right-hand side by some appropriate power of $\lambda_1\lambda_2$, we can express it as a rational function in λ_1,λ_2, with denominator $\lambda_1^n\lambda_2^n$ and numerator a multiple of $c_1(S)^2$. Setting this multiple as $f_n\in\mathbb{C}[\lambda_1,\lambda_2]$ gives the desired expression. □

Example 3.8. The universal series for $\mathcal{N}_S$ are known explicitly in the compact case [Boj21a, Theorem 1.2]. For a smooth projective surface S and α of rank r, we apply [AJL+21, Theorem 17, (16), (17)] for $f(x)=1-ze^x$, $g(x)=\frac{x}{1-e^{-x}}$ and get

$$\mathcal{N}_S(\mathcal{O}_S,\alpha;q,z)=\left[\left(\frac{1-zQ}{1-z}\right)^r\left(\frac{-rzQ(1-Q)}{1-zQ}+1\right)\right]^{c_1(S)^2}\left[\frac{1-zQ}{1-z}\right]^{c_1(S)\cdot c_1(\alpha)}$$

via the substitution

$$q = \frac{1-Q^{-1}}{(1-zQ)^r}.$$

As mentioned in the introduction, when $N=1$, we shall set $y_1=1$, and omit the subscript ξ for $H_{\mu,\nu,\xi}$. The formula above allows us to compute $H_{(1),(0)}(q,z)$ and $H_{(0),(1)}(q,z)$. For a smooth projective surface S and α of rank 0, we have

$$\mathcal{N}_S(\mathcal{O}_S,\alpha,q,z) = \left(\frac{1-q-z}{(1-q)(1-z)}\right)^{c_1(S)\cdot c_1(\alpha)}.$$

The exponent is interpreted as an intersection product, which in the toric case corresponds to the equivariant push-forward

$$\int_S c_1(\alpha)c_1(S).$$

Therefore, for rank 0, we have the following series from expansion (17)

$$H_{(1),(0)}(q,z) = \log\frac{1-q-z}{(1-q)(1-z)}, \quad H_{(0),(1)}(q,z) = 0.$$

We take the Chern limit of Lemma 3.3 by substituting $q \rightsquigarrow -q\varepsilon$, $z \rightsquigarrow (1+\varepsilon)^{-1}$ and get

$$C_{(1),(0)}(q) = 1+q.$$

Replacing α by $-\alpha$ in the Chern series to get the Segre series for α, we see

$$A_{(1),(0)}(q) = C_{(1),(0)}(q)^{-1} = \frac{1}{1+q}.$$

On the other hand, the Verlinde limit yields

$$B_{(1),(0)}(q) = \frac{1}{1-q}.$$

The Segre–Verlinde correspondence of Corollary 3.7 is indeed satisfied.

Example 3.9. Let $S=\mathbb{C}^2$, $n=N=2$, $E=\mathcal{O}_S\langle y_1\rangle \oplus \mathcal{O}_S\langle y_2\rangle$ and $L=\mathcal{O}_S\langle v\rangle$. The T_1-fixed locus of $\mathrm{Quot}_S(E,n)$ is the disjoint union of

$$\mathrm{Hilb}^0(S)\times\mathrm{Hilb}^2(S), \quad \mathrm{Hilb}^1(S)\times\mathrm{Hilb}^1(S), \quad \mathrm{Hilb}^2(S)\times\mathrm{Hilb}^0(S).$$

Denote by Z_μ the point in $\mathrm{Hilb}^i(S)^{\mathsf{T}_0}$ corresponding to a partition μ, then the T-fixed points of $\mathrm{Quot}_S(\mathbb{C}^2,2)$ are

$$(Z_\phi, Z_{(2)}), \quad (Z_\phi, Z_{(1,1)}), \quad (Z_{(1)}, Z_{(1)}), \quad (Z_{(2)}, Z_\phi), \quad (Z_{(1,1)}, Z_\phi).$$

Therefore, by (7), the virtual tangent bundles at these five points are, respectively,

$$\begin{gathered}
(t_1^2+t_2-t_1^2t_2+y_1^{-1}y_2)(1+t_1^{-1}),\\
(t_2^2+t_1-t_1t_2^2+y_1^{-1}y_2)(1+t_2^{-1}),\\
(t_1+t_2-t_1t_2)(1+y_1^{-1}y_2)(1+y_1y_2^{-1}),\\
(t_1^2+t_2-t_1^2t_2+y_1y_2^{-1})(1+t_1^{-1}),\\
(t_2^2+t_1-t_1t_2^2+y_1y_2^{-1})(1+t_2^{-1}).
\end{gathered}$$

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The equivariant Chern roots of $\alpha^{[n]}$ at these points are, respectively,

$$\begin{aligned}
&\{m_2+w, m_2-\lambda_1+w\},\\
&\{m_2+w, m_2-\lambda_2+w\},\\
&\{m_1+w, m_2+w\},\\
&\{m_1+w, m_1-\lambda_1+w\},\\
&\{m_1+w, m_1-\lambda_2+w\}.
\end{aligned}$$

The contribution to the Segre numbers at each of these fixed points are

$$\frac{(2\lambda_1+\lambda_1)(\lambda_1+\lambda_2)}{2(m_1-m_2+\lambda_1)(m_1-m_2)(m_2-\lambda_1+w_1+1)(m_2+w_1+1)(\lambda_2-\lambda_1)\lambda_1^2\lambda_2},$$

$$\frac{(\lambda_1+2\lambda_2)(\lambda_1+\lambda_2)}{2(m_1-m_2+\lambda_2)(m_1-m_2)(m_2-\lambda_2+w_1+1)(m_2+w_1+1)(\lambda_1-\lambda_2)\lambda_1\lambda_2^2},$$

$$\frac{(\lambda_1+\lambda_2+m_1-m_2)(\lambda_1+\lambda_2-m_1+m_2)(\lambda_1+\lambda_2)^2}{(m_1-m_2+\lambda_1)(m_1-m_2-\lambda_1)(m_1-m_2+\lambda_2)(m_1-m_2-\lambda_2)(m_1+w1+1)(m_2+w1+1)\lambda_1^2\lambda_2^2},$$

$$\frac{(2\lambda_1+\lambda_1)(\lambda_1+\lambda_2)}{2(m_1-m_2-\lambda_1)(m_1-m_2)(m_1-\lambda_1+w_1+1)(m_1+w_1+1)(\lambda_2-\lambda_1)\lambda_1^2\lambda_2},$$

$$\frac{(\lambda_1+2\lambda_2)(\lambda_1+\lambda_2)}{2(m_1-m_2-\lambda_2)(m_1-m_2)(m_1-\lambda_2+w_1+1)(m_1+w_1+1)(\lambda_1-\lambda_2)\lambda_1\lambda_2^2}.$$

Summing them up, we have

$$[q^2]S^2(\alpha;q)=\frac{\begin{pmatrix} m_1m_2\lambda_1-m_1\lambda_1^2-m_2\lambda_1^2+\lambda_1^3+m_1m_2\lambda_2-3m_1\lambda_1\lambda_2-3m_2\lambda_1\lambda_2+3\lambda_1^2\lambda_2-m_1\lambda_2^2-m_2\lambda_2^2\\ +3\lambda_1\lambda_2^2+\lambda_2^3+m_1\lambda_1w+m_2\lambda_1w-2\lambda_1^2w+m_1\lambda_2w+m_2\lambda_2w-6\lambda_1\lambda_2w-2\lambda_2^2w\\ +\lambda_1w^2+\lambda_2w^2+m_1\lambda_1+m_2\lambda_1-2\lambda_1^2+m_1\lambda_2+m_2\lambda_2-6\lambda_1\lambda_2-2\lambda_2^2+2\lambda_1w+2\lambda_2w+\lambda_1+\lambda_2\end{pmatrix}(\lambda_1+\lambda_2)}{2(m_1-\lambda_1+w+1)(m_1-\lambda_2+w+1)(m_1+w+1)(m_2-\lambda_1+w+1)(m_2-\lambda_2+w+1)(m_2+w+1)\lambda_1^2\lambda_2^2}.$$

A similar computation yields another complicated expression for the Verlinde number. We are interested in the total degree 0 part of their difference in the variables $\vec{\lambda}, \vec{m}, \vec{w}$. This computes to

$$\begin{aligned}
[q^2](S_{S,0}(E,L;q)-V_{S,0}(E,L;q)) &= -\frac{(3m_1m_2-\lambda_1\lambda_2+3m_1w+3m_2w+3w^2)(\lambda_1+\lambda_2)^2}{3\lambda_1^2\lambda_2^2}\\
&=\left(\frac{1}{3}c_2(S)-c_2(E)-c_1(E)c_1(V)-c_1(V)^2\right)\left(\int_S c_1(S)\right)^2.
\end{aligned}$$

Note that even though the expressions for the Segre and Verlinde numbers are complicated, their difference in degree 0 simplifies tremendously and satisfies Corollary 3.7.

3.4 Reduced virtual classes and invariants

As mentioned previously, the obstruction for $\mathrm{Quot}_S(E,n)$ at Z contains at least one copy of $K_S^\vee$. For K-trivial surfaces, this causes the Euler class of T^{vir} to vanish. Therefore, the virtual Verlinde and Segre numbers both vanish. One can instead study the 'reduced' versions of these invariants. By [Lim20, Proposition 9], when S is a K-trivial surface, $n>0$, and E a torsion-free sheaf, there is a *reduced obstruction theory* that is perfect in the sense of Definition 3.1. The

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reduced (virtual) tangent bundle in this case is given by adding a trivial summand to the usual virtual tangent bundle:

$$T^{\mathrm{red}} = T^{\mathrm{vir}} + \mathcal{O}_{\mathrm{Quot}_S(E,n)}.$$

In this section, we study the equivariant analogue where $S = \mathbb{C}^2$ with the natural action of the 1-dimensional torus

$$\mathsf{T}_0 = \mathbb{C}^* = \{(t_1, t_2) : t_1 t_2 = 1\}.$$

We write

$$H^*_{\mathsf{T}_0}(\mathrm{pt}) = \mathbb{C}[\lambda_1, \lambda_2]/(\lambda_1 + \lambda_2) = \mathbb{C}[\lambda],$$

$$K_{\mathsf{T}_0}(\mathrm{pt}) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]/(t_1 t_2 - 1) = \mathbb{Z}[t^{\pm 1}].$$

Using the argument of [CK17, Lemma 3.1], we see that the $\mathsf{T} = (\mathsf{T}_0 \times \mathsf{T}_1 \times \mathsf{T}_2)$-fixed locus of $\mathrm{Quot}_S(E,n)$ stays unchanged, and the Zariski tangent space at each of the fixed points has no fixed parts, by (9). The *equivariant reduced Segre and Verlinde series* $\mathcal{S}_S^{\mathrm{red}}$ and $\mathcal{V}_S^{\mathrm{red}}$ are defined in the same way as the virtual ones by replacing T^{vir} with T^{red}. Here we omit the subscript since we are only interested in the $S = \mathbb{C}^2$ case.

$$\mathcal{S}^{\mathrm{red}}(E, \alpha; q) := \sum_{n>0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_S(E,n)^{\mathsf{T}}} \frac{c(\alpha^{[n]}|_Z)}{e(T_Z^{\mathrm{red}})},$$

$$\mathcal{V}^{\mathrm{red}}(E, \alpha; q) := \sum_{n>0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_S(E,n)^{\mathsf{T}}} \frac{\mathrm{ch}(\det(\alpha^{[n]}|_Z))}{\mathrm{ch}(\Lambda_{-1}(T_Z^{\mathrm{red}})^\vee)}.$$

Note that we do not include the $n = 0$ term because condition 2 of [Lim20, Proposition 9] is only satisfied when $n > 0$.

The same strategy as used in the previous section can be applied to study these invariants. For $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i \rangle$ and $V = \oplus_{i=1}^r \mathcal{O}_S\langle v_i \rangle$, we define

$$\mathcal{N}^{\mathrm{red}}(E, V; q, z) := \sum_{\mu \neq (0)} q^{|\mu|} \prod_{\square \in \mu} \frac{\prod_{j=1}^N \prod_{i=1}^r (1 - t^{-c(\square) + r(\square)} v_i y_j z)}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{red}}|_Z)^\vee)}.$$

Again note that the $[q^0]$ coefficient is 0. We can think of the reduced obstruction as removing a copy of $K_S^\vee$ from the usual obstruction in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$, then passing to the quotient ring $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]/(t_1 t_2 - 1)$. This gives us the following corollary.

Corollary 3.10. *For $n > 0$, the $[q^n]$ coefficient of $\mathcal{N}^{\mathrm{red}}$ can be obtained from the non-reduced version by taking the following limit:*

$$\begin{aligned}[q^n]\mathcal{N}^{\mathrm{red}}(E, V; q, z) &= [q^n] \frac{\mathcal{N}_S(E, V; q, z)}{1 - e^{-c_1(K_S^\vee)}}\bigg|_{-\lambda_2 \to \lambda_1 = \lambda} \\ &= [q^n] \lim_{-\lambda_2 \to \lambda_1 = \lambda} \frac{\mathcal{N}_S(E, V; q, z)}{\lambda_1 + \lambda_2}.\end{aligned}$$

Using Remark 1.6, we may expand the universal series expression from Theorem 3.5 and obtain

$$\mathcal{S}_S(E, \alpha; q) = \sum_{i=1}^{\infty} \frac{1}{i!} (\lambda_1 + \lambda_2)^i \left(\sum_{\mu,\nu,\xi} \log A_{\mu,\nu,\xi}(q) \cdot \int_S c_\mu(\alpha) c_\nu(S) c_\xi(E) \right)^i.$$

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Using the above corollary to extract the reduced coefficients, we see the terms with $i>1$ all vanish and

$$\mathcal{S}^{\mathrm{red}}(E,\alpha;q)=\sum_{\mu,\nu,\xi}\log A_{\mu,\nu,\xi}(q)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E).$$

The Chern and Verlinde cases are similar, and thus we have the following result.

Theorem 3.11. *When $S=\mathbb{C}^2$, the equivariant reduced Segre, Verlinde and Chern series for $E=\oplus_{i=1}^N\mathcal{O}_S\langle y_i\rangle$ and $\alpha\in K_{\mathsf{T}}(S)$ are*

$$\mathcal{S}^{\mathrm{red}}(E,\alpha;q)=\sum_{\mu,\nu,\xi}\log\left(A_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E),$$
$$\mathcal{V}^{\mathrm{red}}(E,\alpha;q)=\sum_{\mu,\nu,\xi}\log\left(B_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E),$$
$$\mathcal{C}^{\mathrm{red}}(E,\alpha;q)=\sum_{\mu,\nu,\xi}\log\left(C_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)$$

where $A_{\mu,\nu,\xi}$, $B_{\mu,\nu,\xi}$ and $C_{\mu,\nu,\xi}$ are the same series as in Theorem 3.5.

