PROPERTY OF HEREDITARILY LOCALLY CONNECTED CONTINUA RELATED TO ARCWISE ACCESSIBILITY

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Abstract
A continuum (that is, a compact connected Hausdorff space) is hereditarily locally connected if each of its subcontinua is locally connected. It is shown that a continuum $X$ is hereditarily locally connected if and only if for each connected open set $U$ in $X$ and each point $p$ in the boundary of $U$, $U \cup \{p\}$ is locally connected. This result is used to prove that if $X$ is an hereditarily locally connected continuum, $U$ is a connected open subset of $X$, $p$ is an element of the boundary of $U$ and $X$ is first countable at $p$, then $p$ is arcwise accessible from $U$.

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By a continuum we mean a compact connected Hausdorff space. A continuum is hereditarily locally connected if each of its subcontinua is locally connected. A family of subsets of a space $X$ is called a null family if for each two open sets $U$ and $V$ in $X$ with $\overline{U} \cap \overline{V} = \emptyset$, not more than a finite number of elements of the family meet both $U$ and $V$. If $\{F_\alpha \mid \alpha \in A\}$ is a family of disjoint subsets of $X$, then $\{F_\alpha \mid \alpha \in A\}$ is said to have property D if $F_\alpha \cap \overline{\bigcup\{F_\beta \mid \beta \neq \alpha\}} = \emptyset$ for all $\alpha \in A$. Simone (to appear) has shown that a continuum $X$ is hereditarily locally connected if and only if every family of disjoint continua in $X$ with property D is a null family. This characterization of hereditarily locally connected continua will be used in the proofs that follow.

Our first theorem is a generalization of a well-known metric theorem due to Whyburn (1942, p. 90).

**Theorem 1.** If $X$ is an hereditarily locally connected continuum then every family of disjoint connected open subsets of $X$ is a null family.

**Proof.** Since each two points in a connected open set $U$ in a locally connected continuum are contained in a continuum contained in $U$ (Hocking and Young, 1961, p. 110), it follows immediately from the above characterization of hereditarily locally connected continua that every family of disjoint connected open subsets of $X$ is a null family.

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If $X$ is a space and $A \subseteq B \subseteq X$, then we use the notation $\partial B$ to denote the boundary of $B$ in $X$, and the notation $\partial_B A$ to denote the boundary of $A$ in the subspace $B$. Before stating our main result, we prove the following lemma.

**Lemma 2.** Let $X$ be a connected regular space, and let $U$ be a connected proper open subset of $X$. If for each point $p$ in $\partial U$, $U \cup \{p\}$ is locally connected, then for each set $A$ such that $U \subseteq A \subseteq \bar{U}$, $A$ is locally connected.

**Proof.** Let $U \subseteq A \subseteq \bar{U}$, $p \in A$ and $W$ an $A$-open set containing $p$. Suppose that $p \in U$. Since $U$ is a proper subset of $X$ and $X$ is connected, there exists a point $x$ in $\partial U$. Then $U \cup \{x\}$ is locally connected. Now $W \cap U$ is an $X$-open set containing $p$ and contained in $U \cup \{x\}$. Therefore, there exists a connected $U \cup \{x\}$-open set $B$ such that $p \in B \subseteq W \cap U$. However, clearly $B \subseteq U \subseteq A$, and hence $B$ is $A$-open. Since $p \in B \subseteq W$, $A$ is locally connected at $p$.

