Proceedings of the Edinburgh Mathematical Society (1996) 39, 505–523 ©

BOUNDARY VALUE PROBLEMS SINGULAR IN THE SOLUTION VARIABLE WITH NONLINEAR BOUNDARY DATA

by DONAL O'REGAN

(Received 3rd November 1994)

Existence results are established for the equation $y''+f(t,y)=0$, $0<t<1$. Here f may be singular in y and f is allowed to change sign. Our boundary data include $y(0)=y'(1)+ky(1)=0$, $k>-1$ and $y(0)=y'(1)+cy^4(1)=0$, $c>0$.

1991 *Mathematics subject classification*: 34B15.

1. Introduction

This paper discusses problems of the form

$$\begin{cases} y''+f(t,y)=0,\ 0<t<1 \\ y(0)=0 \\ y'(1)+\mu\psi(y(1))=0,\ \mu\geqq 0 \text{ a constant} \end{cases} \tag{1.1}$$

where f is not a Carathéodory function due to the singular behavior of its y variable. Here ψ may be nonlinear and includes for example the Sturm Liouville boundary condition $y'(1)+ky(1)=0$, $k>-1$ and Stefan's condition $y'(1)+cy^4(1)=0$, $c>0$. Also our nonlinearity f is allowed to change sign.

Our study is motivated by the problem

$$\begin{cases} y''+\left(\dfrac{t^2}{32y^2}-\dfrac{\kappa^2}{8}\right)=0,\ 0<t<1 \\ y(0)=0 \\ 2y'(1)-(1+\nu)y(1)=0,\ \kappa>0 \text{ and } 0<\nu<1 \end{cases} \tag{1.2}$$

which arises in nonlinear mechanics; see **[1, 9]** and their references. The problem models the stress in the spherical membrane of a spherical cap.

The literature **[2–4, 7–8, 10–12]** on singular problems of the above type is almost totally devoted to the Dirichlet problem

$$\begin{cases} y''+f(t,y)=0,\ 0<t<1 \\ y(0)=0=y(1), \end{cases}$$

usually when $f(t,y)\geqq 0$ for $t\in(0,1)$ and $y>0$. Very little seems to be known concerning the class of problems (1.1), which includes (1.2). In this paper we obtain a general existence theory for problems of the form (1.1).

The analysis used throughout rely on fixed point methods. We state, for convenience, the two fixed point theorems we will use.

Theorem 1.1. (Schauder [11]). *Let K be a convex subset of a normed linear space E. Then every compact map $F: K\to K$ has at least one fixed point.*

Theorem 1.2. (Nonlinear Alternative [5, 11]). *Assume U is a relatively open subset of a convex set K in a normed linear space E. Let $N:\bar{U}\to K$ be a compact map with $p\in U$. Then either*

(i) *N has a fixed point in $\bar{U}$; or*

(ii) *there is a $u\in\partial U$ and a $\lambda\in(0,1)$ such that $u=\lambda Nu+(1-\lambda)p$.*

Remark. By a map being *compact* we mean it is continuous with relatively compact range. For later purposes, a map is *completely continuous* if it is continuous and the image of every bounded set in the domain is contained in a compact set in the range.

2. Existence

Several existence results are presented for the singular problem

$$\begin{cases} y''+f(t,y)=0,\ 0<t<1 \\ y(0)=0 \\ y'(1)+\mu\psi(y(1))=0,\ \mu\geqq 0 \text{ a constant.} \end{cases} \tag{2.1}$$

Our first two results were motivated by the boundary value problem (1.2); in particular by the boundary condition $2y'(1)-(1+v)y(1)=0$. By a solution to (2.1) we mean a function $y\in C[0,1]\cap C^1(0,1]\cap C^2(0,1)$ which satisfies the differential equation on $(0,1)$ and the stated boundary data.

Theorem 2.1. *Suppose the following conditions are satisfied:*

$$f:(0,1)\times(0,\infty)\to\mathbf{R} \text{ is continuous} \tag{2.2}$$

$$\psi:\mathbf{R}\to\mathbf{R} \text{ is continuous with } \psi(x)\leqq 0 \text{ for } x\geqq 0 \tag{2.3}$$

$$\begin{cases} |f(t,y)| \leqq q_1(t)g(y) + q_2(t)h(y) \textit{ on } (0,1)\times(0,\infty) \textit{ with } g>0 \\ \textit{continuous and nonincreasing on } (0,\infty),\ h \geqq 0 \textit{ continuous} \\ \textit{on } [0,\infty) \textit{ and } \dfrac{h}{g} \textit{ nondecreasing on } (0,\infty); \textit{ here } q_i \in C(0,1), \\ i=1,2 \textit{ with } q_i>0 \textit{ on } (0,1) \textit{ and } \int_0^1 q_i(x)\,dx<\infty \end{cases} \tag{2.4}$$

$$\begin{cases} \textit{let } n\in\{3,4,\ldots\} \textit{ and associated with each } n \textit{ we have a constant} \\ \rho_n \textit{ such that } \{\rho_n\} \textit{ is a nonincreasing sequence with } \lim_{n\to\infty}\rho_n=0 \\ \textit{and such that for } \dfrac{1}{n} \leqq t \leqq 1-\dfrac{1}{n} \textit{ we have } f(t,\rho_n)\geqq 0 \end{cases} \tag{2.5}$$

$$\begin{cases} \textit{there exists a function } \alpha \in C[0,1]\cap C^1(0,1]\cap C^2(0,1) \textit{ with} \\ \alpha(0)=\alpha'(1)+\mu\psi(\alpha(1))=0,\ \alpha>0 \textit{ on } (0,1) \textit{ such that} \\ f(t,y)+\alpha''(t)>0 \textit{ for } (t,y)\in(0,1)\times\{y\in(0,\infty): y<\alpha(t)\} \end{cases} \tag{2.6}$$

$$\int_{1/2}^{1} q_i(x)g(\alpha(x))\,dx<\infty, \quad u=1,2 \tag{2.7}$$

$$\begin{cases} \textit{for any } R>0,\ \dfrac{1}{g} \textit{ is differentiable on } (0,R] \textit{ with } g'<0 \textit{ a.e.} \\ \textit{and } \dfrac{g'}{g^2}\in L^1[0,R]; \textit{ in addition } \displaystyle\int_0^\infty \frac{|g'(t)|^{1/2}}{g(t)}\,dt=\infty \end{cases} \tag{2.8}$$

and

$$\begin{cases} \textit{there exists a constant } M>0 \textit{ such that for } z>0, \\ \displaystyle\int_0^z \frac{du}{g(u)} \leqq \frac{[\rho_3+\mu|\psi^*(z)|]}{g(z)} + \int_0^1 xq_1(x)\,dx + \frac{h(z)}{g(z)}\int_0^1 xq_2(x)\,dx + \int_0^{\rho_3}\frac{du}{g(u)} \\ \textit{implies } z\leqq M; \end{cases} \tag{2.9}$$

here

$$\psi^*(z)=\begin{cases}\psi(z), z\geqq\alpha(1) \\ \psi(\alpha(1)), z<\alpha(1).\end{cases}$$

Then (2.1) *has a solution in* $C[0,1]\cap C^1(0,1]\cap C^2(0,1)$.