The integrals in the above theorem labeled by μ,ν,ξ have degree $|\mu|+|\nu|+|\xi|-2$, which is one degree lower than the integrals in the non-reduced expressions. Therefore, we have a Segre–Verlinde correspondence in degree -1 for the reduced setting. However, the results for degree -1 have no compact analogues since they automatically vanish in the compact setting. In § 4.2, we compute some of the universal series explicitly, giving us some Segre–Verlinde relations in non-negative degrees for the reduced case.

4. Explicit computations of universal series

4.1 Virtual Segre number in rank $r=-1$

In this section, we first consider the non-virtual equivariant Chern series $I^{\mathcal{C}}(\alpha;q)$ at rank $r=\mathrm{rank}(\alpha)=2$ for Hilbert schemes of $S=\mathbb{C}^2$, given by (1). We will assume that $\alpha=[V]$ is the class of a rank 2 vector bundle V. After the proof of Lemma 4.1, a closed expression for $I^{\mathcal{C}}(\alpha;q)$ will be used to compute the equivariant virtual Segre series $\mathcal{S}_S(\mathcal{O}_S,\beta;q)$ at rank $r=\mathrm{rank}(\beta)=-1$. This latter series is equivalent to the virtual Chern series $\mathcal{C}_S(\mathcal{O}_S,\gamma;q)$ at rank $r=\mathrm{rank}(\gamma)=1$, where we can take $\gamma=[L]$ to be the class of some line bundle L.

Consider the Chern series $I^{\mathcal{C}}(V;q)$ for $r=\mathrm{rank}(V)=2$, which has universal series structure

$$I^{\mathcal{C}}(V;q)=\prod_{\mu,\nu\text{ partitions}}G_{\mu,\nu}(q)^{\int_S c_\mu(V)c_\nu(S)},\tag{18}$$

for some series $G_{\mu,\nu}$, as a result of [GM22, (2.6)] and Proposition 2.7. On the other hand, expanding (1) using (4) gives us

$$I^{\mathcal{C}}(V;q)=\sum_{n=0}^{\infty}q^n\sum_{Z\in\left(\mathrm{Hilb}^n(S)\right)^{\mathsf{T}_0}}\frac{c(V^{[n]}|_Z)}{e(T_Z)}.$$

If Z corresponds to a partition μ, the numerator $c(V^{[n]}|_Z)$ in $H^*_{\mathsf{T}}(\mathrm{pt})$ lies in degrees 0 to $2|\mu|=2n$, while the denominator $e(T_Z)$ has degree exactly $2|\mu|=2n$. Hence, the total expression $I^{\mathcal{C}}(V;q)$

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ranges over the degrees $-2n$ to 0. From this observation, we conclude that the series $G_{\mu,\nu}$ must vanish for $|\mu|+|\nu|-2>0$. Otherwise, expanding (18) using Remark 1.6 would give us a positive-degree term in $I^{\mathcal{C}}(V;q)$.

For a smooth projective surface S', let $I^{\mathcal{C}}_{S'}(V;q)=\sum_{n=0}^\infty q^n\int_{\mathrm{Hilb}^n(S')}s(V^{[n]})$ denote the ordinary (non-equivariant, non-virtual) Chern series for Hilbert schemes. By [MOP21, Remark 6], if $\operatorname{rank}(V)=2$, then

$$I^{\mathcal{C}}_{S'}(V;q)=(1+q)^{c_2(V)}.$$

By the argument of § 3.2, the universal series for $I^{\mathcal{C}}_{S'}$ coincides with the degree 0 universal series for $I^{\mathcal{C}}$, which are $G_{\mu,\nu}$ for $|\mu|+|\nu|-2=0$. Therefore $G_{(2),(0)}=(1+q)$ and $G_{(0),(2)}=G_{(1),(1)}=G_{(1,1),(0)}=G_{(0),(1,1)}=0$. We have seen before that $G_{\mu,\nu}$ vanishes for degrees $|\mu|+|\nu|-2>0$. Thus in order to compute $I^{\mathcal{C}}$ for $S=\mathbb{C}^2$, it remains to find the universal series $G_{\mu,\nu}$ with $|\mu|+|\nu|-2<0$, i.e. $G_{(1),(0)},G_{(0),(1)},G_{(0),(0)}$.

As previously mentioned, Göttsche and Mellit used an invariant similar to the Nekrasov genus $\mathcal{N}_S(\mathcal{O}_S,V;q,z)$, denoted by $I_{S',V}(q,z)$ in [GM22, (1.1)], for S' a smooth projective toric surface. To match the notation we use, we shall write this as

$$I^{\mathcal{N}}_{S'}(V;q,z)=\sum_{n=0}^{\infty}(-q)^n\chi\left(\mathrm{Hilb}^n(S'),(\Lambda_{-z}V^{[n]})\otimes\det(\mathcal{O}_{S'}^{[n]})^{-1}\right).$$

The equivariant analogue on $S=\mathbb{C}^2$ is given by $\Omega(q;ze^{w_1},\dots,ze^{w_r};e^{\lambda_1},e^{\lambda_2})$ in [GM22, § 3.2], which we denote by $I^{\mathcal{N}}(V;q,z)=I^{\mathcal{N}}(v_1,\dots,v_r;q,z)$ where $v_i=e^{w_i}$. As in Lemma 3.3, the Chern series $I^{\mathcal{C}}$ is recovered from $I^{\mathcal{N}}$ by taking the Chern limit as in the first equation of [GM22, Proposition 3.5]; that is,

$$I^{\mathcal{C}}(V;q)=\lim_{\varepsilon\to 0}I^{\mathcal{N}}(e^{-\varepsilon w_1},\dots,e^{-\varepsilon w_r};-q\varepsilon^{2-r}(1+\varepsilon)^r,(1+\varepsilon)^{-1})|_{\vec{\lambda}\rightsquigarrow\varepsilon\vec{\lambda}}.$$

Since we are interested in the universal series for $I^{\mathcal{C}}$ in negative degrees, let us compute $I^{\mathcal{N}}$ in negative degrees.

The invariant $I^{\mathcal{N}}$ admits a universal series structure by [GM22, § 3.4] given as follows:

$$\log I^{\mathcal{N}}(v_1,\dots,v_r;q,z)=\sum_{j,k\geqslant -1}H_{j,k}(q;ze^{w_1},\dots,ze^{w_r})\lambda_1^j\lambda_2^k.$$

Here, $H_{j,k}(q;z_1,\dots,z_r)$ are series labeled by integers j,k, symmetric in $z_1,\dots,z_r$. The expansion [GM22, (3.12)–(3.13)] states that

$$\begin{aligned}H_{-1,-1}(q;ze^{w_1},\dots,ze^{w_r})&=C_0(q,z)+C_1(q,z)c_1(V)+C_2(q,z)c_2(V)+C_{1,1}c_1(V)^2+\dots,\\ H_{-1,0}(q;ze^{w_1},\dots,ze^{w_r})&=D_0(q,z)+D_1(q,z)c_1(V)+\dots,\end{aligned}$$

where the omitted terms have positive degrees in $H^*_{\mathsf{T}}(\mathrm{pt})$. The series $C_2,C_{1,1},D_1$ can be computed explicitly using [GM22, (3.16)], which expresses them in terms of another collection of series G_0,G_1,G_3 given by [GM22, Theorem 1.1]. We have

$$C_2=\log(1-qz),\quad D_1=C_{1,1}=\frac{1}{2}\log\frac{1-qz^2}{1-qz}.$$

We are now in a situation to apply Lemma 4.1. We take the series $H(z_1,\dots,z_r)$ in the lemma to be $H_{-1,-1}(q;z_1,\dots,z_r)$ and the variables $x_1,\dots,x_r$ to be $w_1,\dots,w_r$. The above formula for $H_{-1,-1}$ shows that $H(q;ze^{w_1},\dots,ze^{w_r})$ satisfies the expansion $\sum_{\mu\text{ partition}}H_\mu(z)\prod_{i=1}^{\ell(\mu)}e_{\mu_i}(w_1,\dots,w_r)$ for $H_{(0)}=C_0,H_{(1)}=C_1,H_{(2)}=C_2,\ H_{(1,1)}=C_{1,1}$ and

$H_\mu = 0$ for any other index μ. Lemma 4.1 states that

$$D_zC_0(q,z) = 2C_1(q;z) \quad D_zC_1(q,z) = C_2(q,z) + 4C_{1,1}(q,z).$$

Similarly, applying this to $H_{-1,0}$ yields $D_zD_0 = 2D_1$. Therefore

$$C_0(q,z) = C_0(q,0) - \mathrm{Li}_3(qz^2) + 2\mathrm{Li}_3(qz), \quad C_1(q,z) = -\mathrm{Li}_2(qz^2) + \mathrm{Li}_2(qz),$$
$$D_0(q,z) = D_0(q,0) - \frac{1}{2}\mathrm{Li}_2(qz^2) + \mathrm{Li}_2(qz).$$

Here $\mathrm{Li}_s(z) = \mathrm{Li}_s(z) = \sum_{k=1}^\infty z^k/k^s$ denote the polylogarithm functions. The terms $C_0(q,0)$ (respectively $D_0(q,0)$) are obtained by extracting the coefficients of $\lambda_1^{-1}\lambda_2^{-1}$ (respectively λ_1^{-1} or λ_2^{-1}) of $\log I^{\mathcal{N}}$ via the expression [GM22, (2.5)]. We get

$$C_0(q,0) = -\mathrm{Li}_3(q), \quad D_0(q,0) = \frac{1}{2}\mathrm{Li}_2(q).$$

Finally, by the Chern limit (19) for rank $r = 2$, we see

$$\log G_{(0),(0)}(q) = \lim_{\varepsilon\to 0} \varepsilon^{-2} \cdot (-\varepsilon) \cdot C_1(-q(1+\varepsilon)^2, (1+\varepsilon)^{-1}) = \log(1+q).$$

Similar limits for $G_{(1),(0)}$ (respectively $G_{(0),(1)}$) using C_1 (respectively D_0) yield $\log G_{(1),(0)} = \log(1+q)$ and $\log G_{(0),(1)} = 0$. Exponentiating, we get $G_{(0),(1)} = 1$ and $G_{(0),(0)}(q) = G_{(1),(0)} = 1+q$. Hence

$$\begin{aligned} I^{\mathcal{C}}(V;q) =& (1+q)^{\int_S c_0(V)}(1+q)^{\int_S c_1(V)}(1+q)^{\int_S c_2(V)} \\ =& (1+q)^{\int_S c(V)} \end{aligned} \tag{19}$$

for any T-equivariant bundle V of rank 2.

Lemma 4.1. *Suppose $r \geqslant 0$ and let $H(z_1,\dots,z_r)$ be a power series in $z_1,\dots z_r$ whose coefficients are series in some other variables $q_1, q_2, \dots$. If H is symmetric in $z_1,\dots,z_r$ with an expansion*

$$H(ze^{x_1},\dots,ze^{x_r}) = \sum_{\mu \text{ partition}} H_\mu(z) \prod_{i=1}^{\ell(\mu)} e_{\mu_i}(x_1,\dots,x_r),$$

then for any $k \geqslant 0$, we have

$$D_z^k H_{(0)}(z) = k! \sum_{\mu:|\mu|=k} \binom{r}{\mu} H_\mu(z)$$

where $\binom{r}{\mu}$ denotes $\prod_{i=1}^{\ell(\mu)}\binom{r}{\mu_i}$, and $D_z = z\frac{\partial}{\partial z}$.

Proof. We begin by claiming that the statement is closed under polynomial expressions; that is, if the equality holds for both $F(z_1,\dots,z_r)$ and $G(z_1,\dots,z_r)$, then it holds for $F\cdot G$ and $aF+G$ for any $a \in \mathbb{Q}[\![q_1,q_2,\dots]\!]$. The additive part is straightforward, and we shall prove the multiplicative part of this claim. Expand

$$F(ze^{x_1},\dots,ze^{x_r}) = \sum_{\mu \text{ partition}} F_\mu(z)e_\mu(x_1,\dots,x_r),$$
$$G(ze^{x_1},\dots,ze^{x_r}) = \sum_{\mu \text{ partition}} G_\mu(z)e_\mu(x_1,\dots,x_r),$$

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then $H = F \cdot G$ can be expanded as

$$H(ze^{x_1}, \ldots, ze^{x_r}) = \sum_{\mu \text{ partition}} H_\mu(z) e_\mu(x_1, \ldots, x_r) = \sum_{\nu+\xi=\mu} F_\nu(z) G_\xi(z) e_\mu(x_1, \ldots, x_r),$$

where by $\nu + \xi$ we mean combining them as sequences to get a partition of size $|\nu| + |\xi|$ with length $\ell(\nu) + \ell(\xi)$. Suppose the statement holds for both F and G, then we have

$$\begin{aligned}
D_z^k H_{(0)}(z) &= D_z^k \left(F_{(0)}(z) G_{(0)}(z)\right) \\
&= \sum_{i=1}^{k} \binom{k}{i} D_z^i F_{(0)} D_z^{k-i} G_{(0)} \\
&= \sum_{i=1}^{k} \binom{k}{i} i!(k-i)! \sum_{|\nu|=i, |\xi|=k-i} \binom{r}{\nu} F_\nu \binom{r}{\xi} G_\xi \\
&= k! \sum_{|\mu|=k} \binom{r}{\mu} H_\mu.
\end{aligned}$$

Since $H(z_1, \ldots, z_r)$ is symmetric, by the above observation, it suffices to prove the statement when H is the power-sum symmetric polynomial $p_n(z_1, \ldots, z_r) = z_1^n + \ldots + z_r^n$. For each $n \geqslant 0$, we expand

$$H(ze^{x_1}, \ldots, ze^{x_r}) = p_n(ze^{x_1}, \ldots, ze^{x_r}) = \sum_{j=1}^{r} z^n e^{nx_j} = z^n \left(r + \sum_{i>0} \frac{n^i}{i!} p_i(x_1, \ldots, x_r) \right).$$