Next, suppose that $p \in \partial U$. By hypothesis $U \cup \{p\}$ is locally connected. Let $W^* = W \cap (U \cup \{p\})$. $W^*$ is $U \cup \{p\}$-open and $p \in W^*$ and so there exists a connected $U \cup \{p\}$-open set $V^*$ such that $p \in V^* \subseteq W^*$. Let $W_1$ be an open set such that $W = W_1 \cap A$ and let $V_1$ be a $\bar{U}$-open set such that $V^* = V_1 \cup (U \cup \{p\})$. Finally, let $V = V_1 \cap W_1$. Then $V$ is $\bar{U}$-open and $V^* = V \cap (U \cup \{p\})$. Furthermore, it is clear that $p \in V \cap A \subseteq W$ and $V \cap A$ is $A$-open. We claim that $V \cap A$ is connected. Clearly $V^* \subseteq V \cap A$. We will show that $V \cap A \subseteq V^*$. Suppose that $y \in V - V^*$. Then $y \notin U$, and hence $y \notin \partial U$. Now $V$ is $\bar{U}$-open and $y \in V \cap (X - V^*)$. Since $X$ is regular, there exists a $\bar{U}$-open set $B$ such that $y \in B \subseteq V$ and $B \cap V^* = \emptyset$. However, since $y \in \partial U$ it is clear that $U \cup B \neq \emptyset$. Let $z \in U \cap B$. Then $z \in B \subseteq V$ and $z \in U$, and therefore

$$z \in V \cap U \subseteq V^*.$$  

Hence, $z \in B \cap V^*$ which is a contradiction. Therefore, $V \subseteq V^*$, and hence $V \cap A \subseteq V^*$. Since $V^*$ is connected and $V^* \subseteq V \cap A \subseteq V^*$, it follows that $V \cap A$ is connected. Therefore, $A$ is locally connected at $p$, and hence $A$ is locally connected.

**Theorem 3.** The continuum $X$ is hereditarily locally connected if and only if for each connected open set $U$ in $X$ and each point $p$ in $\partial U$, $U \cup \{p\}$ is locally connected.

**Proof.** Let $X$ be hereditarily locally connected, and suppose that there exist a connected open set $U$ in $X$ and a point $p$ in $\partial U$ such that $U \cup \{p\}$ is not locally connected. Since $U \cup \{p\}$ is obviously connected im kleinen at each point of $U$, it follows that $U \cup \{p\}$ is not connected im kleinen at $p$. Let $U_1 = U \cup \{p\}$. There exists a $U_1$-open set $W$ such that $p \in W$ and such that if $p \in B \subseteq W$ and $B$ is $U_1$-open, then the component of some point of $B$ in $W$ does not contain $p$. Let $V$ be a $U_1$-open set such that $p \in V$ and $C_1 V \subseteq W$, and let $\mathcal{W}$ denote the set of all $U_1$-open sets containing $p$ and contained in $V$. For each $B$ in $\mathcal{W}$, there exists a point $x_B$ in
B such that the component $K_B$ of $x_B$ in $W$ does not contain $p$. Now, for each $B$ in $\mathcal{U}$, $p \notin K_B$ and hence $K_B$ is a component of $W - \{p\}$. However, $W - \{p\}$ is $X$-open and $X$ is locally connected so each $K_B$ is open in $X$. Furthermore, since each $K_B$ is a component of $W$, each $K_B$ is $W$-closed.

Consider the net $\{K_B, B \in \mathcal{U}\}$, where $\mathcal{U}$ has been directed by reverse inclusion. Clearly, $p \in \lim \inf \{K_B, B \in \mathcal{U}\}$. We claim that $p \notin K_B$ for each $B$ in $\mathcal{U}$. For suppose that $p \in K_B$ for some $B$ in $\mathcal{U}$. Then $K_B \cup \{p\}$ is connected, contained in $W$ and properly contains $K_B$ which contradicts the fact that $K_B$ is a component of $W$. Since $p \notin K_B$ for each $B$ in $\mathcal{U}$, it follows that $\{K_B | B \in \mathcal{U}\}$ is infinite. Suppose that $K_B \cap \partial U_1 \neq \emptyset$ for each $B$ in $\mathcal{U}$. $V$ is $U_1$-open so there exists an open set $V^*$ such that $V = V^* \cap U_1$. Now, it follows immediately that $\partial U_1 \subset \partial V^*$ and therefore $K_B \cap \partial V^* \neq \emptyset$ for each $B$ in $\mathcal{U}$. Hence, there exists a point $y$ in $\lim \sup \{K_B, B \in \mathcal{U}\} \cap \partial V^*$.