Remark. Note $\psi^*(z)\leqq 0$ for $z\in\mathbf{R}$.

Proof. Fix $n\in\{3,4,\ldots\}$. We begin by showing that

$$\begin{cases} y''+f(t,y)=0, 0<t<1 \\ y(0)=\rho_n \\ y'(1)+\mu\psi(y(1))=\rho_n \end{cases} \tag{2.10^n}$$

has a solution in $C^1[0,1]\cap C^2(0,1)$. To show $(2.10)^n$ has a solution we consider the family of problems

$$\begin{cases} y''+\lambda f^*(t,y)=0, 0<t<1, 0<\lambda<1 \\ y(0)=\rho_n \\ y'(1)+\lambda\mu\psi^*(y(1))=\rho_n \end{cases} \tag{2.11$^n_\lambda$}$$

where

$$f^*(t,y)=\begin{cases} f(t,y), y\geqq\rho_n \\ f(t,\rho_n)+\rho_n-y, y<\rho_n \text{ and } \frac{1}{n}\leqq t\leqq 1-\frac{1}{n} \\ f\left(\frac{1}{n},\rho_n\right)+\rho_n-y, y<\rho_n \text{ and } 0\leqq t\leqq\frac{1}{n} \\ f\left(1-\frac{1}{n},\rho_n\right)+\rho_n-y, y<\rho_n \text{ and } 1-\frac{1}{n}\leqq t\leqq 1. \end{cases}$$

We first show that

$$y(t)\geqq\rho_n, \quad t\in[0,1] \tag{2.12}$$

for any solution $y\in C^1[0,1]\cap C^2(0,1)$ to $(2.11)^n_\lambda$. To see this suppose $y-\rho_n$ has a negative minimum at $t_0\in(0,1]$. If $t_0\in(0,1)$ then $y'(t_0)=0$ and $y''(t_0)\geqq 0$. However

$$y''(t_0)=-\lambda f^*(t_0,y(t_0))=\begin{cases} -\lambda[f(t_0,\rho_n)+\rho_n-y(t_0)] & \text{if } \frac{1}{n}\leqq t_0\leqq 1-\frac{1}{n} \\ -\lambda\left[f\left(\frac{1}{n},\rho_n\right)+\rho_n-y(t_0)\right] & \text{if } 0\leqq t_0\leqq\frac{1}{n} \\ -\lambda\left[f\left(1-\frac{1}{n},\rho_n\right)+\rho_n-y(t_0)\right] & \text{if } 1-\frac{1}{n}\leqq t_0\leqq 1 \end{cases}$$

i.e. $y''(t_0)<0$, a contradiction. It remains to consider the case $t_0=1$. Then $y'(1)\leqq 0$. However

$$y'(1)=\rho_n-\lambda\mu\psi^*(y(1))>0,$$

a contradiction. Thus (2.12) holds.

Suppose the absolute maximum of y occurs at say t_n. Note we take $t_n\in(0,1]$. There are two cases to consider, namely $t_n\in(0,1)$ and $t_n=1$. Notice for $x\in(0,1)$ we have

$$\frac{-y''(x)}{g(y(x))} \leqq q_1(x) + q_2(x)\frac{h(y(x))}{g(y(x))}. \tag{2.13}$$

Case (*i*) $t_n \in (0,1)$.

Then $y'(t_n)=0$. Integrate (2.13) from $t(t<t_n)$ to t_n to obtain

$$\frac{y'(t)}{g(y(t))} + \int_t^{t_n} \left\{\frac{-g'(y(x))}{g^2(y(x))}\right\}[y'(x)]^2\,dx \leqq \int_t^{t_n} q_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\int_t^{t_n} q_2(x)\,dx,$$

and so

$$\frac{y'(t)}{g(y(t))} \leqq \int_t^{t_n} q_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\int_t^{t_n} q_2(x)\,dx.$$

Integrate from 0 to t_n to obtain

$$\int_0^{y(t_n)} \frac{du}{g(u)} \leqq \int_0^1 xq_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\int_0^1 xq_2(x)\,dx + \int_0^{\rho_3}\frac{du}{g(u)}.$$

Consequently (2.9) implies

$$y(t_n) \leqq M. \tag{2.14)*}$$

Case (*ii*) $t_n = 1$.

Now since $y'(1) = \rho_n - \lambda\mu\psi^*(y(1))$ we have

$$|y'(1)| \leqq \rho_3 + \mu|\psi^*(y(1))|. \tag{2.15}$$

Integrate (2.13) from t to 1 to obtain

$$\frac{y'(t)}{g(y(t))} - \frac{y'(1)}{g(y(1))} \leqq \int_t^1 q_1(x)\,dx + \frac{h(y(1))}{g(y(1))}\int_t^1 q_2(x)\,dx,$$

and so

$$\frac{y'(t)}{g(y(t))} \leqq \frac{[\rho_3 + \mu|\psi^*(y(1))|]}{g(y(1))} + \int_t^1 q_1(x)\,dx + \frac{h(y(1))}{g(y(1))}\int_t^1 q_2(x)\,dx.$$

Integrate from 0 to 1 to obtain

$$\int_0^{y(t_n)} \frac{du}{g(u)} \leqq \frac{[\rho_3 + \mu|\psi^*(y(1))|]}{g(y(1))} + \int_0^1 xq_1(x)\,dx + \frac{h(y(1))}{g(y(1))}\int_0^1 xq_2(x)\,dx + \int_0^{\rho_3}\frac{du}{g(u)}.$$