This means $H_{(0)}(z) = rz^n$ and

$$D_z^k H_{(0)}(z) = rn^k z^n.$$

Fixing $r \geqslant 1$, we write

$$p_n(x_1, \ldots, x_r) = \sum_{|\mu|=n} C_\mu e_\mu(x_1, \ldots, x_r)$$

for some constant terms C_μ. Evaluating at $x_1 = \ldots = x_r = 1$, we get

$$r = \sum_{|\mu|=n} \binom{r}{\mu} C_\mu.$$

Hence

$$k! \sum_{|\mu|=k} \binom{r}{\mu} H_\mu(z) = z^n n^k \sum_{|\mu|=k} \binom{r}{\mu} C_\mu = rn^k z^n = D_z^k H_{(0)}(z).$$

A quick calculation for the $k = 0$ and $r = 0$ cases finishes the proof. $\square$

Now we proceed with the virtual case. Recall that, on Hilbert schemes, the obstruction theory at a fixed point $[Z_\mu]$ is given by $(K_S^{[n]})^\vee|_{Z_\mu}$, so

$$\frac{1}{e(T_{Z_\mu}^{\mathrm{vir}})} = \frac{e((K_S^{[n]})^\vee|_{Z_\mu})}{e(T_{Z_\mu})} = (-1)^{|\mu|} \frac{e((K_S^{[n]})|_{Z_\mu})}{e(T_{Z_\mu})}.$$

Let $L = \mathcal{O}_S\langle v_1 \rangle$ be an equivariant line bundle, and $V = L \oplus \mathcal{O}_S\langle v_2 \rangle$. Apply (19) to V and we have

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$$I^{\mathcal{C}}(V;q) = (1+q)^{\int_S c(V)} = (1+q)^{\int_S (1+w_1+w_2+w_1w_2)}.$$

Set $w_2 = c_1(K_S) - 1$ and replace q by $-q$. Then this becomes

$$I^{\mathcal{C}}(V;-q)|_{w_2=c_1(K_S)-1} = (1-q)^{\int_S c(L)c_1(K_S)}.$$

On the other hand, we have by definition

$$\begin{aligned}
&I^{\mathcal{C}}(V;-q)|_{w_2=c_1(K_S)-1}\\
&= \sum_{\mu} (-1)^{|\mu|} q^{|\mu|} \prod_{\Box\in\mu} \frac{(1+w_1-c(\Box)\lambda_1 - r(\Box)\lambda_2)(c_1(K_S)-c(\Box)\lambda_1 - r(\Box)\lambda_2)}{((a(\Box)+1)\lambda_1 - l(\Box)\lambda_2)((l(\Box)+1)\lambda_2 - a(\Box)\lambda_1)}\\
&= \sum_{\mu} (-1)^{|\mu|} q^{|\mu|} \frac{c(L^{[n]}|_{Z_\mu}) e((K_S^{[n]})|_{Z_\mu})}{e(T|_{Z_\mu})}\\
&= \sum_{\mu} q^{|\mu|} \frac{c(L^{[n]}|_{Z_\mu})}{e(T^{\mathrm{vir}}|_{Z_\mu})} = \mathcal{C}_S(\mathcal{O}_S, L; q).
\end{aligned}$$

Therefore we conclude

$$\mathcal{C}_S(\mathcal{O}_S, L; w) = (1-q)^{\int_S c(L)c_1(K_S)} = \left(\frac{1}{1-q}\right)^{\int_S c(L)c_1(S)}.$$

In particular, restricting to the lowest degree part in the variables $\lambda_1, \lambda_2, w_1$, we obtain the following Corollary.

COROLLARY 4.2. *For $S = \mathbb{C}^2$, the following equality holds:*

$$\sum_{n=1}^{\infty} q^n \int_{[\mathrm{Hilb}^n(S)]^{\mathrm{vir}}} 1 := \sum_{Z\in \mathrm{Hilb}^n(S)^{\mathsf{T}}} \frac{1}{e(T_Z^{\mathrm{vir}})} = e^{\frac{(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}q}.$$

4.2 Segre–Verlinde correspondence in non-zero degrees

Recall that we use the notation $H_{\mu,\nu,\xi}$ for the series from (17) describing the virtual Nekrasov genus for Quot schemes on $S = \mathbb{C}^2$. Note that $\mathcal{N}_S$ satisfies

$$\begin{aligned}
\mathcal{N}_S(y_1,\dots,y_N; v_1,\dots,v_r; q, z) &= \mathcal{N}_S(y_1,\dots,y_N; ze^{w_1},\dots,ze^{w_r}; q, 1)\\
&= \mathcal{N}_S(ze^{m_1},\dots,ze^{m_N}; v_1,\dots,v_r; q, 1).
\end{aligned}$$

Applying Lemma 4.1 to $H_{\mu,\nu,\xi}$ in the variables $w_1,\dots,w_r$ gives us, for $r>0$,

$$D_z^k H_{(0),\nu,\xi}(q,z) = rD_z^{k-1} H_{(1),\nu,\xi}(q,z) = k! \sum_{|\mu|=k} \binom{r}{\mu} H_{\mu,\nu,\xi}(q,z),$$

and applying in the variables $m_1,\dots,m_N$ yields

$$D_z^k H_{\mu,\nu,(0)}(q,z) = ND_z^{k-1} H_{\mu,\nu,(1)}(q,z) = k! \sum_{|\xi|=k} \binom{N}{\xi} H_{\mu,\nu,\xi}(q,z). \tag{20}$$

When the rank r is negative, we consider $\alpha = -[V]$ where $V = \oplus_{i=1}^{-r} \mathcal{O}_S\langle v_i\rangle$. Write

$$\log \mathcal{N}_S(E, -[V]; q, z) = \sum_{\mu,\nu,\xi \text{ partitions}} H_{\mu,\nu,\xi}(q,z) \cdot \int_S c_\mu(V) c_\nu(S) c_\xi(E) c_1(S).$$

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Then the same argument as used in Lemma 4.1 applies, and, for all $r \neq 0$,

$$D_z^k H_{(0),\nu,\xi}(q,z) = |r| D_z^{k-1} H_{(1),\nu,\xi}(q,z) = k! \sum_{|\mu|=k} \binom{|r|}{\mu} H_{\mu,\nu,\xi}(q,z). \tag{21}$$

Observe that when $r>0$, the Chern limit of $H_{\mu,\nu,\xi}$ returns the universal series for the Chern invariants $\log C_{\mu,\nu,\xi}$, and the Verlinde limit returns $\log B_{\mu,\nu,\xi}$. On the other hand, when $r<0$, the Chern series retrieves the rank $-r$ Segre invariants $\log A_{\mu,\nu,\xi}$, but the Verlinde limit does not give the Verlinde series. This is because $H_{\mu,\nu,\xi}$ is associated to $c_\mu(V)$, whereas $\log B_{\mu,\nu,\xi}$ is associated to $c_\mu(\alpha) = c_\mu(-[V])$. This results in a change of basis for the symmetric series in the Chern roots of V, and Lemma 4.1 no longer applies for these negative ranks.

Our goal in this section is to apply Chern and Verlinde limits to (20) and (21) for $k>0$, then obtain relations for the Chern and Verlinde series for various μ, ν, and ξ. We may obtain explicit expressions for when $|\mu|+|\nu|+|\xi|=1$ from the compact setting. For a smooth projective surface S', a torsion-free sheaf E' of rank N, and a K-theory class α of rank r, the universal series structure for $\mathcal{N}_S$ can be obtained from [Boj21a, (5.1), Theorem 5.1] by setting

$$f(t) = 1 - ze^t, \quad g(t) = \frac{t}{1-e^{-t}}.$$

This gives us

$$\mathcal{N}_S(E,\alpha;q,z) = \left(\prod_{i=1}^N F(H_i)\right)^{c_1(S)c_1(\alpha)} \left(\prod_{i=1}^N F(H_i)\right)^{\frac{r}{N}c_1(S)c_1(E)} G(R)^{c_1(S)^2},$$

where

$$R = f^r g^N, \quad F = \frac{f}{f(0)}.$$

The series $H_i(q)$ are Newton–Puiseux solutions to

$$H_i^N = qR(H_i),$$

and $G(R)$ is given explicitly by [Boj21a, (4.24)]. Therefore

$$\begin{aligned} &H_{(1),(0),(0)}(q,z) = \frac{r}{|r|}\sum_{i=1}^N \log F(H_i), \\ &H_{(0),(1),(0)}(q,z) = \log G(R), \quad H_{(0),(0),(1)}(q,z) = \frac{r}{N}\sum_{i=1}^N \log F(H_i). \end{aligned} \tag{22}$$

The sign $\frac{r}{|r|}$ in the first line appears as a result of $c_1(-V) = -c_1(V)$.

4.2.1 *The Chern limit.* We begin with the case $\nu=\xi=(0)$. Apply [Boj21a, (4.17)], which gives a term-by-term expression for (22), by setting $f(t)=1-ze^t$ and $g(t)=\frac{t}{1-e^{-t}}$. We have

$$H_{(1),(0),(0)}(q,z) = \frac{r}{|r|}\sum_{n=1}^\infty \frac{1}{n} q^n [t^{nN-1}] \left\{ -ze^t(1-ze^t)^{nr-1}\left(\frac{t}{1-e^{-t}}\right)^{nN}\right\}. \tag{23}$$

By (20) and (21), we have, for $k_1, k_2 \geqslant 0$ and $k := k_1 + k_2 > 0$,

$$\begin{aligned} &k_1!k_2! \sum_{|\mu|=k_1}\sum_{|\xi|=k_2} \binom{|r|}{\mu}\binom{N}{\xi} H_{\mu,(0),\xi} \\ =&|r| D_z^{k-1} H_{(1),(0),(0)} \end{aligned}$$

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$$
\begin{aligned}
&= r\sum_{n=1}^{\infty}\frac{1}{n}q^n[t^{nN-1}]\left\{D_z^{k-1}\left(-ze^t(1-ze^t)^{nr-1}\right)\left(\frac{t}{1-e^{-t}}\right)^{nN}\right\}\\
&= r\sum_{n=1}^{\infty}\frac{1}{n}q^n[t^{nN-1}]\left\{(-1)^k(1-ze^t)^{nr-k}p_{n,k}(ze^t)\left(\frac{t}{1-e^{-t}}\right)^{nN}\right\}
\end{aligned}
\tag{24}
$$

where $p_{n,k}$ is a polynomial of degree k. We may show inductively that $p_{n,k}(1) = (nr-1)_{(k-1)}$. With this expansion, we would like to apply the Chern limit of Lemma 3.3. In the expansion of universal series (17), the series $H_{\mu,(0),(0)}$ are multiplied by a term in $\vec{\lambda},\vec{w},\vec{m}$ of homogeneous degree $|\mu|-1 = k-1$. When taking the Chern limit, we need to make substitutions

$$
\vec{\lambda} \rightsquigarrow -\varepsilon\vec{\lambda}, \quad \vec{w} \rightsquigarrow -\varepsilon\vec{w}, \quad \vec{m} \rightsquigarrow -\varepsilon\vec{m}.
$$

Therefore, we need to multiply by a factor of $(-\varepsilon)^{k-1}$ when taking the limit of $H_{\mu,(0),(0)}$. Furthermore, we substitute $q \rightsquigarrow (-1)^N q\varepsilon^{N-r}(1+\varepsilon)^r$ and $z \rightsquigarrow (1+\varepsilon)^{-1}$, and the right-hand side of the above expansion becomes

$$
r\sum_{n=1}^{\infty}\frac{1}{n}q^n[t^{nN-1}]\left\{(-1)^{nN-1}\varepsilon^{n(N-r)+k-1}(1+\varepsilon-e^t)^{nr-k}(1+\varepsilon)^k p_{n,k}\left(\frac{e^t}{1+\varepsilon}\right)\left(\frac{t}{1-e^{-t}}\right)^{nN}\right\}.
$$

Since we are extracting the $[t^{nN-1}]$ coefficient of the function inside the curly bracket, we may replace $t \rightsquigarrow -\varepsilon t$ and divide the function by $(-\varepsilon)^{nN-1}$, which gives us

$$
r\sum_{n=1}^{\infty}\frac{1}{n}q^n[t^{nN-1}]\left\{\varepsilon^{k-nr}(1+\varepsilon-e^{-\varepsilon t})^{nr-k}(1+\varepsilon)^k p_{n,k}\left(\frac{e^{-\varepsilon t}}{1+\varepsilon}\right)\cdot\left(\frac{-\varepsilon t}{1-e^{\varepsilon t}}\right)^{nN}\right\}.
$$

Taking $\varepsilon \to 0$, the term $p_{n,k}\left(\frac{\exp(-\varepsilon t)}{1+\varepsilon}\right)$ converges to $p_{n,k}(1) = (nr-1)_{(k-1)}$. Thus the limit gives us

$$
r\sum_{n=1}^{\infty}\frac{1}{n}q^n[t^{nN-1}]\left\{(nr-1)_{(k-1)}(1+t)^{nr-k}\right\} = r\sum_{n=1}^{\infty}\frac{(nr-1)_{(k-1)}}{n}\binom{nr-k}{nN-1}q^n.
$$

We further apply the following identity:

$$
(nr-1)_{(k-1)}\binom{nr-k}{nN-1} = (n(r-N))_{(k-1)}\binom{nr-1}{nN-1}.
$$

The Chern limit of the left-hand side of (24) will be the Chern series of rank r for $r>0$ and the Segre series of rank $-r$ for $r<0$. Therefore, for all $r>0$,

$$
\begin{aligned}
k_1!k_2!\sum_{|\mu|=k_1}\sum_{|\xi|=k_2}\binom{r}{\mu}\binom{N}{\xi}\log A_{\mu,(0),\xi}(q) &= -r\sum_{n=1}^{\infty}\frac{(-n(r+N))_{(k-1)}}{n}\binom{-nr-1}{nN-1}q^n\\
&= -r\sum_{n=1}^{\infty}\frac{1}{n}q^n[t^{nN-1}]\left\{(-n(r+N))_{(k-1)}(1+t)^{-nr-1}\right\},
\end{aligned}
\tag{25}
$$

$$
k_1!k_2!\sum_{|\mu|=k_1}\sum_{|\xi|=k_2}\binom{r}{\mu}\binom{N}{\xi}\log C_{\mu,(0),\xi}(q) = r\sum_{n=1}^{\infty}\frac{(n(r-N))_{(k-1)}}{n}\binom{nr-1}{nN-1}q^n.
$$

