PROPERTY OF HEREDITARILY LOCALLY CONNECTED CONTINUA RELATED TO ARCWISE ACCESSIBILITY

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Abstract

A continuum (that is, a compact connected Hausdorff space) is hereditarily locally connected if each of its subcontinua is locally connected. It is shown that a continuum \( X \) is hereditarily locally connected if and only if for each connected open set \( U \) in \( X \) and each point \( p \) in the boundary of \( U \), \( U \cup \{p\} \) is locally connected. This result is used to prove that if \( X \) is an hereditarily locally connected continuum, \( U \) is a connected open subset of \( X \), \( p \) is an element of the boundary of \( U \) and \( X \) is first countable at \( p \), then \( p \) is arcwise accessible from \( U \).

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By a continuum we mean a compact connected Hausdorff space. A continuum is hereditarily locally connected if each of its subcontinua is locally connected. A family of subsets of a space \( X \) is called a null family if for each two open sets \( U \) and \( V \) in \( X \) with \( U \cap V = \emptyset \), not more than a finite number of elements of the family meet both \( U \) and \( V \). If \( \{F_\alpha | \alpha \in A\} \) is a family of disjoint subsets of \( X \), then \( \{F_\alpha | \alpha \in A\} \) is said to have property \( D \) if \( F_\alpha \cap \text{Cl} \bigcup \{F_\beta | \beta \neq \alpha\} = \emptyset \) for all \( \alpha \) in \( A \). Simone (to appear) has shown that a continuum \( X \) is hereditarily locally connected if and only if every family of disjoint continua in \( X \) with property \( D \) is a null family. This characterization of hereditarily locally connected continua will be used in the proofs that follow.

Our first theorem is a generalization of a well-known metric theorem due to Whyburn (1942, p. 90).

**Theorem 1.** If \( X \) is an hereditarily locally connected continuum, then every family of disjoint connected open subsets of \( X \) is a null family.

**Proof.** Since each two points in a connected open set \( U \) in a locally connected continuum are contained in a continuum contained in \( U \) (Hocking and Young, 1961, p. 110), it follows immediately from the above characterization of hereditarily locally connected continua that every family of disjoint connected open subsets of \( X \) is a null family.
If $X$ is a space and $A \subseteq B \subseteq X$, then we use the notation $\partial B$ to denote the boundary of $B$ in $X$, and the notation $\partial_B A$ to denote the boundary of $A$ in the subspace $B$. Before stating our main result, we prove the following lemma.

**Lemma 2.** Let $X$ be a connected regular space, and let $U$ be a connected proper open subset of $X$. If for each point $p$ in $\partial U$, $U \cup \{p\}$ is locally connected, then for each set $A$ such that $U \subseteq A \subseteq \overline{U}$, $A$ is locally connected.

**Proof.** Let $U \subseteq A \subseteq \overline{U}$, $p \in A$ and $W$ an $A$-open set containing $p$. Suppose that $p \in U$. Since $U$ is a proper subset of $X$ and $X$ is connected, there exists a point $x$ in $\partial U$. Then $U \cup \{x\}$ is locally connected. Now $W \cap U$ is an $X$-open set containing $p$ and contained in $U \cup \{x\}$. Therefore, there exists a connected $U \cup \{x\}$-open set $B$ such that $p \in B \subseteq W \cap U$. However, clearly $B \subseteq U \subseteq A$, and hence $B$ is $A$-open. Since $p \in B \subseteq W$, $A$ is locally connected at $p$.

Next, suppose that $p \in \partial U$. By hypothesis $U \cup \{p\}$ is locally connected. Let $W^* = W \cap (U \cup \{p\})$, $W^*$ is $U \cup \{p\}$-open and $p \in W^*$ and so there exists a connected $U \cup \{p\}$-open set $V^*$ such that $p \in V^* \subseteq W^*$. Let $V_1$ be an open set such that $W = V_1 \cap A$ and let $V_2$ be a $\overline{U}$-open set such that $V^* = V_1 \cap (U \cup \{p\})$. Finally, let $V = V_1 \cap W_1$. Then $V$ is $\overline{U}$-open and $V^* = V \cap (U \cup \{p\})$. Furthermore, it is clear that $p \in V \cap A \subseteq W$ and $V \cap A$ is $A$-open. We claim that $V \cap A$ is connected. Clearly $V^* \subseteq V \cap A$. We will show that $V \cap A \subseteq V^*$. Suppose that $y \in V - V^*$. Then $y \not\in U$, and hence $y \in \partial U$. Now $V$ is $\overline{U}$-open and $y \in V \cap (X - V^*)$. Since $X$ is regular, there exists a $\overline{U}$-open set $B$ such that $y \in B \subseteq V$ and $B \cap V^* = \emptyset$. However, since $y \in \partial U$ it is clear that $U \cap B \neq \emptyset$. Let $z \in U \cap B$. Then $z \in B \subseteq V$ and $z \in U$, and therefore