Consequently (2.9) implies

$$y(t_n) = y(1) \leqq M. \tag{2.14)**}$$

https://doi.org/10.1017/S0013091500023269 Published online by Cambridge University Press

Thus in both cases

$$\rho_n \leqq y(t) \leqq M \quad \text{for} \quad t \in [0,1]. \tag{2.16}$$

Also the mean value theorem implies that there exists $\tau \in (0,1)$ with $|y'(\tau)| = |y(1) - y(0)| \leqq 2M$. For $t \in [0,1]$ we have

$$|y'(t)| \leqq |y'(\tau)| + \left| \int_\tau^t \right| f^*(x, y(x)) \, |dx|$$

$$\leqq 2M + g(\rho_n) \int_0^1 \left[q_1(x) + q_2(x) \frac{h(M)}{g(M)} \right] dx \equiv M_1.$$

Define the mappings

$$L, F: C^1_{\rho_n}[0,1] \to C_0[0,1] \times \mathbf{R}$$

by

$$Ly(t) = (y'(t) - y'(0), \rho_n - y'(1)) \quad \text{and} \quad Fy(t) = \left(-\int_0^t f^*(x, y(x)) \, dx, \mu\psi^*(y(1)) \right).$$

Here $C_0[0,1] = \{u \in C[0,1] : u(0) = 0\}$ and $C^1_{\rho_n}[0,1] = \{u \in C^1[0,1] : u(0) = \rho_n\}$. Now F is completely continuous by the Arzela–Ascoli theorem. Also if $Ly = (u(t), \gamma)$ then

$$y(t) = \rho_n + (\rho_n - \gamma - u(1))t + \int_0^t u(x) \, dx;$$

hence L^{-1} exists and is continuous.

Solving $(2.11)^n_\lambda$ is equivalent to finding a fixed point of $y = \lambda L^{-1} F y = \lambda N y$ where $N = L^{-1}F$: $C^1_{\rho_n}[0,1] \to C^1_{\rho_n}[0,1]$ is completely continuous. Let

$$U = \{u \in C^1_{\rho_n}[0,1] : |u|_1 < \max\{M, M_1\} + 1\}, \; K = C^1_{\rho_n}[0,1] \text{ and } E = C^1[0,1];$$

here $|u|_1 = \max\{|u|_0, |u'|_0\}$ and $|u|_0 = \sup_{[0,1]} |u(t)|$. Now Theorem 1.2 implies that $(2.11)^n_1$ has a solution $y_n \in C^1[0,1] \cap C^2(0,1)$. Also $\rho_n \leqq y_n(t) \leqq M$ for $t \in [0,1]$. Next we obtain a sharper lower bound on y_n, namely we will show

$$\alpha(t) \leqq y_n(t) \leqq M \quad \text{for} \quad t \in [0,1]. \tag{2.17}$$

If this is not true then $y_n - \alpha$ would have a negative minimum at say $t_0 \in (0,1]$. If $t_0 \in (0,1)$ then $y_n''(t_0) - \alpha''(t_0) \geqq 0$. However since $0 < y_n(t_0) < \alpha(t_0)$ and $y_n(t_0) \geqq \rho_n$ we have

$$y_n''(t_0) - \alpha''(t_0) = -[f(t_0, y_n(t_0)) + \alpha''(t_0)] < 0,$$

a contradiction. It remains to consider the case $t_0=1$. Then $y_n'(1)\leqq\alpha'(1)$ and $0<y_n(1)<\alpha(1)$. However

$$y_n'(1)-\alpha(1)=\rho_n-\mu\psi^*(y(1))+\mu\psi(\alpha(1))$$
$$=\rho_n-\mu\psi(\alpha(1))+\mu\psi(\alpha(1))=\rho_n>0,$$

a contradiction. Hence (2.17) is true. In particular $y_n(1)\geqq\alpha(1)$ and consequently $y_n\in C^1[0,1]\cap C^2(0,1)$ is a solution of $(2.10)^n$.

We shall now obtain a solution to (2.1) by means of the Arzela–Ascoli theorem, as a limit of solutions of $(2.10)^n$. To this end, we will show

$$\{y_n\}_{n=3}^{\infty} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{2.18}$$

Of course $\{y_n\}$ is uniformly bounded by (2.17). To show equicontinuity, some more estimates are needed.

The differential equation yields

$$-y_n''(x)\leqq g(y_n(x))\left\{q_1(x)+q_2(x)\frac{h(M)}{g(M)}\right\} \quad \text{for} \quad x\in(0,1). \tag{2.19}$$

Also $y_n'(1)=\rho_n-\mu\psi(y_n(1))$ together with (2.17) yields

$$|y_n'(1)|\leqq\rho_3+\mu\max_{z\in[0,M]}|\psi(z)|\equiv K_0.$$

Divide (2.19) by $g(y_n(x))$ and integrate from 0 to 1 to obtain

$$\frac{-y_n'(1)}{g(y_n(1))}+\frac{y_n'(0)}{g(\rho_n)}+\int_0^1\left\{\frac{-g'(y_n(x))}{g^2(y_n(x))}\right\}[y_n'(x)]^2\,dx\leqq\int_0^1\left[q_1(x)+q_2(x)\frac{h(M)}{g(M)}\right]dx. \tag{2.20}$$

Then since $y_n'(0)\geqq 0$ (note $y_n(0)=\rho_n$ and $y_n\geqq\rho_n$ on $[0,1]$) we have

$$\int_0^1\left\{\frac{-g'(y_n(x))}{g^2(y_n(x))}\right\}[y_n'(x)]^2\,dx\leqq\frac{K_0}{g(M)}+\int_0^1\left[q_1(x)+q_2(x)\frac{h(M)}{g(M)}\right]dx\equiv K_1. \tag{2.21}$$

Now consider

$$I(z)=\int_0^z\frac{[-g'(u)]^{1/2}}{g(u)}\,du.$$

Notice I is an increasing map from $[0,\infty)$ onto $[0,\infty)$ with I continuous on $[0,\Omega]$ for any $\Omega>0$. For $t,s\in[0,1]$ we have from Hölder's inequality that

$$|I(y_n(t))-I(y_n(s))|=\left|\int_{y_n(s)}^{y_n(t)}\frac{[-g'(u)]^{1/2}}{g(u)}\,du\right|=\left|\int_s^t\frac{[-g'(y_n(x))]^{1/2}}{g(y_n(x))}\,y_n'(x)\,dx\right|$$

$$\leqq|t-s|^{1/2}\left(\int_0^1\left\{\frac{-g'(y_n(x))}{g^2(y_n(x))}\right\}[y_n'(x)]^2\,dx\right)^{1/2}\leqq K_1^{1/2}|t-s|^{1/2}.$$

It follows from this inequality, the uniform continuity of I^{-1} on $[0,I(M)]$ and

$$|y_n(t)-y_n(s)|=|I^{-1}(I(y_n(t)))-I^{-1}(I(y_n(s)))|$$

that $\{y_n\}$ is equicontinuous on $[0,1]$. Thus (2.18) is established.