Let $\{K_{B_a}, \alpha \in E\}$ be a convergent subnet of $\{K_B, B \in \mathcal{U}\}$ such that $y \in \lim \{K_{B_a}, \alpha \in E\}$ (Frolík, 1960, p. 173). Clearly, $p \in \lim \{K_{B_a}, \alpha \in E\}$. Recall now that $V^*$ is an open set containing $p$. Let $G_1$ and $G_2$ be open sets containing $p$ such that

$$p \in G_1 \subseteq G_2 \subseteq \overline{G_2} \subseteq V^*.$$

Then $G_1$ and $X - \overline{G_2}$ are open sets and $\overline{G_1} \cap (X - \overline{G_2}) = \emptyset$. Furthermore, $p \in G_1$ and $y \in X - \overline{G_2}$ and hence there exists an $\alpha_0$ in $E$ such that $K_{B_{\alpha}} \cap G_1 \neq \emptyset$ and $K_{B_{\alpha}} \cap (X - \overline{G_2}) \neq \emptyset$ for all $\alpha \geq \alpha_0$. Since $p \notin K_{B_{\alpha}}$ for all $\alpha$ in $E$, it follows, as before, that $\{K_{B_{\alpha}} | \alpha \geq \alpha_0\}$ is infinite. However, this contradicts Theorem 1.

Hence, there exists a $B_0$ in $\mathcal{U}$ such that $K_{B_0} \cap \partial U_1 \neq \emptyset$. Since $K_{B_0}$ is connected and $K_{B_0} \cap V \neq \emptyset$, it follows immediately that $K_{B_0} \subseteq \text{Cl}_{U_1} V$. Furthermore, since $\text{Cl}_{U_1} V \subseteq W$ we have that $K_{B_0}$ is a component of $\text{Cl}_{U_1} V$, and therefore, that $K_{B_0}$ is closed in $\text{Cl}_{U_1} V$. However, $\text{Cl}_{U_1} V$ is closed in $U_1$, and hence $K_{B_0}$ is closed in $U_1$. But $K_{B_0}$ is $X$-open, and therefore $K_{B_0}$ is open and closed in $U_1$ which contradicts the fact that $U_1$ is connected. We conclude that $U \cup \{p\}$ is locally connected.

Suppose, now, that for each connected open set $U$ in $X$ and each point $p$ in $\partial U$, $U \cup \{p\}$ is locally connected. We claim that $X$ is locally connected. For since $X$ is a continuum, $X$ contains a noncut point $p$. Then $X - \{p\}$ is a connected open set and $p \in \partial (X - \{p\})$, and hence $X = (X - \{p\}) \cup \{p\}$ is locally connected.

Assume that $X$ is not hereditarily locally connected. Then there exist an infinite family $\{G_n\}_{n=1}^{\infty}$ of disjoint sub continua of $X$ with property $D$, and open sets $W$ and $V$ such that $W \cap V = \emptyset$ and $G_n \cap W \neq \emptyset$ and $G_n \cap V \neq \emptyset$ for all $n$. Since $\{G_n\}_{n=1}^{\infty}$ has property $D$, it follows immediately that there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of disjoint open sets such that $G_n \subseteq U_n$ for each $n$. Now $\overline{W}$ is compact, and therefore there exists a point $p$ in $\overline{W} \cap \lim \sup G_n$. Let $W^*$ be a connected open set such that $p \in W^*$ and $\overline{W^*} \subseteq X - \overline{V}$. Since $U_n \cap U_m = \emptyset$ if $n \neq m$, it is clear that $p \notin G_n$ for each $n$, and therefore that infinitely many $G_n$ meet $W^*$. Let $\{G_n\}_{i=1}^{\infty}$ be the
subsequence of elements of $\{G_n\}_{n=1}^\infty$ which meet $W^*$. Now $X$ is locally connected, so for each $n$ there exists a connected open set $V_n$ such that $G_n \subseteq V_n$ and $\overline{V}_n \subseteq U_n$. Let

$$U = W^* \cup \bigcup_{i=1}^\infty V_n,$$

$U$ is a connected open set. Since $\overline{V}$ is compact and each $G_{n_i}$ meets $V$, there exists a point $q$ in $\overline{V} \cap \limsup G_{n_i}$. Clearly $q \notin W^*$. Furthermore, for each $i \neq j$, $V_{n_i} \cap G_{n_j} = \emptyset$, and hence $q \notin \bigcup_{i=1}^\infty V_{n_i}$. Therefore, $q \notin U$. By Lemma 2 and hypothesis, it follows that $\overline{U}$ is a locally connected continuum.