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4.2.2 *The Verlinde limit.* We apply a similar argument for the Verlinde limit using (22). To simplify the computation, we consider a different change of variable. Let

$$\tilde{H}_i = 1 - e^{-H_i}.$$

Then the $\tilde{H}_i$ are the Newton–Puiseux solutions to

$$\tilde{H}_i^N = q\frac{(1-\tilde{H}_i - z)^r}{(1-\tilde{H}_i)^r}.$$

Also,

$$F(H_i) = \frac{1-ze^{H_i}}{1-z} = \frac{1-\tilde{H}_i - z}{(1-\tilde{H}_i)(1-z)},$$

so

$$H_{(1),(0),(0)}(q,z) = \frac{r}{|r|}\sum_{i=1}^N \log F(H_i) = \frac{r}{|r|}\sum_{i=1}^N \log\frac{1-\tilde{H}_i - z}{(1-\tilde{H}_i)(1-z)}.$$

Apply the Lagrange inversion theorem of [Boj23, Corollary 5] with $\varphi(t) = \log((1-t-z)/(1-t)(1-z))$ and $R = (1-t-z)^r/(1-t)^r$, we have

$$\begin{aligned}
&k_1!k_2!\sum_{|\mu|=k_1}\sum_{|\xi|=k_2}\binom{|r|}{\mu}\binom{N}{\xi}H_{\mu,(0),\xi}\\
=&|r|D_z^{k-1}H_{(1),(0),(0)}\\
=&rD_z^{k-1}\sum_{n=1}^\infty \frac{1}{n}q^n[t^{nN-1}]\left\{\varphi'(t)R(t)^n\right\}\\
=&r\sum_{n=1}^\infty \frac{1}{n}q^n[t^{nN-1}]\left\{D_z^{k-1}\left(-z(1-t-z)^{nr-1}\right)(1-t)^{-nr-1}\right\}\\
=&r\sum_{n=1}^\infty \frac{1}{n}q^n[t^{nN-1}]\left\{(-1)^k(1-t-z)^{nr-k}q_{n,k}(z)(1-t)^{-nr-1}\right\}.
\end{aligned}$$

Here $q_{n,k}(z)$ is a polynomial of degree k in z whose coefficients involve the variable t, and its leading coefficient is $(nr)^{k-1}$. To take the Verlinde limit for $r>0$, we substitute

$$q \rightsquigarrow (-1)^r q\varepsilon^r, \quad z \rightsquigarrow \varepsilon^{-1},$$

then take $\varepsilon \to 0$ and get

$$k_1!k_2!\sum_{|\mu|=k_1}\sum_{|\xi|=k_2}\binom{r}{\mu}\binom{N}{\xi}\log B_{\mu,(0),\xi} = r\sum_{n=1}^\infty \frac{1}{n}q^n[t^{nN-1}]\left\{(nr)^{k-1}(1-t)^{-nr-1}\right\}.$$

Observe that the Verlinde series for $\alpha \in K_{\mathsf{T}}(S)$ only depends on $c_1(\alpha)$ by definition, so the universal series are non-trivial only when $\mu = (1)_k := (1,\dots,1)$ has k copies of 1. We can therefore simplify the left-hand side of the above equation and get

$$\begin{aligned}
k_1!k_2!r^{k_1}\sum_{|\xi|=k_2}\binom{N}{\xi}\log B_{(1)_{k_1},(0),\xi}(q) &= r\sum_{n=1}^\infty \frac{1}{n}q^n[t^{nN-1}]\left\{(nr)^{k-1}(1-t)^{-nr-1}\right\}\\
&= r^k\sum_{n=1}^\infty (-1)^{nN-1}n^{k-2}\binom{-nr-1}{nN-1}q^n. \qquad (26)
\end{aligned}$$

By [GM22, Lemma 3.3], the universal series are polynomials in r for $r \geqslant 0$. Therefore, the above results hold for the rank $r=0$ case as well.

4.2.3 *The degree -1 case.* When $k=0$, the argument from above still applies, where $D_z^{-1}(-)$ denotes the anti-derivative of $\frac{1}{z}(-)$ with respect to z. However, this would result in an undetermined constant term from the integration, so we deal with this case separately. We compute $A_{(0),(0),(0)}, B_{(0),(0),(0)}, C_{(0),(0),(0)}$ using expressions for $H_{(0),(0),(0)}$, which we shall denote as A, B, C and H, respectively. Since these series are associated to the part of their respective invariants independent of the weights of α, we have

$$A^{N,r}_{(0),(0),(0)} = C^{N,-r}_{(0),(0),(0)},$$

and by the second part of [GM22, Lemma 3.3], A, B, C, H are polynomials with respect to r for all $r \in \mathbb{Z}$.

By Lemma 4.1, $D_z H = |r| H_{(1),(0),(0)}$. Taking D_z^{-1} of (23) with respect to z, we get

$$\begin{aligned} H(q,z) &= H_0(q) + \sum_{n=1}^{\infty} \frac{1}{n^2} q^n [t^{nN-1}] \left\{ (1-ze^t)^{nr} \left(\frac{t}{1-e^{-t}} \right)^{nN} \right\} \\ &=: H_0(q) + H_1(q,z) \end{aligned}$$

for some $H_0(q)$ independent of the variable z.

We apply the result of § 4.2.1 with $k=0$. We see that $H_1(q)$ admits the following Chern limit:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \binom{nr}{nN-1} q^n.$$

Write $H_0(q) = \sum_{n=1}^{\infty} h_n q^n$. Then its Chern limit is

$$\lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} h_n (-1)^{nN-1} \varepsilon^{n(N-r)-1} (1+\varepsilon)^{rn} q^n.$$

When $N - r \leqslant 0$, we must have $h_n = 0$ for all n, since otherwise we would have negative powers on ϵ and the limit does not make sense. As each h_n is polynomial in r, we conclude that $H_0(q) = 0$. Hence, for all $r \in \mathbb{Z}$,

$$\log A(q) = \sum_{n=1}^{\infty} \frac{1}{n^2} \binom{-nr}{nN-1} q^n,$$

$$\log C(q) = \sum_{n=1}^{\infty} \frac{1}{n^2} \binom{nr}{nN-1} q^n.$$

When $0 \leqslant r \leqslant N-1$, the formula for $C(q)$ is consistent with Conjectures 1.20 and 1.21.

Similarly by § 4.2.2, the Verlinde limit of $H_{(0),(0),(0)}$ is

$$\log B(q) = \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^{nN-1} \binom{-nr-1}{nN-1} q^n.$$

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Combining the results of the above sections, we have the following theorem.

Theorem 4.3. *For rank $r \geqslant 0$ and integers $k_1, k_2 \geqslant 0$ with $k := k_1 + k_2$, the universal series of Theorem 3.5 satisfy*

$$k_1!k_2! \sum_{|\mu|=k_1} \sum_{|\xi|=k_2} \binom{r}{\mu}\binom{N}{\xi} \log A_{\mu,(0),\xi}(q) = -r \sum_{n=1}^{\infty} \frac{(-n(r+N))_{(k-1)}}{n} \binom{-nr-1}{nN-1} q^n,$$

$$k_1!k_2!r^{k_1} \sum_{|\xi|=k_2} \binom{N}{\xi} \log B_{(1)_{k_1},(0),\xi}(q) = -r^k \sum_{n=1}^{\infty} n^{k-2} \binom{-nr-1}{nN-1} \left((-1)^N q\right)^n.$$

Furthermore, we have

$$k_1!k_2! \sum_{|\mu|=k_1} \sum_{|\xi|=k_2} \binom{r}{\mu}\binom{N}{\xi} \log C_{\mu,(0),\xi}(q) = r \sum_{n=1}^{\infty} \frac{(n(r-N))_{(k-1)}}{n} \binom{nr-1}{nN-1} q^n,$$

which can be compared to the identities above by replacing r with $-r$.

When $k=2$, we have the following Segre–Verlinde correspondences in degree 1.

Corollary 4.4. *For rank $r \geqslant 0$, the universal series of Theorem 3.5 satisfy the following correspondences*

$$A_{(1,1),(0),(0)}(q)^{-r} A_{(2),(0),(0)}(q)^{\frac{-(r-1)}{2}} = B_{(1,1),(0),(0)}\left((-1)^N q\right)^{r+N},$$

$$A_{(1),(0),(1)}(q)^{-r} = B_{(1),(0),(1)}\left((-1)^N q\right)^{r+N}, \text{ and}$$

$$A_{(0),(0),(1,1)}(q)^{-rN} A_{(0),(0),(2)}(q)^{\frac{-r(N-1)}{2}} = B_{(0),(0),(1,1)}\left((-1)^N q\right)^{N(r+N)} B_{(0),(0),(2)}\left((-1)^N q\right)^{\frac{(N-1)(r+N)}{2}}.$$

Remark 4.5. As mentioned in the introduction, combining Theorem 4.3 with Theorem 3.11 yields the corresponding relations for reduced invariants. In particular, Corollary 4.4 implies a correspondence in degree 0 for reduced invariants, which could provide insight into the reduced invariants for K3-surfaces in the compact setting.

4.3 Universal series via Lagrange inversion

Let $r > 0$. Consider (25) and (26) from the previous section. The right-hand sides of these identities are linear combinations of series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n} q^n [t^{nN-1}] \left\{ n^{k-1}(1+t)^{-nr}\varphi' \right\}$$

for $\varphi(t) = \log(1+t)$ and $k > 0$. The goal of this section is to express these series without the process of extracting coefficients.

By Lagrange inversion theorem [Boj23, Corollary 2], we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} q^n [t^{nN-1}] \left\{ n^{k-1}(1+t)^{-nr}\varphi' \right\} \\
=&D_q^{k-1} \sum_{n=1}^{\infty} \frac{1}{n} q^n [t^{nN-1}] \left\{ (1+t)^{-nr}\varphi' \right\} \\
=&D_q^{k-1} \sum_{i=1}^{N} \left(\varphi(H_i) - \varphi(0)\right) = D_q^{k-1} \sum_{i=1}^{N} \varphi(H_i)
\end{aligned}$$

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where $D_q = q\frac{d}{dq}$ and H_i are the Newton–Puiseux solutions to

$$H_i^N = q(1+H_i)^{-r}. \tag{27}$$

Note that

$$D_q\varphi(H_i) = q\frac{d}{dq}\varphi(H_i) = q\varphi'(H_i)\cdot\frac{dH_i}{dq} = \varphi'(H_i)\cdot D_qH_i.$$

Here $H_i(q^{\frac{1}{N}})$ is a Puiseux series, and by dH_i/dq we mean $(dq/dH_i)^{-1}$. Differentiating both sides of (27) with respect to H_i yields

$$\begin{aligned} NH_i^{N-1}\frac{dH_i}{dq} &= (1+H_i)^{-r} - rq(1+H_i)^{-r-1}\frac{dH_i}{dq} \\ NqH_i^{-1}\frac{dH_i}{dq} &= 1 - rq(1+H_i)^{-1}\frac{dH_i}{dq} \\ D_qH_i &= \frac{1}{NH_i^{-1}+r(1+H_i)^{-1}}. \end{aligned}$$

Let $\psi(t) = Nt^{-1}+r(1+t)^{-1}$, then $D_qH_i = \frac{1}{\psi(H_i)}$. Define $D_\psi = \frac{1}{\psi}\cdot\frac{d}{dt}$. We conclude that

$$D_q\varphi(H_i) = (D_\psi\varphi)(H_i)$$

for arbitrary power series $\varphi(t)$. Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n}q^n[t^{nN-1}]\left\{n^{k-1}(1+t)^{-nr}\varphi'\right\} = \sum_{i=1}^{N}(D_\psi^{k-1}\varphi)(H_i).$$

Theorem 4.6 follows directly by applying this to (25) and (26).

Theorem 4.6. *Let $\varphi(t) = \log(1+t)$ and $\psi(t) = Nt^{-1}+r(1+t)^{-1}$. Define the differential operator*

$$D_\psi = \frac{1}{\psi}\cdot\frac{d}{dt}.$$

Furthermore, use the notation

$$D_{\mathcal{S}}^{(k)} = (-(r+N)D_\psi)_{(k-1)}, \quad D_{\mathcal{V}}^{(k)} = r^{k-1}D_\psi^{k-1}$$

for $k \geqslant 0$, where $D_\psi^{-1}(-)$ denotes integrating $\psi\cdot(-)$ assuming a constant term 0. In the setting of Theorem 4.3, we have the following relations

$$\begin{aligned} k_1!k_2!\sum_{|\mu|=k_1}\sum_{|\xi|=k_2}\binom{r}{\mu}\binom{N}{\xi}\log A_{\mu,(0),\xi}(q) &= -r\sum_{i=1}^{N}\left(D_{\mathcal{S}}^{(k)}\varphi\right)(H_i), \\ k_1!k_2!r^{k_1}\sum_{|\xi|=k_2}\binom{N}{\xi}\log B_{(1)_{k_1},(0),\xi}\left((-1)^Nq\right) &= -r\sum_{i=1}^{N}\left(D_{\mathcal{V}}^{(k)}\varphi\right)(H_i), \end{aligned}$$

where the H_i are the Newton–Puiseux solutions to $H_i^N = q(1+H_i)^{-r}$.

Example 4.7. We shall compute $\log B_{(1,1,1),(0),(0)}$ using the above theorem. Set $k_1 = 3, k_2 = 0$. We have

$$D_F^2\varphi = \frac{Nt\,(1+t)}{((N+r)\,t+N)^3}.$$

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Hence

$$\log B_{(1,1,1),(0),(0)}((-1)^N q) = \frac{-1}{3!}\sum_{n=1}^{\infty}\left(D_F^2\varphi\right)(H_i)$$
$$= \frac{-N}{6}\sum_{i=1}^{N}\frac{H_i\,(1+H_i)}{((N+r)\,H_i+N)^3}.$$

Example 4.8. Similarly, we may compute $\sum_{|\mu|=3}\binom{r}{\mu}\log A_{\mu,(0),(0)}$:

$$\sum_{|\mu|=3}\binom{r}{\mu}\log A_{\mu,(0),(0)}$$
$$=\frac{-r}{3!}\sum_{n=1}^{\infty}\left(\left((r+N)^2D_F^2+(r+N)D_F\right)\varphi\right)(H_i)$$
$$=\frac{-r(r+N)}{6}\sum_{i=1}^{N}\frac{H_i\left((N+r)^2H_i^2+3N(N+r)H_i+N(2N+r)\right)}{((N+r)H_i+N)^3}.$$

Again, the H_i are the Newton–Puiseux solutions to $H_i^N = q(1+H_i)^{-r}$.