The Arzela–Ascoli theorem guarantees the existence of a subsequence N of integers and a function $y\in C[0,1]$ with y_n converging uniformly on $[0,1]$ to y as $n\to\infty$ through N. Also $y(0)=0$ and $\alpha(t)\leqq y(t)\leqq M$ for $t\in[0,1]$. Now y_n, $n\in N$, satisfies the integral equation

$$y_n(t)=y_n(1)+(\mu\psi(y_n(1))-\rho_n)(1-t)-\int_t^1(x-t)f(x,y_n(x))\,dx \quad \text{for} \quad t\in[0,1]. \tag{2.22}$$

We would like to let $n\to\infty$ through N in (2.22). First notice

$$\int_{1/2}^1|f(x,y_n(x))|\,dx\leqq\int_{1/2}^1 g(\alpha(x))\left\{q_1(x)+q_2(x)\frac{h(M)}{g(M)}\right\}dx<\infty.$$

Fix $t\in(0,1]$. Let $n\to\infty$ through N in (2.22), and so the Lebesgue dominated convergence theorem implies

$$y(t)=y(1)+\mu\psi(y(1))(1-t)-\int_t^1(x-t)f(x,y(x))\,dx.$$

Also for $t\in(0,1]$ we have

$$y'(t)=-\mu\psi(y(1))+\int_t^1 f(x,y(x))\,dx \tag{2.23}$$

so $y\in C^1(0,1]$. In addition $-y''(t)=f(t,y(t))$ for $t\in(0,1)$ and $y\in C^2(0,1)$. Finally (2.23) implies $y'(1)+\mu\psi(y(1))=0$. □

The next theorem is a "general upper and lower solution theorem" for singular problems with nonlinear boundary data.

Theorem 2.2. *Suppose* (2.2)–(2.8) *hold. In addition assume*

$$\begin{cases} \textit{there exists a function } \beta \in C[0,1] \cap C^1(0,1] \cap C^2(0,1) \textit{ with } \beta(0) \geqq \rho_3, \\ \beta'(1) + \mu\psi(\beta(1)) > \rho_3, \beta \geqq \rho_3 \textit{ on } (0,1) \textit{ and } \beta(1) > \alpha(1) \\ \textit{such that } f(t,\beta(t)) + \beta''(t) \leqq 0 \textit{ for } t \in (0,1) \end{cases} \tag{2.24}$$

is satisfied. Then (2.1) *has a solution in* $C[0,1] \cap C^1(0,1] \cap C^2(0,1)$.

Proof. Fix $n \in \{3,4,\dots\}$. We first show

$$\begin{cases} y'' + f(t,y) = 0, 0 < t < 1 \\ y(0) = \rho_n \\ y'(1) + \mu\psi(y(1)) = \rho_n \end{cases} \tag{2.25}^n$$

has a solution. The idea is to look at

$$\begin{cases} y'' + f^{**}(t,y) = 0, 0 < t < 1 \\ y(0) = \rho_n \\ y'(1) + \mu\psi^{**}(y(1)) = \rho_n \end{cases} \tag{2.26}^n$$

where

$$f^{**}(t,y) = \begin{cases} f(t,\beta(t)) + r(\beta(t) - y), y \geqq \beta(t) \\ f(t,y), \rho_n \leqq y \leqq \beta(t) \\ f(t,\rho_n) + r(\rho_n - y), y < \rho_n \text{ and } \dfrac{1}{n} \leqq t \leqq 1 - \dfrac{1}{n} \\ f\left(\dfrac{1}{n}, \rho_n\right) + r(\rho_n - y), y < \rho_n \text{ and } 0 \leqq t \leqq \dfrac{1}{n} \\ f\left(1 - \dfrac{1}{n}, \rho_n\right) + r(\rho_n - y), y < \rho_n \text{ and } 1 - \dfrac{1}{n} \leqq t \leqq 1 \end{cases}$$

$$\psi^{**}(z) = \begin{cases} \psi(\beta(1)), z > \beta(1) \\ \psi(z), \alpha(1) \leqq z \leqq \beta(1) \\ \psi(\alpha(1)), z < \alpha(1). \end{cases}$$

and $r: \mathbf{R} \to [-1,1]$ is the radial retraction defined by

$$r(u) = \begin{cases} u & \text{if } |u| \leqq 1 \\ \dfrac{u}{|u|} & \text{otherwise.} \end{cases}$$

Remark. Notice $\psi^{**}(z) \leqq 0$ for $z \in \mathbf{R}$.

Let $C_0[0,1], C^1_{\rho_n}[0,1]$ be as in Theorem 2.1 and define mappings

$$L, F: C^1_{\rho_n}[0,1] \to C_0[0,1] \times \mathbf{R}$$

by

$$Ly(t) = (y'(t) - y'(0), \rho_n - y'(1)) \quad \text{and} \quad Fy(t) = \left(-\int_0^t f^{**}(x, y(x))\, dx, \mu\psi^{**}(y(1)) \right).$$