$$z \in V \cap U \subseteq V^*.$$  

Hence, $z \in B \cap V^*$ which is a contradiction. Therefore, $V \subseteq V^*$, and hence $V \cap A \subseteq V^*$. Since $V^*$ is connected and $V^* \subseteq V \cap A \subseteq V^*$, it follows that $V \cap A$ is connected. Therefore, $A$ is locally connected at $p$, and hence $A$ is locally connected.

**Theorem 3.** The continuum $X$ is hereditarily locally connected if and only if for each connected open set $U$ in $X$ and each point $p$ in $\partial U$, $U \cup \{p\}$ is locally connected.

**Proof.** Let $X$ be hereditarily locally connected, and suppose that there exist a connected open set $U$ in $X$ and a point $p$ in $\partial U$ such that $U \cup \{p\}$ is not locally connected. Since $U \cup \{p\}$ is obviously connected in the klein at each point of $U$, it follows that $U \cup \{p\}$ is not connected in the klein at $p$. Let $U_1 = U \cup \{p\}$. There exists a $U_i$-open set $W$ such that $p \in W$ and such that if $p \in B \subseteq W$ and $B$ is $U_i$-open, then the component of some point of $B$ in $W$ does not contain $p$. Let $V$ be a $U_i$-open set such that $p \in V$ and $C_i V \subseteq W$, and let $V$ denote the set of all $U_i$-open sets containing $p$ and contained in $V$. For each $B$ in $\mathcal{W}$, there exists a point $x_B$ in

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B such that the component $K_B$ of $x_B$ in $W$ does not contain $p$. Now, for each $B$ in $\mathcal{U}$, $p \notin K_B$ and hence $K_B$ is a component of $W - \{p\}$. However, $W - \{p\}$ is $X$-open and $X$ is locally connected so each $K_B$ is open in $X$. Furthermore, since each $K_B$ is a component of $W$, each $K_B$ is $W$-closed.

Consider the net $\{K_B, B \in \mathcal{U}\}$, where $\mathcal{U}$ has been directed by reverse inclusion. Clearly, $p \in \liminf \{K_B, B \in \mathcal{U}\}$. We claim that $p \notin \overline{K_B}$ for each $B$ in $\mathcal{U}$. For suppose that $p \in \overline{K_B}$ for some $B$ in $\mathcal{U}$. Then $K_B \cup \{p\}$ is connected, contained in $W$ and properly contains $K_B$ which contradicts the fact that $K_B$ is a component of $W$. Therefore, $p \notin \overline{K_B}$ for each $B$ in $\mathcal{U}$, it follows that $\{K_B | B \in \mathcal{U}\}$ is infinite. Suppose that $K_B \cap \partial U_1 \neq \emptyset$ for each $B$ in $\mathcal{U}$. Now, it follows immediately that $\partial U_1 \subseteq \partial V^*$ and therefore $K_B \cap \partial V^* \neq \emptyset$ for each $B$ in $\mathcal{U}$. Hence, there exists a point $y$ in

$$\limsup \{K_B, B \in \mathcal{U}\} \cap \partial V^*.$$

Let $\{K_{B\alpha}, \alpha \in E\}$ be a convergent subnet of $\{K_B, B \in \mathcal{U}\}$ such that $y \in \lim \{K_{B\alpha}, \alpha \in E\}$ (Frolík, 1960, p. 173). Clearly, $p \in \lim \{K_{B\alpha}, \alpha \in E\}$. Recall now that $V^*$ is an open set containing $p$. Let $G_1$ and $G_2$ be open sets containing $p$ such that

$$p \in G_1 \subseteq \overline{G_1} \subseteq G_2 \subseteq \overline{G_2} \subseteq V^*.$$

Then $G_1$ and $X - \overline{G_2}$ are open sets and $\overline{G_1} \cap (X - \overline{G_2}) = \emptyset$. Furthermore, $p \in G_1$ and $y \in X - \overline{G_2}$ and hence there exists an $\alpha_0$ in $E$ such that $K_{B\alpha} \cap G_1 \neq \emptyset$ and $K_{B\alpha} \cap (X - \overline{G_2}) \neq \emptyset$ for all $\alpha \geq \alpha_0$. Since $p \notin \overline{K_{B\alpha}}$ for all $\alpha$ in $E$, it follows, as before, that $\{K_{B\alpha} | \alpha \geq \alpha_0\}$ is infinite. However, this contradicts Theorem 1.