4.4 Strong Segre symmetry and weak Verlinde symmetry

Let $r>0$. According to Theorem 4.3, the following identity holds for the Segre series:

$$[q^n]\sum_{|\mu|=k}\binom{r}{\mu}\log A^{r,N}_{\mu,(0),(0)}((-1)^Nq) = [q^n]\sum_{|\xi|=k}\binom{N}{\xi}\log A^{r,N}_{(0),(0),\xi}((-1)^Nq)$$
$$=(-1)^{nN-1}\frac{r}{n}(-n(r+N))_{(k-1)}\binom{-nr-1}{nN-1}$$
$$=(-1)^{nN-1}\frac{r}{n}\cdot\frac{(-nr-1)\cdots(-n(r+N)-k+2)}{(nN-1)!}$$
$$=(-1)^{k-1}rN\cdot(n(r+N)+k-2)_{(k-2)}\binom{n(r+N)}{nN}.$$

The right-hand side does not change if we swap r and N; that is

$$\sum_{|\mu|=k}\binom{r}{\mu}\log A^{r,N}_{\mu,(0),(0)}((-1)^Nq) = \sum_{|\mu|=k}\binom{r}{\mu}\log A^{N,r}_{(0),(0),\mu}((-1)^rq).$$

This is consistent with the strong Segre symmetry of Conjecture 1.15. We have checked that this conjecture holds when

$$\begin{cases} n=1, \text{for } N\leqslant 5, r\leqslant 3,\\ n=2, \text{for } N\leqslant 3, r\leqslant 3,\\ n=3, \text{for } N\leqslant 3, r\leqslant 2,\\ n=4,5, \text{for } N\leqslant 2, r=1.\end{cases} \tag{28}$$

As for the Verlinde series, the weak Segre symmetry of Corollary 1.14 together with the weak Segre–Verlinde correspondence of Corollary 1.9 would imply a weak Verlinde symmetry. However, calculations using a computer program show the 'strong' Verlinde symmetry does not

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hold for $S=\mathbb{C}^2$. This can also be observed from the fact that the equivariant Verlinde series only depends on V through $c_1(V)$ but depends on E through $c_1(E), c_2(E), \dots, c_N(E)$, thus breaking the symmetry.

5. Segre and Verlinde invariants on $\mathbb{C}^4$

Consider $X=\mathbb{C}^4$ with a $(\mathbb{C}^4)^*$-action by scaling coordinates

$$(t_1,t_2,t_3,t_4)\cdot(x_1,x_2,x_3,x_4)=(t_1x_1,t_2x_3,t_3x_3,t_4x_4).$$

Let $\mathsf{T}_0=\{(t_1,t_2,t_3,t_4): t_1t_2t_3t_4=1\}\subseteq(\mathbb{C}^4)^*$ be the subtorus which preserves the usual volume form on X, making X a smooth quasi-projective toric Calabi–Yau 4-fold. As in the surface case, we include two additional tori

$$\mathsf{T}_1=(\mathbb{C}^*)^N,\quad \mathsf{T}_2=(\mathbb{C}^*)^{r+s},$$

where T_1 acts naturally on $\mathbb{C}^N$, and T_2 acts naturally on $\mathbb{C}^r\times\mathbb{C}^s$. Set $\mathsf{T}=\mathsf{T}_0\times\mathsf{T}_1\times\mathsf{T}_2$, and consider $E=\oplus_{i=1}^N\mathcal{O}_X\langle y_i\rangle$, $\alpha=[\oplus_{i=1}^r\mathcal{O}_X\langle v_i\rangle]-[\oplus_{i=r+1}^{r+s}\mathcal{O}_X\langle v_i\rangle]$. Write

$$K_{\mathsf{T}}(\mathrm{pt})=\mathbb{Z}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1};y_1^{\pm1},\dots,y_N^{\pm1};v_1^{\pm1},\dots,v_{r+s}^{\pm1}]/(t_1t_2t_3t_4-1),$$
$$H^*_{\mathsf{T}}(\mathrm{pt})=\mathbb{C}[\lambda_1,\lambda_2,\lambda_3,\lambda_4;m_1,\dots,m_N;w_1,\dots,w_{r+s}]/(\lambda_1+\lambda_2+\lambda_3+\lambda_4).$$

By [HT08, Theorem 4.1] the truncated Atiyah class of the universal subsheaf $\mathcal{I}$ defines an obstruction theory

$$\mathbf{R}\mathscr{H}om_p(\mathcal{I},\mathcal{I})_0^\vee[-1]\to L^\bullet_{\mathrm{Quot}_X(E,n)},$$

where $\mathbf{R}\mathscr{H}om_q=\mathbf{R}q_*\circ\mathbf{R}\mathscr{H}om$, and $(\cdot)_0$ denotes the trace-free part. Note that the obstruction theory is T-equivariant by [Ric21, Theorem B]. The virtual tangent bundle is then

$$T^{\mathrm{vir}}=-\mathbf{R}\mathscr{H}om_p(\mathcal{I},\mathcal{I})_0\in K_{\mathsf{T}}(\mathrm{Quot}_X(E,n)).$$

5.1 Cohomological virtual invariants

When X is a projective Calabi–Yau 4-fold, the virtual fundamental class involves a choice of orientation on $\mathrm{Quot}_X(E,n)$. Let $\mathcal{L}=\det\mathbf{R}\mathscr{H}om_q(\mathcal{I},\mathcal{I})$ be the determinant line bundle. An *orientation* $o(\mathcal{L})$ is a choice of square root of the isomorphism

$$Q:\mathcal{L}\otimes\mathcal{L}\to\mathcal{O}_{\mathrm{Quot}_X(E,n)}$$

induced by Serre duality. A virtual class $[\mathrm{Quot}_X(E,n)]^{\mathrm{vir}}_{o(\mathcal{L})}\in H_{2nN}(\mathrm{Quot}_X(E,n),\mathbb{Z})$ was constructed in [Boj21a, § 2.1] for X a strict Calabi–Yau 4-fold and E a simple rigid locally free sheaf. For $\gamma\in H^{2nN}(\mathrm{Quot}_X(E,n))$, the holomorphic Donaldson invariants are defined to be

$$\mathcal{Q}(\gamma)=\int_{[\mathrm{Quot}_X(E,n)]^{\mathrm{vir}}_{o(\mathcal{L})}}\gamma.$$

For the non-compact $X=\mathbb{C}^4$, similar to the surface case, we have, by [CK17, Lemma 3.6], that T^{vir} has no fixed parts and the T-fixed locus of $\mathrm{Hilb}^n(X)$ consists of only finitely many reduced points. Thus one can define these invariants equivariantly using Oh and Thomas' localization formula [OT20, Theorem 7.1]. For a fixed orientation and any $SO(2k)$-bundle B, one can define its Edidin–Graham square-root Euler class $\sqrt{e}(B)$ [OT20, § 3]. Consider the self-dual resolution (33) of T^{vir}

$$\phi: B^\bullet\to L^\bullet_{\mathrm{Quot}_X(E,n)}$$

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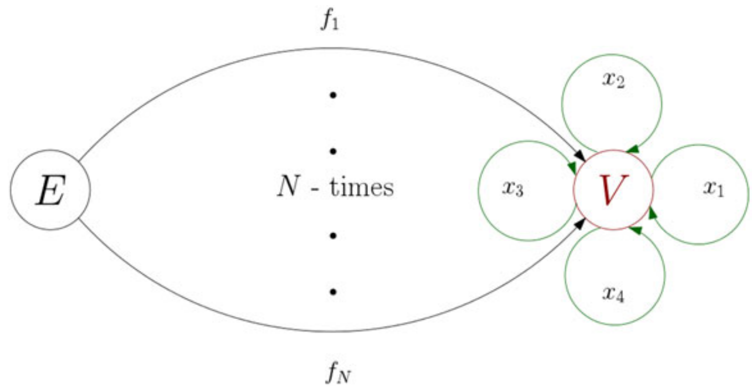

Figure 3. Framed quiver with four loops at one node.

where $B^\bullet = (T \to B \to T^*)$ and B is a $SO(6n^2)$-bundle. Set [OT20, (115)]

$$\sqrt{e_\mathsf{T}}(T^{\mathrm{vir}}) := \sqrt{e_\mathsf{T}}(B^\bullet) := \frac{e_\mathsf{T}(T)}{\sqrt{e_\mathsf{T}}(B)}.$$

For computational purposes, we consider a square root $\sqrt{T^{\mathrm{vir}}|_Z} \in K_\mathsf{T}(\mathrm{pt})$ for each fixed point Z such that

$$T^{\mathrm{vir}}|_Z = \sqrt{T^{\mathrm{vir}}|_Z} + \overline{\sqrt{T^{\mathrm{vir}}|_Z}}, \tag{29}$$

where $\overline{(\cdot)}$ denotes the involution $t_i \mapsto t_i^{-1}$. The existence of this square root follows from $T^{\mathrm{vir}}|_Z$ being self-dual and containing no fixed part (see discussion before [CK17, Definition 3.15]). This allows us to compute the square-root Euler class at the cost of a sign dependent on the choice of orientation $o(\mathcal{L})$

$$\sqrt{e}(T^{\mathrm{vir}}|_Z) = \pm e\left(\sqrt{T^{\mathrm{vir}}|_Z}\right).$$

As each fixed point is reduced, Kool and Rennemo [KR] show that their virtual fundamental classes constructed with respect to the fixed parts of the obstruction theory are given by further signs determined by an orientation on $B^\bullet$. We will denote the product of the two signs at each $Z \in \mathrm{Quot}_X(E,n)^\mathsf{T}$ by $(-1)^{o(\mathcal{L})|_Z}$.

Definition 5.1. For $n > 0$, $\gamma \in H^*_\mathsf{T}(\mathrm{Quot}_X(E,n))$, the holomorphic Donaldson invariants are

$$\mathcal{Q}(X,E,n,\gamma) := \sum_{Z \in \mathrm{Quot}_X(E,n)^\mathsf{T}} (-1)^{o(\mathcal{L})|_Z} \frac{\gamma|_Z}{e(\sqrt{T^{\mathrm{vir}}|_Z})}.$$

The authors of [KR] also gave an explicit description of the moduli space $\mathrm{Quot}_X(E,n)$ when $X = \mathbb{C}^4$ as a vanishing locus of an isotropic section. This allowed them to derive the signs $(-1)^{o(\mathcal{L})|_Z}$ by knowing $\mathcal{L}$ and T^{vir}. For the purpose of the Verlinde invariants from the introduction, we recall their approach here.

Consider the quiver with four loops and N framings as in Figure 3. After imposing the relations

$$[x_i, x_j] = 0\,, \tag{30}$$

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its representations with dimension vector $(1,n)$ consist of a one-dimensional complex vector space $\mathbb{C}$ and an n-dimensional vector space V together with N morphisms $f_i:\mathbb{C}\to V$ for $i=1,\dots,N$ and 4 morphisms $x_i:V\to V$ satisfying (30).

The space of all representations without requiring the relations is

$$R=\mathrm{End}(V)^{\oplus 4}\oplus \mathrm{Hom}(E,V),$$

where we use the suggestive notation $E=\mathbb{C}^N$. This carries an $SO(6n^2)$ vector bundle B that is trivial with fiber

$$\Lambda^2\mathbb{C}^4\otimes \mathrm{End}(V)$$

and pairing

$$b:\Lambda^2\mathbb{C}^4\otimes \mathrm{End}(V)\otimes\Lambda^2\mathbb{C}^4\otimes \mathrm{End}(V)\xrightarrow{(-\wedge-)\otimes\mathrm{tr}(-\circ-)}\Lambda^4\mathbb{C}^4\otimes \mathrm{End}(V)=\mathrm{End}(V)\,.$$

The existence of an isotropic section

$$s:R\to B\,,\qquad (\vec{x},\vec{f})\mapsto\sum_{i\neq j}(e_i\wedge e_j)\otimes x_i\circ x_j$$

connects it to the local toy model of [OT20, (1)].

To construct the Quot scheme, we need to take a quotient by the $GL(V)$-action on R defined for each $g\in GL(V)$ by $(\vec{x},\vec{f})\mapsto(g\circ\vec{x}\circ g^{-1},g\circ\vec{f})$. One can also extend it in a natural way to an action on B. In particular, after defining $R^0\subset R$ as the open subscheme of representations satisfying

$$\mathbb{C}[x_1,\dots,x_4]\cdot\mathbb{C}\langle f_1(1),\dots,f_N(1)\rangle=V\,,$$

the vector bundle B and its section s restrict and then descend to the non-commutative Quot scheme

$$A=\mathrm{NCQuot}_X(E,n)=[R^0/GL(V)]\,.$$

The usual Quot scheme is identified with the zero locus

$$\mathrm{Quot}_X(E,n)=s^{-1}(0)\subset A\,. \tag{31}$$

We will always use the letters B and s to denote the vector bundle and section on R, R^0, their descent to $\mathrm{NCQuot}_X(E,n)$ and their restriction to $\mathrm{Quot}_X(E,n)$ without distinguishing the four cases.

To make it all equivariant, one uses the action of $\mathsf{T}_0\times\mathsf{T}_1$ on R by

$$(\vec{t};\vec{y})\cdot(\vec{x};\vec{f})=(t_1\cdot x_1,t_2\cdot x_2,t_3\cdot x_3,t_4\cdot x_4;y_1\cdot f_1,\dots,y_N\cdot f_N),$$

which commutes with the action of $\mathrm{GL}(V)$ so that it descends to one on $\mathrm{Quot}_X(E,n)$. If we use T to denote the tangent bundle of A, which at each point is the cokernel of some injective map

$$\mathrm{End}(V)\hookrightarrow R \tag{32}$$

obtained by differentiating the action of $\mathrm{GL}(V)$ on R, then there is a $\mathsf{T}_0\times\mathsf{T}_1$-equivariant resolution of $\mathbf{R}\mathscr{H}om_p(\mathcal{I},\mathcal{I})_0^\vee[-1]$ given by

$$B^\bullet=(T\xrightarrow{ds^*}B\xrightarrow{ds}T^*),$$

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which gives the natural $\mathsf{T}_0 \times \mathsf{T}_1$-equivariant obstruction theory

$$\phi : B^\bullet \to L^\bullet_{\mathrm{Quot}_X(E,n)} \, .$$

Note that the first term in (32) has trivial weights.

The choice of orientations $o(\mathcal{L})$ was shown in [OT20, Proposition 4.2] to be equivalent in this setting to choosing a positive isotropic subbundle I of B. This is done explicitly in [KR] by constructing I as a trivial bundle with the fiber

$$\langle v \rangle \wedge \langle v \rangle^{\perp} \otimes \mathrm{End}(V) \tag{33}$$

for some non-zero vector $v \in \mathbb{C}^4$.[1] We will not go further in recalling the explicit derivation of $o(\mathcal{L})|_Z$ in [KR], as we only do computations up to a fixed order n to formulate conjectures, the proof of which we leave for the future.