Now L^{-1} exists and is continuous as in Theorem 2.1. Notice also that F is compact. Hence solving $(2.26)^n$ is equivalent to finding a fixed point of $y = L^{-1}Fy = Ny$ where $N = L^{-1}F: C^1_{\rho_n}[0,1] \to C^1_{\rho_n}[0,1]$ is compact. Theorem 1.1 implies $(2.26)^n$ has a solution $y_n \in C^1[0,1] \cap C^2(0,1)$. Essentially the same reasoning as in Theorem 2.1 yields

$$y_n(t) \geqq \rho_n \quad \text{for} \quad t \in [0,1]. \tag{2.27}$$

Next we claim

$$y_n(t) \leqq \beta(t) \quad \text{for} \quad t \in [0,1]. \tag{2.28}$$

If (2.28) is not true then $y_n - \beta$ would have a positive maximum at say $t_0 \in (0,1]$. If $t_0 \in (0,1)$ then $y_n''(t_0) - \beta''(t_0) \leqq 0$. However since $y_n(t_0) > \beta(t_0)$ we have

$$y_n''(t_0) - \beta''(t_0) = -[f(t_0, \beta(t_0)) + r(\beta(t_0) - y_n(t_0)) + \beta''(t_0)] > 0,$$

a contradiction. If $t_0 = 1$ then $y_n'(1) \geqq \beta'(1)$. However since $y_n(1) > \beta(1)$ we have

$$y_n'(1) - \beta'(1) = \rho_n - \mu\psi^{**}(y_n(1)) - \beta'(1) = \rho_n - [\mu\psi(\beta(1)) + \beta'(1)] < \rho_n - \rho_3 \leqq 0,$$

a contradiction. Thus (2.28) is true. Consequently

$$\rho_n \leqq y_n(t) \leqq \beta(t) \quad \text{for} \quad t \in [0,1]. \tag{2.29}$$

Essentially the same reasoning as in Theorem 2.1 establishes

$$\alpha(t) \leqq y_n(t) \leqq \beta(t) \quad \text{for} \quad t \in [0,1]. \tag{2.30}$$

In particular $\alpha(1) \leqq y_n(1) \leqq \beta(1)$ so $y_n \in C^1[0,1] \cap C^2(0,1)$ is a solution of $(2.25)^n$. The reasoning in Theorem 2.1 (from (2.18) onwards) now establishes that (2.1) has a solution. □

We now discuss condition (2.6). One can usually "construct" α explicitly from the differential equation; see [2, 10, 11]. However if (2.5) is replaced by the conditions

$$\begin{cases} \text{let } n \in \{3,4,\dots\} \text{ and associated with each } n \text{ we have a constant} \\ \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing sequence with } \lim_{n\to\infty} \rho_n = 0, \\ \text{and there exists a constant } k_0 > 0 \text{ such that for } \dfrac{1}{n} \leqq t \leqq 1 - \dfrac{1}{n} \text{ and} \\ 0 < y \leqq \rho_n \text{ we have } f(t,y) \geqq k_0 \end{cases} \tag{2.31}$$

and

$$\psi(0) = 0 \tag{2.32}$$

then we can construct an explicit α off the sequence of constants $\{\rho_n\}$; this is a standard argument, see [6] for example.

The details are as follows. Let $0 \leqq x \leqq \frac{1}{3}$ and

$$r_0(x) = \begin{cases} \rho_k\left(x - \dfrac{1}{k}\right) + \displaystyle\sum_{m=k+1}^{\infty} \rho_m\left(\dfrac{1}{m-1} - \dfrac{1}{m}\right), & x \in \left(\dfrac{1}{k}, \dfrac{1}{k-1}\right], \quad k = 4,5,\dots \\ 0, \quad x = 0. \end{cases}$$

Remark. Notice $r_0(x) = \int_0^x \phi(s)\,ds$ for $0 \leqq x \leqq \frac{1}{3}$ where $\phi:[0,\frac{1}{3}] \to [0,\infty)$ is the step function defined by

$$\phi(t) = \begin{cases} 0, \quad t = 0 \\ \rho_k, k \in \left(\dfrac{1}{k}, \dfrac{1}{k-1}\right], \quad k = 4,5 \end{cases}$$

Here $r_0 \in C[0,\frac{1}{3}]$ and notice

$$r_0(t) \leqq \phi(t) \leqq \rho_k \quad \text{for} \quad t \in \left(\frac{1}{k}, \frac{1}{k-1}\right], \quad k = 4,5,\dots.$$

Next define

$$\theta(t) = \int_0^t \int_0^s r_0(x)\,dx\,ds \quad \text{for} \quad 0 \leqq t \leqq \frac{1}{3}.$$

Notice $\theta(t) \leqq \rho_k$ for $t \in \left(\frac{1}{k}, \frac{1}{k-1}\right]$, $k = 4,5,\dots$ and so

$$f(t,y) \geqq k_0 \quad \text{for} \quad (t,y) \in \left(0, \frac{1}{3}\right] \times \{y \in (0,\infty) : y \leqq \theta(t)\}. \tag{2.33}$$

Let

https://doi.org/10.1017/S0013091500023269 Published online by Cambridge University Press

$$\alpha^*(t)=\begin{cases}\theta(t), 0\leqq t<\frac{1}{3}\\ q(t), \frac{1}{3}\leqq t\leqq\frac{2}{3}\\ \theta(1-t), \frac{2}{3}<t\leqq 1.\end{cases}$$

Here $q:[\frac{1}{3},\frac{2}{3}]\to(0,\rho_3]$ is such that $q\in C^2[\frac{1}{3},\frac{2}{3}]$ with $q(\frac{1}{3})=\theta(\frac{1}{3})=q(\frac{2}{3})$, $q'(\frac{1}{3})=\theta'(\frac{1}{3})=-q'(\frac{2}{3})$ and $q''(\frac{1}{3})=\theta''(\frac{1}{3})=q''(\frac{1}{3})$.

Notice since $0<q(t)\leqq\rho_3$ for $t\in[\frac{1}{3},\frac{2}{3}]$ we have

$$f(t,y)\geqq k_0 \quad \text{for} \quad (t,y)\in\left[\frac{1}{3},\frac{2}{3}\right]\times\{y\in(0,\infty): y\leqq q(t)\}. \tag{2.34}$$

Consequently (2.33) and (2.34) imply

$$f(t,y)\geqq k_0 \quad \text{for} \quad (t,y)\in(0,1)\times\{y\in(0,\infty): y\leqq\alpha^*(t)\}. \tag{2.35}$$

Finally define

$$\alpha(t)=\eta\alpha^*(t) \tag{2.36}$$

where

$$\eta=\min\left\{1,\frac{k_0}{|(\alpha^*)''|_0+1}\right\}.$$

Now $\alpha\in C^2[0,1]$ with $\alpha(0)=0$ and $\alpha>0$ on $(0,1)$. Also since $\alpha(t)\leqq\alpha^*(t)$ we have

$$f(t,y)\geqq k_0 \quad \text{for} \quad (t,y)\in(0,1)\times\{y\in(0,\infty): y<\alpha(t)\}.$$

In addition for $(t,y)\in(0,1)\times\{y\in(0,\infty): y<\alpha(t)\}$ we have

$$f(t,y)+\alpha''(t)\geqq k_0+\alpha''(t)\geqq k_0-\frac{k_0|(\alpha^*)''(t)|}{|(\alpha^*)''|_0+1}>k_0-k_0=0.$$

Finally since $(\alpha^*)'(t)=-\int_0^{1-t}r_0(x)\,dx$ for $\frac{2}{3}\leqq t\leqq 1$ we have $(\alpha^*)'(1)=0$ and so $\alpha'(1)+\mu\psi(\alpha(1))=0+\mu\psi(0)=0$ since (2.32) holds.