Hence, there exists a $B_0$ in $\mathcal{U}$ such that $K_{B_0} \cap \partial U_1 = \emptyset$. Since $K_{B_0}$ is connected and $K_{B_0} \cap V \neq \emptyset$, it follows immediately that $K_{B_0} \subseteq \text{Cl} U_1, V$. Furthermore, since $\text{Cl} U_1 \subseteq W$ we have that $K_{B_0}$ is a component of $\text{Cl} U_1, V$, and therefore, that $K_{B_0}$ is closed in $\text{Cl} U_1 V$. However, $\text{Cl} U_1 V$ is closed in $U_1$, and hence $K_{B_0}$ is closed in $U_1$.

But $K_{B_0}$ is $X$-open, and therefore $K_{B_0}$ is open and closed in $U_1$ which contradicts the fact that $U_1$ is connected. We conclude that $U \cup \{p\}$ is locally connected.

Suppose, now, that for each connected open set $U$ in $X$ and each point $p$ in $\partial U, U \cup \{p\}$ is locally connected. We claim that $X$ is locally connected. For since $X$ is a continuum, $X$ contains a noncut point $p$. Then $X - \{p\}$ is a connected open set and $p \in \partial (X - \{p\})$, and hence $X = (X - \{p\}) \cup \{p\}$ is locally connected.

Assume that $X$ is not hereditarily locally connected. Then there exist an infinite family $\{G_n\}_{n=1}^\infty$ of disjoint subcontinua of $X$ with property $D_1$ and open sets $W$ and $V$ such that $\overline{W} \cap V = \emptyset$ and $G_n \cap W \neq \emptyset$ and $G_n \cap V \neq \emptyset$ for all $n$. Since $\{G_n\}_{n=1}^\infty$ has property $D_1$ it follows immediately that there exists a sequence $\{U_n\}_{n=1}^\infty$ of disjoint open sets such that $G_n \subseteq U_n$ for each $n$. Now $\overline{W}$ is compact, and therefore there exists a point $p$ in $\overline{W} \cap \limsup G_n$. Let $W^*$ be a connected open set such that $p \in W^*$ and $\overline{W^*} \subseteq X - \overline{V}$. Since $U_n \cap U_m = \emptyset$ if $n \neq m$, it is clear that $p \notin G_n$ for each $n$, and therefore that infinitely many $G_n$ meet $W^*$. Let $\{G_n\}_{i=1}^\infty$ be the
subsequence of elements of \( \{G_n\}_{n=1}^{\infty} \) which meet \( \mathcal{W}^* \). Now \( X \) is locally connected, so for each \( n \) there exists a connected open set \( V_n \) such that \( G_n \subseteq V_n \) and \( \overline{V}_n \subseteq U_n \). Let

\[
U = \mathcal{W}^* \cup \bigcup_{i=1}^{\infty} V_n.
\]

\( U \) is a connected open set. Since \( \overline{V} \) is compact and each \( G_n \) meets \( V \), there exists a point \( q \) in \( \overline{V} \cap \limsup G_n \). Clearly \( q \notin \mathcal{W}^* \). Furthermore, for each \( i \neq j \), \( V_n \cap G_j = \emptyset \), and hence \( q \notin \bigcup_{i=1}^{\infty} V_n \). Therefore, \( q \notin U \). By Lemma 2 and hypothesis, it follows that \( U \) is a locally connected continuum.

Now, it is easily shown that

\[
\text{Cl}\left( \bigcup_{i=1}^{\infty} V_n \right) = \bigcup_{i=1}^{\infty} \overline{V}_n \cup \limsup \overline{V}_n,
\]

and hence

\[
\overline{U} = \bigcup_{i=1}^{\infty} \overline{V}_n \cup \limsup \overline{V}_n \cup \mathcal{W}^*.
\]

However, each \( \overline{V}_n \) is contained in \( U_n \) and \( \{U_n\}_{n=1}^{\infty} \) is a sequence of disjoint open sets, and hence

\[
\overline{V}_n \cap \limsup \overline{V}_n = \emptyset
\]

for each \( j \).