Definition 5.2. Let $X = \mathbb{C}^4$, $\alpha = [\oplus_{i=1}^{r} \mathcal{O}_X\langle v_i \rangle] - [\oplus_{i=r+1}^{r+s} \mathcal{O}_X\langle v_i \rangle] \in K_{\mathsf{T}}(X)$, and $E = \oplus_{i=1}^{N} \mathcal{O}_X\langle y_i \rangle$. The equivariant Segre and Chern series for a choice of signs $o(\mathcal{L})$ are, respectively,

$$\begin{aligned}
\mathcal{S}_X(E,\alpha;q) &:= \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z} \frac{s(\alpha^{[n]}|_Z)}{e\left(\sqrt{T^{\mathrm{vir}}|_Z}\right)} \\
&\in \frac{\mathbb{C}(\lambda_1,\lambda_2,\lambda_3,\lambda_4; m_1,\dots,m_N; w_1,\dots,w_{r+s})}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}[q], \\
\mathcal{C}_X(E,\alpha;q) &:= \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z} \frac{c(\alpha^{[n]}|_Z)}{e\left(\sqrt{T^{\mathrm{vir}}|_Z}\right)}.
\end{aligned}$$

5.2 K-theoretic virtual invariants

In the setting where the moduli space M is a zero locus of an isotropic section s of an $SO(2m)$ bundle B on some ambient space A, just as in (31) above, [OT20] give a simpler construction of $\hat{\mathcal{O}}^{\mathrm{vir}}$ relying on their equivariant *localized K-theoretic square-root Euler class*

$$\sqrt{\mathfrak{e}_{\mathsf{T}}}(E,s) : K_0(A,\mathbb{Z}) \to K_0(M,\mathbb{Z}[2^{-1}])$$

defined after choosing orientations on M.

When B admits an isotropic subbundle I compatible with the choice of orientation, then the push-forward under the inclusion

$$\iota : M = s^{-1}(0) \hookrightarrow A$$

becomes just tensoring with the equivariant *K-theoretic square-root Euler class*

$$\iota_*(\sqrt{\mathfrak{e}_{\mathsf{T}}}(E,s)) = \otimes \sqrt{\mathfrak{e}_{\mathsf{T}}}(E) = (-1)^m \mathfrak{e}_{\mathsf{T}}(\mathsf{I}^*) \sqrt{\det}(\mathsf{I}^*),$$

where $\mathfrak{e}_{\mathsf{T}}(\mathsf{I}^*) = \Lambda_{-1}\mathsf{I}^*$. In fact, $\sqrt{\mathfrak{e}_{\mathsf{T}}}(E,s)$ can also be written as the product

$$\sqrt{\mathfrak{e}_{\mathsf{T}}}(E,s) = (-1)^m \mathfrak{e}_{\mathsf{T}}(\mathsf{I}^*, s) \sqrt{\det}(\mathsf{I}^*),$$

where $\mathfrak{e}_{\mathsf{T}}(\mathsf{I}^*, s)$ is some localization of $\mathfrak{e}_{\mathsf{T}}(\mathsf{I}^*)$ to $M = s^{-1}(0)$ constructed using Kiem and Li's cosection localization in K-theory [KL20]. Their equivariant *twisted virtual structure sheaf* $\hat{\mathcal{O}}^{\mathrm{vir}}$

[1] In fact, the particular choice of signs compatible with the existing literature corresponds to setting $v = e_4$ for the fourth vector of the canonical basis of $\mathbb{C}^4$.

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is then constructed as

$$\begin{aligned}\hat{\mathcal{O}}^{\mathrm{vir}} &= \sqrt{\mathfrak{e}_{\mathsf{T}}}(B,s)([\![\mathcal{O}_A]\!])\sqrt{\det}(T^*)\\ &= (-1)^m \mathfrak{e}_{\mathsf{T}}(\mathsf{I}^*, s)([\![\mathcal{O}_A]\!])\sqrt{\det}(T^* + \mathsf{I}^*)\,.\end{aligned}$$

For our case of $M = \mathrm{Quot}_{\mathbb{C}^4}(E,n)$, we can use (32) and (33) to show that

$$\det(T) = \det\left(\mathrm{End}(V)\right)^{\otimes 4}\prod_{i=1}^{4} t_i^{n^2}\det\left(\mathrm{Hom}(E,V)\right) = \det\left(\mathrm{Hom}(E,V)\right)$$

and $\det(\mathsf{I}) = t_4^{2n^2}$. This implies that the only term in the construction of $\hat{\mathcal{O}}^{\mathrm{vir}}$ that does not admit a square root is

$$\begin{aligned}\det{}^{-1}(\mathrm{Hom}(E,V)) &= \det{}^{-1}((E^\vee)^{[n]})\\ &= (y_1\dots y_N)^{-n}\det{}^{-N}(V)\,.\end{aligned}$$

This gives further motivation for the definition of the *untwisted virtual structure sheaf*

$$\mathcal{O}^{\mathrm{vir}} := \hat{\mathcal{O}}^{\mathrm{vir}} \otimes \mathsf{E}^{\frac{1}{2}}\,,\quad \mathsf{E} = \det((E^\vee)^{[n]}) \tag{34}$$

that appeared in [Boj21b, Definition 5.10], [Boj21a, § 1.4]. Here, the superscript $(-)^{\frac{1}{2}}$ refers to taking the uniquely defined square root of a line bundle (see [OT20, Remark 5.2]). From the above discussion it is clear that the following integrality statement holds.

Proposition 5.3. *The untwisted virtual structure sheaf is an integral class:*

$$\mathcal{O}^{\mathrm{vir}} \in K_0(\mathrm{Quot}_{\mathbb{C}^4}(E,n),\mathbb{Z})\,.$$

Using $\hat{\mathcal{O}}^{\mathrm{vir}}$ and $\mathcal{O}^{\mathrm{vir}}$, we define the following *twisted* and *untwisted Euler characteristics:*

$$\begin{aligned}\hat{\chi}^{\mathrm{vir}}(\mathrm{Quot}_Y(E,n),-) &= \chi(\mathrm{Quot}_Y(E,n), \hat{\mathcal{O}}^{\mathrm{vir}}\otimes(-)),\\ \chi^{\mathrm{vir}}(\mathrm{Quot}_Y(E,n),-) &= \chi(\mathrm{Quot}_Y(E,n), \mathcal{O}^{\mathrm{vir}}\otimes(-)).\end{aligned}$$

For compact X and $\alpha\in K^0(X)$, the *Verlinde series* [Boj21a, § 1.3] is then defined by

$$\begin{aligned}\mathcal{V}_X(E,\alpha;q) &:= \sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}\left(\mathrm{Quot}_X(E,n), \det(\alpha^{[n]})\right)\\ &= \sum_{n=0}^{\infty} q^n \hat{\chi}^{\mathrm{vir}}\left(\mathrm{Quot}_X(E,n), \det(\alpha^{[n]})\otimes\sqrt{\det}((E^\vee)^{[n]})\right).\end{aligned}$$

Using the virtual Riemann–Roch formula and the equivariant localization of Oh and Thomas [OT20, Theorem 6.1, Theorem 7.3], we have the following *twisted* equivariant virtual Euler characteristic for $X = \mathbb{C}^4$:

$$\hat{\chi}^{\mathrm{vir}}_{\mathsf{T}}(\mathrm{Quot}_X(E,n),\alpha) := \sum_{Z\in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z} e\left(-\sqrt{T^{\mathrm{vir}}|_Z}\right)\sqrt{\mathrm{td}}\left(T^{\mathrm{vir}}|_Z\right)\mathrm{ch}_{\mathsf{T}}(\alpha)$$

where $\sqrt{\mathrm{td}}$ is the the square-root Todd class satisfying

$$\begin{aligned}\sqrt{\mathrm{td}}(T^{\mathrm{vir}}|_Z) =& \mathrm{td}\left(\sqrt{T^{\mathrm{vir}}|_Z}\right)\mathrm{ch}\left(\sqrt{\det}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)\\ =& \frac{e\left(\sqrt{T^{\mathrm{vir}}|_Z}\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)}\mathrm{ch}\left(\sqrt{\det}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right).\end{aligned}$$

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Following the notation used in [CKM22, § 0.1], we write

$$K^{\mathrm{vir}} = \det(T^{\mathrm{vir}})^{\vee}, \quad \sqrt{K^{\mathrm{vir}}} = \det\sqrt{T^{\mathrm{vir}}}^{\vee}, \quad \sqrt{K^{\mathrm{vir}}}^{\frac{1}{2}} = \sqrt{\det}\sqrt{T^{\mathrm{vir}}}^{\vee}.$$

Substituting into the above equation, we have

$$\hat{\chi}^{\mathrm{vir}}(\mathrm{Quot}_X(E,n),\alpha) = \sum_{Z\in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z} \frac{\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}|_Z}^{\frac{1}{2}}\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)} \mathrm{ch}(\alpha^{[n]}|_Z).$$

Now including the twist of (34), we may define the equivariant Verlinde series as follows.

Definition 5.4. The equivariant Verlinde series for a choice of signs $o(\mathcal{L})$ is

$$\mathcal{V}_X(E,\alpha;q) := \sum_{n=0}^{\infty} q^n \sum_{Z\in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z} \frac{\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}|_Z}^{\frac{1}{2}}\right)\mathrm{ch}\left(\sqrt{\det}((E^{\vee})^{[n]}|_Z)\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)} \mathrm{ch}\left(\det(\alpha^{[n]}|_Z)\right)$$
$$\in \frac{\mathbb{Q}(t_1,t_2,t_3,t_4;y_1,\dots,y_N;v_1,\dots,v_{r+s})}{(t_1t_2t_3t_4)}[\![q]\!].$$

The relation between Segre and Verlinde numbers in the compact case is studied in [Boj21a] using the *Nekrasov genus* for Hilbert schemes, introduced for the 3-fold case by [NO14]. We consider the following Quot scheme version, originally defined as an instanton partition function in [NP19, (2.7)], then rephrased as follows in [CKM22, Remark 0.4]:

$$\mathcal{N}_X(E,\alpha;q) := \sum_{n=0}^{\infty} q^n \sum_{Z\in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z} \frac{\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}|_Z}^{\frac{1}{2}}\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)} \mathrm{ch}\left(\frac{\Lambda_{-1}}{\sqrt{\det}}\alpha^{[n]}|_Z\right). \tag{35}$$

Remark 5.5. Recall that in [CKM22], the Nekrasov genus for Hilbert schemes involves a variable y coming from a trivial $\mathbb{C}^*$-action on X. This is exactly the $N=1$ case for the above definition, where we have the parameter y_1 from the T_1-action on E.

5.3 Vertex formalism

The invariants in the previous section can be calculated for Hilbert schemes using a vertex formalism developed by [CK20], based on the method introduced in [MNOP06a] for Calabi–Yau 3-folds. We generalize this to Quot schemes using the computations from [Boj21a, § 2.1]. First, when $N=1$, the T-fixed points of $\mathrm{Hilb}^n(X)$ correspond to monomial ideals of $\mathbb{C}[x_1,x_2,x_3,x_4]$ [CK17, Lemma 3.1], which are labeled by solid partitions π of size n where

$$\mathcal{O}_{Z_\pi} = \mathbb{C}[x_1,x_2,x_3,x_4]/I_{Z_\pi} = \mathrm{span}\{x_1^a x_2^b x_3^c x_4^d : (a,b,c,d)\in\pi\}.$$

We denote by Q_π the character of $\mathcal{O}_{Z_\pi}$:

$$Q_\pi = \sum_{(i,j,k,l)\in\pi} t_1^{-a}t_2^{-b}t_3^{-c}t_4^{-d} \in K^*_{\mathsf{T}}(\mathrm{pt}) = \frac{\mathbb{Z}\left[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}\right]}{(t_1t_2t_3t_4-1)}.$$

As in the surface case, for $E=\oplus_{i=1}^N \mathcal{O}_X\langle y_i\rangle$ the T-fixed points for $\mathrm{Quot}_X(E,n)$ are labeled by N-colored solid partitions $\pi=(\pi^{(1)},\dots,\pi^{(n)})$ of size n, i.e. sequences of the form

$$Z_\pi = ([Z_1],[Z_2],\dots,[Z_N]) \in \mathrm{Hilb}^{n_1}(X)\times\dots\times\mathrm{Hilb}^{n_N}(X)$$

such that each Z_i corresponds to the solid partition $\pi^{(i)}$.