Consequently we have constructed an α which satisfies (2.6). However since (2.7) must also be satisfied it is desirable to construct the "best" α. Usually it is possible to obtain an explicit α from the differential equation. We now provide a general result for the boundary value problem

$$\begin{cases}y''+f(t,y)=0, 0<t<1\\ y(0)=0\\ y'(1)-\mu y(1)=0, 0\leqq\mu<1.\end{cases} \tag{2.37}$$

Remark. Note (2.37) is a special case of (2.1); here $\psi(u)=-u$.

Theorem 2.3. *Suppose* (2.2), (2.4), (2.8) *and* (2.31) *are satisfied. In addition assume*

$$\begin{cases} \text{there exists a function } \beta\in C[0,1]\cap C^1(0,1]\cap C^2(0,1) \text{ with } \beta(0)\geqq\rho_3, \\ \beta'(1)-\mu\beta(1)>\rho_3, \beta\geqq\rho_3 \text{ on } (0,1) \text{ and } \beta(1)>\rho_3 \\ \text{such that } f(t,\beta(t))+\beta''(t)\leqq 0 \text{ for } t\in(0,1), \end{cases} \tag{2.38}$$

and

$$\text{there exists } \tau\in(0,1) \text{ with } f(t,y)>0 \text{ for } t\in[\tau,1) \text{ and } 0<y\leqq\frac{\rho_3}{1-\mu(1-\tau)} \tag{2.39}$$

are satisfied. Then (2.37) *has a solution in* $C[0,1]\cap C^1(0,1]\cap C^2(0,1)$.

Proof. Let γ denote the α given above (i.e. as in (2.36)). Without loss of generality assume $\frac{1}{3}<\tau\leqq\frac{2}{3}$. Define

$$\mu_1(t)=\begin{cases} \gamma(t), 0\leqq t\leqq\frac{1}{3} \\ \omega(t), \frac{1}{3}<t<\tau \\ \dfrac{\gamma(\tau)[1-\mu(1-t)]}{1-\mu(1-\tau)}, t\geqq\tau \end{cases}$$

and

$$\alpha(t)=\eta\mu_1(t) \tag{2.40}$$

where

$$\eta=\min\left\{1,\frac{k_0}{|\mu_1''|_0+1}\right\}.$$

Here $\omega:[\frac{1}{3},\tau]\to(0,\rho_3]$ is such that $\omega\in C^2[\frac{1}{3},\tau]$ with

$$\omega(\tfrac{1}{3})=\gamma(\tfrac{1}{3}), \omega'(\tfrac{1}{3})=\gamma'(\tfrac{1}{3}), \omega''(\tfrac{1}{3})=\gamma''(\tfrac{1}{3}), \omega(\tau)=\gamma(\tau), \omega'(\tau)=\frac{\mu\gamma(\tau)}{1-\mu(1-\tau)} \text{ and } \omega''(\tau)=0.$$

Remark. Notice

$$\alpha(1)\leqq\frac{\gamma(\tau)}{1-\mu(1-\tau)}\leqq\frac{\rho_3}{1-\mu(1-\tau)}.$$

We now claim that α satisfies (2.6). First notice $\alpha(0)=\alpha'(1)-\mu\alpha(1)=0$. Also $\alpha>0$ on $(0,1)$ since $0\leqq\mu<1$. Also from above

$$f(t,y)+\alpha''(t)>0 \quad \text{for} \quad (t,y)\in(0,\tau)\times\{y\in(0,\infty): y<\alpha(t)\}. \tag{2.41}$$

Now for $t>\tau$ we have $\alpha''(t)=0$ and also

$$\alpha(t)\leqq\frac{\gamma(\tau)}{1-\mu(1-\tau)}\leqq\frac{\rho_3}{1-\mu(1-\tau)}.$$

Consequently

$$f(t,y)+\alpha''(t)=f(t,y)>0 \quad \text{for} \quad (t,y)\in[\tau,1)\times\{y\in(0,\infty): y<\alpha(t)\}. \tag{2.42}$$

Thus (2.41) and (2.42) imply that α satisfies (2.6). Notice also that (2.7) is trivially satisfied since $\mu_1(t)\geqq\gamma(\tau)$ for $t>\tau$. Existence of a solution is now guaranteed from Theorem 2.2. □

Example 2.1. (Membrane response of a spherical cap).
The boundary value problem

$$\begin{cases} y''+\left(\dfrac{t^2}{32y^2}-\dfrac{\kappa^2}{8}\right)=0, 0<t<1 \\ y(0)=0 \\ y'(1)-\mu y(1)=0, 0\leqq\mu<1, \kappa>0 \end{cases} \tag{2.43}$$

has a solution.

We will apply Theorem 2.3. First choose $n_0\in\{3,4,\dots\}$ such that

$$\frac{1}{2n_0(8+\kappa^2)}<\frac{1-\mu}{2(1+\mu)\kappa}.$$

Notice that (2.31), with

$$\rho_n=\frac{1}{2(n_0+n)(\kappa^2+8)^{1/2}} \quad \text{and} \quad k_0=1,$$

is true since for $\frac{1}{n}\leqq t\leqq 1-\frac{1}{n}$ and $0<y\leqq\rho_n$ we have

$$f(t,y)\geqq\frac{1}{32n^2\rho_n^2}-\frac{\kappa^2}{8}\geqq\frac{(\kappa^2+8)}{8}-\frac{\kappa^2}{8}=1.$$

Now let

$$\beta(t)=\frac{t}{2\kappa}+\rho_3.$$

Notice $\beta\geqq\rho_3$ on $[0,1]$ and $\beta'(1)-\mu\beta(1)>\rho_3$

$$\left(\text{i.e. } \frac{1}{2\kappa}-\mu\left(\frac{1}{2\kappa}+\rho_3\right)>\rho_3\right) \text{ since } \rho_3<\frac{1-\mu}{2(1+\mu)\kappa}.$$