Now, it is easily shown that

$$\mathrm{Cl} \left( \bigcup_{i=1}^\infty V_{n_i} \right) = \bigcup_{i=1}^\infty \overline{V}_n \cup \limsup \overline{V}_{n_i},$$

and hence

$$\overline{U} = \bigcup_{i=1}^\infty \overline{V}_n \cup \limsup \overline{V}_{n_i}.$$
The point \( p \) is said to be **arcwise accessible from** \( A \) if for each point \( x \) in \( A \) there exists an arc \( K \) from \( x \) to \( p \) such that \( K \subseteq A \cup \{p\} \). Closely related to Theorem 3 is the following question.

**Question.** If \( X \) is an hereditary locally connected continuum and \( U \) is a connected open subset of \( X \), then is each point \( p \) in \( \partial U \) arcwise accessible from \( U \)?

Although this general problem remains open, by assuming that \( X \) is first countable at the point in question, we can answer the above problem in the affirmative for a large class of hereditarily locally connected continua.

**Theorem 5.** If \( X \) is an hereditarily locally connected continuum, \( U \) is a connected open subset of \( X \), \( p \in \partial U \) and \( X \) is first countable at \( p \), then \( p \) is arcwise accessible from \( U \).

**Proof.** Let \( x \in U \) and let \( U_1 = U \cup \{p\} \). By Theorem 3, \( U_1 \) is locally connected, and hence there exists a decreasing countable base \( \{B_n\}_{n=1}^\infty \) at \( p \) of connected \( U \)-open sets. Since \( U_1 \) is locally connected and \( B_n \) is \( U_1 \)-open, it follows that each \( B_n \) is locally connected.

Let \( x_1 \in B_1 - \{p\} \). Now \( x_1 \in U \) and hence there exists a continuum \( K_1 \) in \( U \) containing \( x_1 \) and \( x \). Let \( V_1 \) be the component of \( x_1 \) in \( B_1 - \{p\} \). Since \( B_1 \) is connected and locally connected, it follows that \( p \in \text{Cl}_{B_1} V_1 \) (Kuratowski, 1968, Theorem 19, p. 235). Thus \( p \in \text{Cl}_{U_1} V_1 \). However, \( B_2 \) is a \( U_1 \)-open set containing \( p \) and therefore \( V_1 \cap B_2 \neq \emptyset \). Let \( x_2 \in V_1 \cap B_2 \). Now \( V_1 \) is open in \( U_1 \). However, \( p \notin V_1 \) and therefore \( V_1 \subseteq U \). Hence, \( V_1 \) is a connected open set. Therefore, as before, there exists a continuum \( K_2 \) in \( V_1 \) containing \( x_1 \) and \( x_2 \). Let \( V_2 \) be the component of \( x_2 \) in \( B_2 - \{p\} \). As above, \( p \in \text{Cl}_{U_1} V_2 \) and hence there exists a point \( x_3 \in V_2 \cap B_3 \). Again, \( V_2 \) is a connected \( X \)-open set and therefore, there exists a continuum \( K_3 \) in \( V_2 \) containing \( x_2 \) and \( x_3 \).

Continue in this way by induction. Let \( K^* = \text{Cl} \bigcup_{n=1}^\infty K_n \). \( K^* \) is a continuum, and clearly \( x \) and \( p \) are in \( K^* \). Furthermore, since for each \( n \), \( K_{n+1} \subseteq V_n \subseteq B_n \), it is obvious that \( K^* \subseteq U \cup \{p\} \). Let \( K \) be an irreducible continuum in \( K^* \) from \( x \) to \( p \). It is well known that every such irreducible continuum in an hereditarily locally connected continuum is an arc, and hence \( K \) is an arc from \( x \) to \( p \). Since \( K \subseteq U \cup \{p\} \), the proof is complete.

**References**


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