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Let Q_i be the character of $\mathcal{O}_{Z_i}$. The virtual tangent bundle at Z_π is

$$\begin{aligned} T^{\mathrm{vir}}_{Z_\pi} =& \mathrm{Ext}\left(\bigoplus_{i=1}^N I_{\mathcal{Z}_i}\langle y_i\rangle, \bigoplus_{j=1}^N I_{\mathcal{Z}_j}\langle y_j\rangle\right)_0 \\ =& \sum_{i,j=1}^N \mathcal{O}_X \otimes (1-\overline{P(I_{Z_i})}P(I_{Z_j}))y_i^{-1}y_j \\ =& \sum_{i,j=1}^N \left(Q_j + t_1t_2t_3t_4\overline{Q_i} - t_1t_2t_3t_4P_{1234}\overline{Q_i}Q_j\right) y_i^{-1}y_j \end{aligned} \tag{36}$$

where $P(I)$ is the Poincaré polynomial of I defined analogously to (8), and $P_I := \prod_{i\in I}(1-t_i^{-1})$ for the set of indices I. Specializing to $t_1t_2t_3t_4 = 1$, we get the following (non-unique) square root

$$\sqrt{T^{\mathrm{vir}}_{Z_\pi}} := \sum_{i,j=1}^N \left(Q_j - \overline{P_{123}Q_i}Q_j\right) y_i^{-1}y_j$$

in the sense that it satisfies (29). The reason for this choice of square root is that

$$\begin{aligned} \mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}|_{Z_\pi}}^{\frac{1}{2}}\right) &= \mathrm{ch}\left(\prod_{i,j}\det((\overline{Q_j - \overline{P}_{123}Q_j\overline{Q_i}})y_i^{-1}y_j)^{\frac{1}{2}})\right) \\ &= \mathrm{ch}\left(\prod_{i,j}\sqrt{\det}(\overline{Q_j})y_iy_j^{-1})\right) \\ &= \frac{1}{\mathrm{ch}\left(\sqrt{\det}((E^\vee)^{[n]}|_{Z_\pi})\right)} \end{aligned}$$

matches our twist in (34), and this simplifies our computation as we now have

$$\mathcal{V}_X(E,\alpha;q) = \sum_{\pi}^{\infty} q^{|\pi|}(-1)^{o(\mathcal{L})|_{Z_\pi}} \frac{\mathrm{ch}\left(\det(\alpha^{[n]}|_{Z_\pi})\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}^\vee\right)}.$$

In this case, the signs $(-1)^{o(\mathcal{L})}$ are described in [Mon22, KR] as follows: for any solid partition π,

$$o(\mathcal{L})|_{Z_\pi} := |\pi| + \#\{(i,i,i,j)\in\pi : i<j\};$$

and for any N-colored solid partition π,

$$o(\mathcal{L})|_{Z_\pi} := \sum_{i=1}^N o(\mathcal{L})|_{Z_i}.$$

The fiber of $V^{[n]} = \oplus_{i=1}^r \mathcal{O}_X^{[n]}\langle v_i\rangle$ over $Z_\pi = (Z_1,\dots Z_N)$ is the rn-dimensional representation

$$V^{[n]}|_{Z_\pi} = \bigoplus_{i=1}^r\bigoplus_{j=1}^N \mathcal{O}_{Z_j}\langle v_iy_j\rangle = \left(\sum_{i=1}^r\sum_{j=1}^N\sum_{(a,b,c,d)\in\pi^{(j)}} v_iy_jt_1^{-a}t_2^{-b}t_3^{-c}t_4^{-d}\right) \in K_{\mathsf{T}}(\mathrm{pt}).$$

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Therefore, for any point Z corresponding to an N-colored solid partition π, we have

$$c(V^{[n]}|_{Z_\pi}) = \prod_{j=1}^{N} \prod_{(a,b,c,d)\in\pi^{(j)}} \prod_{i=1}^{r} (1 + w_i + m_j - a\lambda_1 - b\lambda_2 - c\lambda_3 - d\lambda_4),$$

$$\det(V^{[n]}|_{Z_\pi}) = \prod_{j=1}^{N} \prod_{(a,b,c,d)\in\pi^{(j)}} \prod_{i=1}^{r} v_i y_j t_1^{-a} t_2^{-b} t_3^{-c} t_4^{-d},$$

$$\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}|_{Z_\pi}}^{\frac{1}{2}}\right) \mathrm{ch}\left(\frac{\Lambda_{-1}}{\det^{\frac{1}{2}}} V^{[n]}|_{Z_\pi}\right) = \prod_{j=1}^{N} \prod_{(a,b,c,d)\in\pi^{(j)}} t_1^{\frac{a}{2}} t_2^{\frac{b}{2}} t_3^{\frac{c}{2}} t_4^{\frac{d}{2}} \cdot \prod_{i=1}^{r} \left(v_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}} t_1^{\frac{a}{2}} t_2^{\frac{b}{2}} t_3^{\frac{c}{2}} t_4^{\frac{d}{2}} - v_i^{\frac{1}{2}} y_j^{\frac{1}{2}} t_1^{-\frac{a}{2}} t_2^{-\frac{b}{2}} t_3^{-\frac{c}{2}} t_4^{-\frac{d}{2}} \right).$$

Using these expressions, we see that the Chern and Verlinde series can be extracted by taking limits of the Nekrasov genus, as in the surface case. Also, it follows that

$$\mathcal{N}_X(E,V;q) \in \frac{\mathbb{Q}\left(t_1^{\frac{1}{2}}, t_2^{\frac{1}{2}}, t_3^{\frac{1}{2}}, t_4^{\frac{1}{2}}\right)}{(t_1t_2t_3t_4 - 1)} \left[\!\left[q, y_1^{\pm\frac{1}{2}}, \dots, y_N^{\pm\frac{1}{2}}, v_1^{\pm\frac{1}{2}}, \dots, v_r^{\pm\frac{1}{2}} \right]\!\right].$$

The argument of [CKM22, Propositions 1.13 and 1.15] can be applied to show that $\mathcal{N}_X(E,V;q)$ in fact lives in $\frac{\mathbb{Q}(t_1,t_2,t_3,t_4)}{(t_1t_2t_3t_4-1)}[\![q, y_1^{\pm\frac{1}{2}}, \dots, y_N^{\pm\frac{1}{2}}, v_1^{\pm\frac{1}{2}}, \dots, v_r^{\pm\frac{1}{2}}]\!]$. This enables us to talk about admissibility (up to specializing to $t_1t_2t_3t_4 = 1$) in the sense of Definition 2.6.

5.4 Factor of $c_3(X)$

In the surface case, we saw that the powers in the universal series of virtual invariants are multiples of $c_1(S)$. In the $X = \mathbb{C}^4$ case, we shall show that if the universal expressions exist, then they are multiples of $c_3(X)$, by showing that

$$e\left(-\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}\right)$$

has $c_3(X) = -(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)$ in its numerator. This factor of $c_3(X)$ and the weak Segre–Verlinde correspondence and Segre symmetry of Corollaries 1.7 and 1.14 in the surface case motivate Conjecture 1.19. This is because the only degree zero contribution linear in $c_3(X)$ is expected to come from $\int_X c_3(X)c_1(\alpha)$, which was already studied in the compact case in [Boj21a, § 5.3]. We do not expect any additional terms with exponential $\int_X c_3(X)c_1(\alpha)$ coming from the equivariant setting, just as we did not have any in the case of a surface.

It suffices to show that this term vanishes when we set $\lambda_i = -\lambda_j$ for $i \neq j$ in $\{1,2,3\}$. By symmetry, we may assume $i=1, j=2$. Recall that e is the top equivariant Chern class, which vanishes if its input has a trivial summand. The process of setting $\lambda_1 = -\lambda_2$ in cohomology is the same as setting $t_1 = t_2^{-1}$ in K-theory. Therefore, we would like to show that $-\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}$ has a trivial summand when we set $t_1 = t_2^{-1}$, i.e. the character of $\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}$ in $K_{\mathsf{T}}(\mathrm{pt})$ having a strictly negative constant term. This occurs if and only if the image of $T^{\mathrm{vir}}_{Z_\pi}$ in

$$\mathbb{Z}\left[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}\right] / (t_1t_2 - 1, t_3t_4 - 1)$$

has a strictly negative constant term (which is necessarily a negative even integer). From (36), we see that it suffices to show this for the term

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$$Q_\pi + t_1t_2t_3t_4\overline{Q_\pi} - t_1t_2t_3t_4P_{1234}\overline{Q_\pi}Q_\pi$$

whenever π is a non-trivial solid partition.

Lemma 5.6. *For any non-trivial solid partition π, the expression*

$$Q_\pi + t_1t_2t_3t_4\overline{Q_\pi} - t_1t_2t_3t_4P_{1234}\overline{Q_\pi}Q_\pi$$

has a strictly negative constant term when viewed in the quotient ring

$$\mathbb{Z}\left[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}\right]/(t_1t_2-1, t_3t_4-1).$$

Proof. Let $x = t_1 = \frac{1}{t_2}$, $y = t_3 = \frac{1}{t_4}$, so that

$$\mathbb{Z}\left[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}\right]/(t_1t_2-1, t_3t_4-1) = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}].$$

Let P_π be the image of Q_π in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$. Then

$$T^{\mathrm{vir}}|_{Z_\pi} = P_\pi + \overline{P_\pi} - P_\pi\overline{P_\pi}(1-x)\left(1-\frac{1}{x}\right)(1-y)\left(1-\frac{1}{y}\right). \tag{37}$$

Write

$$P_\pi = \sum_{i,j\in\mathbb{Z}} p_{i,j}x^iy^j.$$

The image of $\overline{Q_\pi}$ is then

$$\overline{P_\pi} = \sum_{i,j\in\mathbb{Z}} p_{i,j}x^{-i}y^{-j}.$$

We see that the constant terms of P_π and $\overline{P_\pi}$ are both $p_{0,0}$. By definition, all monomial terms in Q_π have positive coefficients, and Q_π has constant term 1, so $p_{0,0} > 0$. We need to find the constant term of $P_\pi\overline{P_\pi}(1-x)(1-\frac{1}{x})(1-y)(1-\frac{1}{y})$.

Observe that

$$(1-x)\left(1-\frac{1}{x}\right)(1-y)\left(1-\frac{1}{y}\right) = 4 - 2\left(x+y+\frac{1}{x}+\frac{1}{y}\right) + \left(xy + \frac{1}{xy} + \frac{x}{y} + \frac{y}{x}\right).$$

We write $F = \sum f_{i,j}x^iy^j$. The constant term of $F\cdot(1-x)(1-\frac{1}{x})(1-y)(1-\frac{1}{y})$ is equal to

$$4f_{0,0} - 2(f_{0,1}+f_{1,0}+f_{0,-1}+f_{-1,0}) + (f_{1,1}+f_{1,-1}+f_{-1,1}+f_{-1,-1}). \tag{38}$$

If we set $F = P_\pi\overline{P_\pi}$, then

$$f_{i,j} = \sum_{\substack{a-c=i\\ b-d=j}} p_{a,b}p_{c,d}.$$

In particular,

$$\begin{aligned}
f_{0,0} &= \sum_{a,b\in\mathbb{Z}} p_{a,b}^2,\\
f_{0,1}+f_{1,0}+f_{0,-1}+f_{-1,0} &= \sum_{a,b\in\mathbb{Z}} p_{a,b}(p_{a-1,b}+p_{a+1,b}+p_{a,b-1}+p_{a,b+1}),\\
f_{1,1}+f_{1,-1}+f_{-1,1}+f_{-1,-1} &= \sum_{a,b\in\mathbb{Z}} p_{a,b}(p_{a+1,b+1}+p_{a+1,b-1}+p_{a-1,b-1}+p_{a-1,b+1}).
\end{aligned}$$

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We write

$$\begin{aligned}
s_{a,b} =& 4p_{a,b} - 2(p_{a-1,b} + p_{a+1,b} + p_{a,b-1} + p_{a,b+1}) \\
&+ (p_{a+1,b+1} + p_{a+1,b-1} + p_{a-1,b-1} + p_{a-1,b+1}), \\
s^{++}_{a,b} =& p_{a,b} - (p_{a+1,b} + p_{a,b+1}) + p_{a+1,b+1}, \\
s^{+-}_{a,b} =& p_{a,b} - (p_{a+1,b} + p_{a,b-1}) + p_{a+1,b-1}, \\
s^{-+}_{a,b} =& p_{a,b} - (p_{a-1,b} + p_{a,b+1}) + p_{a-1,b+1}, \\
s^{--}_{a,b} =& p_{a,b} - (p_{a-1,b} + p_{a,b-1}) + p_{a-1,b-1};
\end{aligned}$$

$$\begin{aligned}
S^{++} &= \sum_{a,b\geqslant 0} p_{a,b}s^{++}_{a,b} + p_{a+1,b}s^{-+}_{a+1,b} + p_{a,b+1}s^{+-}_{a,b+1} + p_{a+1,b+1}s^{--}_{a+1,b+1}, \\
S^{+-} &= \sum_{a\geqslant 0, b\leqslant 0} p_{a,b}s^{+-}_{a,b} + p_{a+1,b}s^{--}_{a+1,b} + p_{a,b-1}s^{++}_{a,b-1} + p_{a+1,b-1}s^{-+}_{a+1,b-1}, \\
S^{-+} &= \sum_{a\leqslant 0, b\geqslant 0} p_{a,b}s^{-+}_{a,b} + p_{a+1,b}s^{++}_{a+1,b} + p_{a,b+1}s^{--}_{a,b+1} + p_{a+1,b+1}s^{+-}_{a+1,b+1}, \\
S^{--} &= \sum_{a,b\leqslant 0} p_{a,b}s^{--}_{a,b} + p_{a+1,b}s^{+-}_{a+1,b} + p_{a,b+1}s^{-+}_{a,b+1} + p_{a+1,b+1}s^{++}_{a+1,b+1}.
\end{aligned}$$

Then (38) becomes

$$\begin{aligned}
\sum_{a,b\in\mathbb{Z}} p_{a,b}s_{a,b} =& \sum_{a,b\in\mathbb{Z}} p_{a,b}\cdot\left(s^{++}_{a,b} + s^{+-}_{a,b} + s^{-+}_{a,b} + s^{--}_{a,b}\right) \\
=& S^{++} + S^{+-} + S^{-+} + S^{--}.
\end{aligned}$$

For the remainder of this proof, we shall show that $S^{++} \geqslant p_{0,0}$. The same will hold for the summands S^{+-}, S^{-+}, S^{--} by symmetry. We conclude that the value of (38) is at least $4p_{0,0}$. Hence, by (37), the constant term of $T^{\mathrm{vir}}|_{Z_\pi}$ is at most $-2p_{0,0} < 0$, and we are done.

Recall that

$$Q_\pi = \sum_{(i,j,k,l)\in\pi} t_1^i t_2^j t_3^k t_4^l,$$

so

$$P_\pi = \sum_{(i,j,k,l)\in\pi} x^{i-j}y^{k-l},$$

and

$$p_{a,b} = \#\{(i,j,k,l)\in\pi : i-j=a, k-l=b\}.$$

Fix k and l, and then the set $\{(i,j) : (i,j,k,l)\in\pi\}$ is a plane partition. By property (5) of solid partitions, for fixed $b=k-l$, we have $p_{a,b}\geqslant p_{a+1,b}$ when $a\geqslant 0$. For the same reason, we have $p_{a,b}\geqslant p_{a,b+1}$ when $b\geqslant 0$. Therefore, the numbers $(p_{a,b})_{a,b\geqslant 0}$ are non-increasing as the pair (a,b) move away from the origin.

We apply induction on $\max\{a : p_{a,0}\neq 0\}$. Suppose for all sequences $(q_{a,b})$ with $\max\{a : q_{a,0}\neq 0\} < \max\{a : p_{a,0}\neq 0\}$, we have

$$S^{++}(q_{a,b}) \geqslant q_{0,0}$$

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whenever the sequence $(q_{a,b})_{a,b\geqslant 0}$ satisfies the property that $q_{a,b}$ is non-increasing in a,b. The base case is simply when $q_{a,b}=0$ for all a,b, which sums to 0. Let $q_{a,b}=p_{a+1,b}$, then

$$\begin{aligned} S^{++}(p_{a,b}) = &\sum_{a,b\geqslant 0}(p_{a,b}-p_{a+1,b}-p_{a,b+1}+p_{a+1,b+1})^2 \\ =&S^{++}(q_{a,b})+\sum_{b\geqslant 0}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2 \\ \geqslant& p_{1,0}+\sum_{b\geqslant 0}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2 \end{aligned} \tag{39}$$

where the first equality follows from the definition and the inequality is by the induction hypothesis.