In addition $f(t,\beta(t))+\beta''(t)=f(t,\beta(t))\leqq 0$ on (0, 1) and so (2.38) holds. Also let

$$q_1(t)=\frac{t^2}{32}, g(y)=y^{-2}, q_2(t)=\frac{\kappa^2}{8} \text{ and } h(y)=1$$

so (2.4) and (2.8) are clearly true. Let

$$\tau=\frac{1}{(n_0+3)(1-\mu)} \text{ so for } \tau\leqq t<1 \text{ and } 0<y\leqq\frac{\rho_3}{1-\mu(1-\tau)}$$

we have

$$f(t,y)\geqq\frac{\tau^2(1-\mu(1-\tau))^2}{32\rho_3^2}-\frac{\kappa^2}{8}\geqq\frac{4\tau^2(1-\mu)^2(n_0+3)^2(8+\kappa^2)}{32}-\frac{\kappa^2}{8}=1>0$$

and so (2.39) is satisfied. Existence of a solution to (2.43) is now guaranteed from Theorem 2.3.

Our next two results are modelled on the Stefan boundary condition $y'(1)+cy^4(1)=0, c>0$.

Theorem 2.4. *Suppose* (2.2), (2.4), (2.5), (2.6), (2.7) *and* (2.8) *hold. In addition suppose the following conditions are satisfied:*

$\psi:\mathbf{R}\to\mathbf{R}$ *is continuous and nondecreasing with* $\psi(0)=0$ *and* $\psi(u)>0$ *for* $u>0$ (2.44)

and

$$\begin{cases}\text{there exists a constant } M>0 \text{ such that for } z>0,\\ \int_0^z \frac{du}{g(u)}\leqq\int_0^1 xq_1(x)\,dx+\frac{h(z)}{g(z)}\int_0^1 xq_2(x)\,dx+\int_0^{\rho_3}\frac{du}{g(u)}\\ \text{implies } z\leqq M.\end{cases} \tag{2.45}$$

Then (2.1), *with* $\mu>0$, *has a solution in* $C[0,1]\cap C^1(0,1]\cap C^2(0,1)$.

Remark. The case $\mu=0$, is included in Theorems 2.1 and 2.2.

Proof. Fix $n\in\{3,4,\ldots\}$ and look at

$$\begin{cases} y'' + f(t,y) = 0, 0 < t < 1 \\ y(0) = \rho_n \\ y'(1) + \mu\psi(y(1)) = \mu\psi(\rho_n). \end{cases} \tag{2.46}^n$$

To show $(2.46)^n$ has a solution we consider the family of problems

$$\begin{cases} y'' + \lambda f^*(t,y) = 0, 0 < t < 1 \\ y(0) = \rho_n \\ y'(1) + y(1) + \lambda\psi_n^*(y(1)) = \rho_n \end{cases} \tag{2.47}_\lambda^n$$

where f^* is as in Theorem 2.1 and

$$\psi_n^*(z) = \begin{cases} \mu[\psi(z) - \psi(\rho_n)] + \rho_n - z, z \geqq \rho_n \\ 0, \quad z < \rho_n. \end{cases}$$

We first show

$$y(t) \geqq \rho_n, \quad t \in [0,1] \tag{2.48}$$

for any solution $y \in C^1[0,1] \cap C^2(0,1)$ to $(2.47)_\lambda^n$. To see this suppose $y - \rho_n$ has a negative minimum at $t_0 \in (0,1]$. If $t_0 \in (0,1)$ then we obtain a contradiction as in Theorem 2.1. It remains to consider the case $t_0 = 1$. Then $y'(1) \leqq 0$. However since $y(1) < \rho_n$ we have

$$y'(1) = \rho_n - y(1) - \lambda\psi_n^*(y(1)) = \rho_n - y(1) > 0,$$

a contradiction. Thus (2.48) holds.

Next suppose the absolute minimum of y occurs at $t_n \in [0,1]$. In fact we may take $t_n \in (0,1)$, and so $y'(t_n) = 0$. To see this notice if $y(t_n) = \rho_n$ then $y \equiv \rho_n$. Next if $y(t_n) > \rho_n$ then if $t_n = 1$ we have $y'(1) \geqq 0$ and so

$$y'(1) = \rho_n - y(1) - \lambda\psi_n^*(y(1)) = (1-\lambda)(\rho_n - y(1)) + \lambda\mu[\psi(\rho_n) - \psi(y(1))] < 0,$$

a contradiction.

For $x \in (0,1)$ we have

$$\frac{-y''(x)}{g(y(x))} \leqq q_1(x) + q_2(x)\frac{h(y(x))}{g(y(x))}$$

Integrate from $t(t < t_n)$ to t_n and then from 0 to t_n to obtain

$$\int_0^{y(t_n)} \frac{du}{g(u)} \leqq \int_0^1 xq_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\int_0^1 xq_2(x)\,dx + \int_0^{\rho_3} \frac{du}{g(u)}.$$

Consequently (2.45) implies

$$\rho_n \leqq y(t) \leqq M \quad \text{for} \quad t \in [0,1]. \tag{2.49}$$

Define the mappings

$$L, F: C^1_{\rho_n}[0,1] \to C_0[0,1] \times \mathbf{R}$$

by

$$Ly(t) = (y'(t) - y'(0), \rho_n - y'(1) - y(1)) \quad \text{and} \quad Fy(t) = \left(-\int_0^t f^*(x, y(x))\, dx, \psi_n^*(y(1)) \right).$$

If $Ly = (u(t), \gamma)$ then

$$y(t) = \rho_n - \frac{t}{2}\left(\gamma + u(1) + \int_0^1 u(x)\, dx \right) + \int_0^t u(x)\, dx;$$

hence L^{-1} exists and is continuous. Also F is completely continuous. Essentially the same reasoning as in Theorem 2.1 implies $(2.47)_1^n$ has a solution $y_n \in C^1[0,1] \cap C^2(0,1)$. Also $\rho_n \leqq y_n(t) \leqq M$ for $t \in [0,1]$, so y_n is a solution of $(2.46)^n$.