Now \( q \in \overline{U} - \mathcal{W}^* \) and \( \overline{U} \) is locally connected, so there exists a connected \( \overline{U} \)-open set \( B \) such that \( q \in B \) and \( B \subseteq \overline{U} - \mathcal{W}^* \). But then,

\[
B = (B \cap \limsup \overline{V}_n) \cup \bigcup_{i=1}^{\infty} B \cap \overline{V}_n.
\]

However, \( q \in B \cap \limsup \overline{V}_n \) and since \( B \) is a \( \overline{U} \)-open set containing \( q \), \( B \cap \overline{V}_n \neq \emptyset \) for infinitely many \( i \). Therefore, \( B \) can be written as a nondegenerate countable union of disjoint compact sets which contradicts the fact that \( B \) is a continuum. Hence, \( X \) is hereditarily locally connected.

Tymchatyn (to appear) has recently obtained some characterizations of hereditarily locally connected continua related to the above theorem. By combining Lemma 2 and Theorem 3, we immediately obtain the following corollary which generalizes the equivalence of parts (1) and (3) of Theorem 2.2 of Nishiura and Tymchatyn (1976, p. 586) to nonmetric continua.

**Corollary 4.** The continuum \( X \) is hereditarily locally connected if and only if for each connected open set \( U \) in \( X \) and each set \( A \) such that \( U \subseteq A \subseteq \overline{U} \), \( A \) is locally connected.

An arc is a continuum with exactly two noncut points. If \( K \) is an arc with noncut points \( p \) and \( q \), then \( K \) is called an arc from \( p \) to \( q \). Let \( X \) be a space, \( A \subseteq X \) and
p \in X - A. The point p is said to be arcwise accessible from A if for each point x in A there exists an arc K from x to p such that K \subseteq A \cup \{p\}. Closely related to Theorem 3 is the following question.

**Question.** If X is an hereditary locally connected continuum and U is a connected open subset of X, then is each point p in \( \partial U \) arcwise accessible from U?

Although this general problem remains open, by assuming that X is first countable at the point in question, we can answer the above problem in the affirmative for a large class of hereditarily locally connected continua.

**Theorem 5.** If X is an hereditarily locally connected continuum, U is a connected open subset of X, p \in \partial U and X is first countable at p, then p is arcwise accessible from U.

**Proof.** Let x \in U and let \( U_1 = U \cup \{p\} \). By Theorem 3, \( U_1 \) is locally connected, and hence there exists a decreasing countable base \( \{B_n\}_{n=1}^\infty \) at p of connected \( U_1 \)-open sets. Since \( U_1 \) is locally connected and \( B_n \) is \( U_1 \)-open, it follows that each \( B_n \) is locally connected.

Let \( x_1 \in B_1 - \{p\} \). Now there exists a continuum \( K_1 \) in \( U \) containing x and \( x_1 \). Let \( V_1 \) be the component of \( x_1 \) in \( B_1 - \{p\} \). Since \( B_1 \) is connected and locally connected, it follows that \( p \in \text{Cl}_{B_1} V_1 \) (Kuratowski, 1968, Theorem 19, p. 235). Thus \( p \in \text{Cl}_{U_1} V_1 \). However, \( B_2 \) is a \( U_1 \)-open set containing p and therefore \( V_1 \cap B_2 \neq \emptyset \). Let \( x_2 \in V_1 \cap B_2 \). Now \( V_1 \) is open in \( U_1 \). However, \( p \notin V_1 \) and therefore \( V_1 \subseteq U_1 \). Hence, \( V_1 \) is a connected open set. Therefore, as before, there exists a continuum \( K_2 \) in \( V_1 \) containing \( x_1 \) and \( x_2 \). Let \( V_2 \) be the component of \( x_2 \) in \( B_2 - \{p\} \). As above, \( p \in \text{Cl}_{U_1} V_2 \) and hence there exists a point \( x_3 \in V_2 \cap B_2 \). Again, \( V_2 \) is a connected \( X \)-open set and therefore, there exists a continuum \( K_3 \) in \( V_2 \) containing \( x_2 \) and \( x_3 \).

Continue in this way by induction. Let \( K^* = \text{Cl} \bigcup_{n=1}^\infty K_n \). \( K^* \) is a continuum, and clearly x and p are in \( K^* \). Furthermore, since for each n, \( K_{n+1} \subseteq V_n \subseteq B_n \), it is obvious that \( K^* \subseteq U \cup \{p\} \). Let K be an irreducible continuum in \( K^* \) from x to p. It is well known that every such irreducible continuum in an hereditarily locally connected continuum is an arc, and hence K is an arc from x to p. Since \( K \subseteq U \cup \{p\} \), the proof is complete.

**References**


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