Now apply another induction on the value of $\max\{b : p_{0,b}\neq 0\}$. The induction hypothesis is that for any sequences $(q_{a,b})$ with $q_{a,b}$ non-increasing in a,b and $\max\{b : q_{0,b}\neq 0\} < \max\{b : p_{0,b}\neq 0\}$, we have

$$\sum_{b\geqslant 0}(q_{0,b}-q_{1,b}-q_{0,b+1}+q_{1,b+1})^2\geqslant q_{0,0}-q_{1,0}.$$

Again, the base case is trivial, and we can apply the hypothesis to $q_{a,b}=p_{a,b+1}$, giving us

$$\sum_{b\geqslant 1}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2\geqslant p_{0,1}-p_{1,1}.$$

So we have the following inequalities

$$\begin{aligned} &(p_{0,0}-p_{1,0}-p_{0,1}+p_{1,1})^2+\sum_{b\geqslant 1}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2-(p_{0,0}-p_{1,0}) \\ \geqslant&(p_{0,0}-p_{1,0}-p_{0,1}+p_{1,1})^2-(p_{0,0}-p_{1,0}-p_{0,1}+p_{1,1}) \\ \geqslant&0 \end{aligned}$$

where the last inequality is due to $p_{0,0}-p_{1,0}-p_{0,1}+p_{1,1}$ being an integer. Therefore,

$$\sum_{b\geqslant 0}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2\geqslant p_{0,0}-p_{1,0},$$

which finishes the second induction. By (39),

$$S^{++}(p_{a,b})\geqslant p_{1,0}+(p_{0,0}-p_{1,0})=p_{0,0}$$

which finishes the first induction and the proof. □

We checked the statements in Conjecture 1.19 using a computer program. The source code is available at [Hua23]. The correspondence part was checked with a computer program for

$$\begin{cases} n\leqslant 6, & \text{for } N,r\leqslant 1,\\ n\leqslant 3, & \text{for } N,r\leqslant 2,\\ n\leqslant 2, & \text{for } N,r\leqslant 3,\\ n\leqslant 2, & \text{for } N,r\leqslant 4. \end{cases} \tag{40}$$

The symmetry part was checked for

$$\begin{cases} n\leqslant 4, & \text{for } N=r=1,\\ n\leqslant 3, & \text{for } N=1,r=2,\\ n\leqslant 2, & \text{for } N=r=2,\\ n\leqslant 2, & \text{for } N=1,r=3. \end{cases} \tag{41}$$

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5.5 Cohomological limits

Recall that the proof of Theorem 3.5 mainly involved showing that the genus $\mathcal{N}_S$ on the surface $S=\mathbb{C}^2$ is admissible in the sense of Definition 2.6. Also, by Proposition 2.7, universal series expressions for the Nekrasov genus, and therefore the Segre and Verlinde series, can be obtained if and only if the Nekrasov genus is admissible. Thus one might ask when the Nekrasov genus $\mathcal{N}_X$ for the Calabi–Yau 4-fold $X=\mathbb{C}^4$ is admissible. For the rank $r=N$ case, we shall show that admissibility is a consequence of the following explicit formula, as conjectured by Nekrasov and Piazzalunga [NP19, § 2.5]. We write

$$[x]=x^{\frac{1}{2}}-x^{-\frac{1}{2}}.$$

The conjecture is in regards to any toric Calabi–Yau 4-fold X (see [CKM22, p. 3, footnote 1]). Say the maximal dense open torus $(\mathbb{C}^4)^*\subseteq X$ contains the torus $\mathsf{T}_0\cong(\mathbb{C}^*)^3$ that preserves the volume form. This gives a T-action on X similar to the $\mathbb{C}^4$ case with weights labeled by $t_1,\dots,t_4,y_1,\dots,y_N,v_1,\dots v_r$ as before.

Conjecture 5.7 (Nekrasov–Piazzalunga). Let X be a toric Calabi–Yau 4-fold. There exists some choice of signs $o(\mathcal{L})$ such that for $E=\oplus_{i=1}^N\mathcal{O}_X\langle y_i\rangle$, $V=\oplus_{i=1}^N\mathcal{O}_X\langle v_i\rangle$,

$$\mathcal{N}_X(E,V;q)=\mathrm{Exp}\left(\frac{[t_1t_2][t_2t_3][t_1t_3]}{[t_1][t_2][t_3][t_4]}\frac{[s]}{[s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}\right)$$

with a change of variables $s=\prod_{i=1}^N y_iv_i$.

Proposition 5.8. *Nekrasov and Piazzalunga's Conjecture 5.7 implies that the Nekrasov genus $\mathcal{N}_X$ of rank $r=N$ is admissible with respect to the variables t_1,t_2,t_3,t_4.*

Proof. Expanding the term inside the plethystic exponential, and specializing with the relation $t_1t_2t_3t_4=1$, we have

$$\frac{[t_1t_2][t_2t_3][t_1t_3][s]}{[t_1][t_2][t_3][t_4][s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}=\frac{(1-t_1t_2)(1-t_2t_3)(1-t_1t_3)}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)}\cdot\frac{[s]}{[s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}.$$

Recalling Definition 2.6, we have that

$$L=(1-t_1t_2)(1-t_2t_3)(1-t_1t_3)\frac{[s]}{[s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}$$

is a series in $q,y_1^{\pm\frac{1}{2}},\dots,y_N^{\pm\frac{1}{2}},v_1^{\pm\frac{1}{2}},\dots,v_r^{\pm\frac{1}{2}}$ whose coefficients are polynomials in t_1,t_2,t_3,t_4, as required. ☐

Lastly, we prove the claim made in the introduction that Conjecture 1.20 is a consequence of Conjecture 5.7 in the $X=\mathbb{C}^4$ case.

Proposition 5.9. *Let $X=\mathbb{C}^4$. If Conjecture 5.7 holds for some choice of signs, then Conjecture 1.20 holds for $Y=X$.*

In particular, we may retrieve the following well-known identity from Conjecture 5.7:

$$\sum_{n=0}^{\infty}q^n\int_{[\mathrm{Quot}_{\mathbb{C}^4}(E,n)]^{\mathrm{vir}}_{o(\mathcal{L})}}1:=\sum_{n=0}^{\infty}q^n\sum_{Z\in\mathrm{Quot}_X(E,n)^{\mathsf{T}}}(-1)^{o(\mathcal{L})|_Z}\frac{1}{e_{\mathsf{T}}\left(\sqrt{T_Z^{\mathrm{vir}}}\right)}$$
$$=\begin{cases} e^{\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)}q}, & \text{when} N=1\\ 1, & \text{otherwise.}\end{cases}$$

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Remark 5.10. One can compare this to the 3-fold case, where [FMR21, Theorem 7.2] states

$$\sum_{n=0}^{\infty} q^n \int_{[\mathrm{Quot}_{\mathbb{C}^3}(E,n)]^{\mathrm{vir}}} 1 = M((-1)^N q)^{-N\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3}}.$$

Here M denotes the MacMahon function.

Proof. We shall compute the following limit using both the definition and the expression from Conjecture 5.7, and then compare the two sides:

$$\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \mathcal{N}_X\left(E, V^\vee; \frac{Q}{w_N}\right)\Big|_{\lambda_i\rightsquigarrow\varepsilon\lambda_i, m_i\rightsquigarrow\varepsilon(1+m_i), w_i\rightsquigarrow\varepsilon w_i}.$$

Let $V=\oplus_{i=1}^N \mathcal{O}_X\langle v_i\rangle$ be a rank N bundle, then for any $Z_\pi\in \mathrm{Quot}_X(E,n)^{\mathsf{T}}$, we have

$$\begin{aligned}
&\frac{\mathrm{ch}_{\mathsf{T}}\left(\sqrt{K^{\mathrm{vir}}|_{Z_\pi}}^{\frac{1}{2}}\right)}{\mathrm{ch}_{\mathsf{T}}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}^{\vee}\right)}\mathrm{ch}_{\mathsf{T}}\left(\frac{\Lambda_{-1}}{\sqrt{\det}}V^{[n]}|_{Z_\pi}^{\vee}\right)\Big|_{\lambda_i\rightsquigarrow\varepsilon\lambda_i, m_i\rightsquigarrow\varepsilon(1+m_i), w_i\rightsquigarrow\varepsilon w_i}\\
&=\varepsilon^{Nn-Nn}\frac{e_{\mathsf{T}}(V^{[n]}|_{Z_\pi})+O(\varepsilon)}{e_{\mathsf{T}}\left(\sqrt{T^{\mathrm{vir}}_{Z_\pi}}\right)+O(\varepsilon)}\\
&=\frac{\prod_{i=1}^N\prod_{j=1}^N\prod_{(a,b,c,d)\in\pi^{(j)}}(1+w_i+m_j-a\lambda_1-b\lambda_2-c\lambda_3-d\lambda_4)+O(\varepsilon)}{e_{\mathsf{T}}\left(\sqrt{T^{\mathrm{vir}}_{Z_\pi}}\right)+O(\varepsilon)}.
\end{aligned}$$

Take the limit $\varepsilon\to 0$ and let $Q=m_N q$, then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\mathrm{ch}_{\mathsf{T}}\left(\sqrt{K^{\mathrm{vir}}|_{Z_\pi}}^{\frac{1}{2}}\right)}{\mathrm{ch}_{\mathsf{T}}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}^{\vee}\right)}\mathrm{ch}_{\mathsf{T}}\left(\frac{\Lambda_{-1}}{\sqrt{\det}}V^{[n]}|_{Z_\pi}\right)\Big|_{\lambda_i\rightsquigarrow\varepsilon\lambda_i, m_i\rightsquigarrow\varepsilon(1+m_i), w_i\rightsquigarrow\varepsilon w_i}\cdot q^n\\
&=\frac{\prod_{i=1}^N\prod_{j=1}^N\prod_{(a,b,c,d)\in\pi^{(j)}}(1+w_i+m_j-a\lambda_1-b\lambda_2-c\lambda_3-d\lambda_4)}{e_{\mathsf{T}}\left(\sqrt{T^{\mathrm{vir}}_{Z_\pi}}\right)}\cdot\frac{Q^n}{m_N^n}\\
&=\frac{\prod_{i=1}^{N-1}\prod_{j=1}^N\prod_{(a,b,c,d)\in\pi^{(j)}}(1+w_i+m_j-a\lambda_1-b\lambda_2-c\lambda_3-d\lambda_4)}{e_{\mathsf{T}}\left(\sqrt{T^{\mathrm{vir}}_{Z_\pi}}\right)}\\
&\cdot\prod_{j=1}^N\prod_{(a,b,c,d)\in\pi^{(j)}}\left(1+\frac{m_j}{w_N}-\frac{a\lambda_1}{w_N}-\frac{b\lambda_2}{w_N}-\frac{c\lambda_3}{w_N}-\frac{d\lambda_4}{w_N}\right)Q^n.
\end{aligned}$$

Now take $w_N\to\infty$ and substitute into (35). Let $V'=\oplus_{i=1}^{N-1}\mathcal{O}_X\langle v_i\rangle$, then

$$\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \mathcal{N}_X\left(E, V^\vee; \frac{Q}{w_N}\right)\Big|_{\lambda_i\rightsquigarrow\varepsilon\lambda_i, m_i\rightsquigarrow\varepsilon(1+m_i), w_i\rightsquigarrow\varepsilon w_i} = \mathcal{C}_X(E,V';Q). \tag{42}$$

On the other hand, we apply the same procedure to

$$\mathcal{N}_X(E,V;q)=\mathrm{Exp}\left(\frac{[t_1t_2][t_2t_3][t_1t_3]}{[t_1][t_2][t_3][t_4]}\frac{[s]}{[s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}\right).$$

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For $n \geqslant 1$, we have

$$\begin{aligned}
&\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \frac{[t_1^n t_2^n][t_2^n t_3^n][t_1^n t_3^n]}{[t_1^m][t_2^n][t_3^n][t_4^n]} \frac{[\prod y_i^n v_i^n]}{[\prod y_i^{\frac{n}{2}} v_i^{\frac{n}{2}} q^n][\prod y_i^{-\frac{n}{2}} v_i^{-\frac{n}{2}} q^n]}\Big|_{\lambda_i\rightsquigarrow\varepsilon\lambda_i, m_i\rightsquigarrow\varepsilon(1+m_i), w_i\rightsquigarrow\varepsilon w_i} \\
&= \lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \frac{(\varepsilon n)^3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)+O(\varepsilon^5)}{(\varepsilon n)^4\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)+O(\varepsilon^5)} \cdot \frac{(\varepsilon n)\sum_i(1+m_i+w_i)+O(\varepsilon)}{(q^{\frac{n}{2}}-q^{-\frac{n}{2}})^2} \\
&= \lim_{w_N\to\infty} \frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)} \cdot \frac{\sum_i(1+m_i+w_i)(\frac{Q}{w_N})^n}{(1-(\frac{Q}{w_N})^n)^2} \\
&= \begin{cases} \frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)}Q, & \text{when } n=1 \\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Note that the right-hand side is independent of the weights on V. Together with (42), we have

$$\begin{aligned}
\mathcal{C}_X(E,V';Q) &= e^{\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)}Q} \\
&= \exp\left(Q\int_X c_3(X)\right).
\end{aligned}$$

This is exactly Conjecture 1.20.

With the same method, we can take limits

$$\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \mathcal{N}_X\left(E,V;\frac{Q}{w_N w_{N-1}\dots w_{N-i+1}}\right)$$

for $1<i\leqslant N$ and V of rank $N-i$, and get

$$\mathcal{C}_X(E,V;Q)=1.$$

In particular, when $i=N$ and $N>1$, we have

$$\sum_{n=0}^{\infty} Q^n \int_{[\mathrm{Quot}_{\mathbb{C}^4}(E,n)]^{\mathrm{vir}}_{o(\mathcal{L})}} 1 = 1.$$

□

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Conflicts of Interest

None.

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Arkadij Bojko abojko@simis.cn
Department of Mathematics, ETH Zürich, Zürich, Switzerland

Jiahui Huang j346huan@uwaterloo.ca
Department of Mathematics, ETH Zürich, Zürich, Switzerland

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