Next we show

$$\alpha(t) \leqq y_n(t) \leqq M \quad \text{for} \quad t \in [0,1]. \tag{2.50}$$

If this is not true then $y_n - \alpha$ would have a negative minimum at say $t_0 \in (0,1]$. If $t_0 \in (0,1)$ then we obtain a contradiction as in Theorem 2.1. If $t_0 = 1$ then $y_n'(1) \leqq \alpha'(1)$ and $0 < y_n(1) < \alpha(1)$. However

$$y_n'(1) - \alpha'(1) = \mu\psi(\rho_n) - \mu\psi(y_n(1)) + \mu\psi(\alpha(1)) \geqq \mu\psi(\rho_n) > 0,$$

a contradiction. Hence (2.50) is true.

Essentially the same reasoning as in Theorem 2.1 (from (2.18) onwards) now establishes the result. □

Theorem 2.5. *Suppose* (2.2), (2.4), (2.5), (2.6), (2.7), (2.8) *and* (2.44) *hold. In addition assume*

$$\begin{cases} \text{there exists a function } \beta \in C[0,1] \cap C^1(0,1] \cap C^2(0,1) \text{ with} \\ \beta \geqq \rho_3 \text{ on } [0,1],\ \beta'(1) + \mu\psi(\beta(1)) \geqq \mu\psi(\rho_3) \text{ and such that} \\ f(t, \beta(t)) + \beta''(t) \leqq 0 \text{ for } t \in (0,1). \end{cases} \tag{2.51}$$

Then (2.1), *with* $\mu > 0$, *has a solution in* $C[0,1] \cap C^1(0,1] \cap C^2(0,1)$.

Proof. Fix $n \in \{3,4,\dots\}$ and look at (2.46)^n. The idea is to first consider

$$\begin{cases} y'' + f^{**}(t,y) = 0, 0<t<1 \\ y(0) = \rho_3 \\ y'(1) + y(1) + \psi_n^{**}(y(1)) = \rho_n \end{cases} \tag{2.52^n}$$

where f^{**} is as in Theorem 2.2 and

$$\psi_n^{**}(z) = \begin{cases} \mu[\psi(\beta(1)) - \psi(\rho_n)] + \rho_n - \beta(1), z > \beta(1) \\ \mu[\psi(z) - \psi(\rho_n)] + \rho_n - z, \rho_n \leqq z \leqq \beta(1) \\ 0, \quad z < \rho_n. \end{cases}$$

Let

$$L, F: C^1_{\rho_n}[0,1] \to C_0[0,1] \times \mathbf{R}$$

be defined by

$$Ly(t) = (y'(t) - y'(0), \rho_n - y'(1) - y(1)) \quad \text{and} \quad Fy(t) = \left(-\int_0^t f^{**}(x, y(x))\, dx, \psi_n^{**}(y(1)) \right).$$

Now F is compact so essentially the same reasoning as in Theorem 2.2 implies (2.52)^n has a solution $y_n \in C^1[0,1] \cap C^2(0,1)$.

The reasoning in Theorem 2.4 yields

$$y_n(t) \geqq \rho_n \quad \text{for} \quad t \in [0,1]. \tag{2.53}$$

Next we claim

$$y_n(t) \leqq \beta(t) \quad \text{for} \quad t \in [0,1]. \tag{2.54}$$

If (2.54) is not true then $y_n - \beta$ would have a positive maximum at say $t_0 \in (0,1]$. If $t_0 \in (0,1)$ then we obtain a contradiction as in Theorem 2.2. If $t_0 = 1$ then $y_n'(1) \geqq \beta'(1)$ and $y_n(1) > \beta(1)$. However

$$\begin{aligned} y_n'(1) - \beta'(1) &\leqq [\rho_n - y_n(1) - \psi^{**}(y_n(1))] + \mu\psi(\beta(1)) - \mu\psi(\rho_3) \\ &= (\beta(1) - y_n(1) - \mu[\psi(\beta(1)) - \psi(\rho_n)]) + \mu\psi(\beta(1)) - \mu\psi(\rho_3) \\ &= \beta(1) - y_n(1) + \mu[\psi(\rho_n) - \psi(\rho_3)] < 0, \end{aligned}$$

a contradiction. Thus (2.54) is true.

Hence y_n is a solution of (2.46)^n. The same reasoning as in Theorem 2.4 establishes

$$\alpha(t) \leqq y_n(t) \leqq \beta(t) \quad \text{for} \quad t \in [0,1].$$

Essentially the same argument as in Theorem 2.1 (from (2.18) onwards) now establishes the result. □

REFERENCES

1. J. V. Baxley, A singular nonlinear boundary value problem: membrane response of a spherical cap, *SIAM J. Appl. Math.* **48** (1988), 497–505.

2. L. E. Bobisud, J. E. Calvert and W. D. Royalty, Some existence results for singular boundary value problems, *Differential Integral Equations* **6** (1993), 553–571.

3. L. E. Bobisud, D. O'Regan and W. D. Royalty, Solvability of some nonlinear boundary value problems, *Nonlinear Anal.* **12** (1988), 855–869.

4. J. A. Gatica, V. Oliker and P. Waltman, Singular nonlinear boundary value problems for second order ordinary differential equations, *J. Differential Equations* **79** (1989), 62–78.

5. A. Granas, R. B. Guenther and J. W. Lee, Some general existence principles in the Carathéodory theory of nonlinear differential systems. *J. Math. Pures Appl.* **70** (1991), 153–196.

6. P. Habets and F. Zanolin, Upper and lower solutions for a generalized Emden–Fowler equation, *J. Math. Anal. Appl.* **181** (1994), 684–700.

7. C. D. Luning and W. L. Perry, Iterative solutions of negative exponent Emden–Fowler problems, *Internat. J. Math. Sci.* **13** (1990), 159–164.

8. J. W. Mooney, Solution of a Thomas–Fermi problem using linear approximants, *Comput. Phys. Comm.* **96** (1993), 51–57.

9. T. Y. Na, *Computational methods in engineering boundary value problems* (Academic Press, New York, 1979).

10. D. O'Regan, Positive solutions to singular boundary value problems with at most linear growth, *Appl. Anal.* **49** (1993), 171–196.

11. D. O'Regan, *Theory of singular boundary value problems* (World Scientific, Singapore, 1994).

12. S. Taliaferro, A nonlinear singular boundary value problem, *J. Nonlinear Anal.* **3** (19790), 897–904.

Department of Mathematics
University College
Galway
Ireland