BULL. AUSTRAL. MATH. SOC. 90C30, 90C25
VOL. 20 (1979), 77-86.

On stationary points and the complementarity problem

Sribatsa Nanda and Sudarsan Nanda

Let S be a closed convex cone in C^n, S^* the polar cone, g a continuous map from C^n into itself, and e a fixed vector in S^*. In this paper we prove that there is a connected set T in S of stationary points of $(D_r(e), g)$ where $D_r(e)$ is the set of all x in S with $\text{re}(e, x) \le r$. This extends the results of Lemke and Eaves to the complex nonlinear case and arbitrary closed convex cones in C^n. We show that if g is strictly monotone on S, then T is both unique as well as arcwise connected. This partly solves the open problems raised by Eaves in this more general setting. We also show that if x is a stationary point of $(D_r(e), g)$ and $\text{re}(e, x) < r$ then x is a stationary point of (S, g).

1. Introduction

Let C^n (R^n) denote the n-dimensional complex (real) space with hermitian (euclidean) norm and the usual inner product and let R^n_+ be the nonnegative orthant of R^n. Let S denote a closed convex cone in C^n. The polar of S, denoted by S^*, is defined by

$$S^* = \{y \in C^n : \text{re}(x, y) \ge 0 \text{ for all } x \in S\} .$$

Since $e \in S^*$ and $r \ge 0$, we write

Received 1 December 1978.

$$D_r(e) = \{x \in S : \text{re}(e, x) \le r\} ,$$

$$D_r^0(e) = \{x \in D_r(e) : \text{re}(e, x) < r\} ,$$

and

$$S_r(e) = \{x \in D_r(e) : \text{re}(e, x) = r\} .$$

Note that $D_r(e)$ is the disjoint union of $D_r^0(e)$ and $S_r(e)$. A mapping $g : C^n \to C^n$ is said to be monotone on S if $\text{re}\big(g(x)-g(y), x-y\big) \ge 0$ for each $(x, y) \in S \times S$ and strictly monotone if strict inequality holds whenever $x \ne y$.

Given a continuous function $g : C^n \to C^n$, the nonlinear complementarity problem in C^n consists of finding a z such that

(1.1) $$z \in S , \quad g(z) \in S^* , \text{ and } \quad \text{re}\big(g(z), z\big) = 0 ,$$

where S is a closed convex cone in C^n. Given a convex set $K \subset C^n$ and a continuous map $g : C^n \to C^n$, a point $x \in K$ is said to be a stationary point of the pair (K, g) if

(1.2) $$x \in \arg\min_{y \in K} \text{re}\big(g(x), y\big)$$

or equivalently, if

(1.3) $$x \in \arg\min_{y \in K} \|y-x+g(x)\| ,$$

where arg min means the set of all $y \in K$ which minimize the objective. Notice that the set in (1.3) is either empty or singleton, whereas in (1.2) it may contain many elements. Note that the existence of a solution to the complementarity problem (1.1) is equivalent to the existence of a stationary point of (S, g).

Several authors including Bazaraa, Goode, and Nashed [1], Habetler and Price [4], Karamardian [6], Mond [8], and Parida and Sahoo [10] have discussed the solution of the complementarity problems in different contexts. Lemke [7] and Eaves [3] have discussed the existence of stationary points of (S, g) in the real case by taking S to be R_+^n. A basic theorem of Lemke [7] asserts that given an affine function

$g : R_+^n \to R^n$ and a $d \in R_+^n$, there is a piecewise affine function $x : R_+^1 \to R_+^n$ such that $x(t)$ is a stationary point of $\left(D_t^n, g\right)$ with $d \cdot x(t) = t$, where D_t^n is the set of all x in R_+^n for which $d \cdot x \le t$. The set $T = \{x(t)\}$ thus becomes connected (indeed, arcwise connected). While extending this result to the nonlinear case, Eaves [3] has shown that the arcwise connectedness property of T is false in general and has asked the following two questions. What conditions give the arcwise connectedness property and what properties give a unique T?

In this paper we extend the results of Lemke [7] and Eaves [3] to the complex nonlinear case by taking S to be any closed convex cone in C^n. We show that there is a connected set T in S such that each $x \in T$ is a stationary point of $\left(D_r(e), g\right)$ for some $r \ge 0$. We also show that if g is strictly monotone on S, then T is both unique as well as arcwise connected and this partly answers the questions raised by Eaves [3] in this more general setting. Finally we prove the existence of stationary points of (S, g) under certain suitable conditions.

2. Preliminary results

We start by mentioning some useful results which will be needed in the proof of our main theorems. The following proposition is the complex version of a lemma of Hartman and Stampacchia [5]; we omit its proof since it involves the same arguments as in the real case. For an outline of the proof in the real case, see [3].

PROPOSITION 2.1. *Let $g : C^n \to C^n$ be a continuous map on a non-empty, compact, convex set $K \subset C^n$. Then (K, g) has a stationary point.*

The following lemma has been proved in [9].

LEMMA 2.2. *Let S be a closed convex cone in C^n and let $e \in \operatorname{int}(S^*)$ be fixed. Then for any $r \ge 0$, the set $D_r(e)$ is compact and convex.*

The following proposition is the complex version of a theorem of Browder (see [2]).

https://doi.org/10.1017/S0004972700009102 Published online by Cambridge University Press

PROPOSITION 2.3. *Let* $G : K \times [0, r] \to K$ *be a continuous function, where* K *is a nonempty, compact, convex set in* C^n *and* $[0, r]$ *denotes the closed interval in* R^1 . *Then there is a connected set* W *in* $K \times [0, r]$ *intersecting both* $K \times \{0\}$ *and* $K \times \{r\}$ *such that* $G(x, t) = x$ *for all* $(x, t) \in W$.

Proof. Suppose that K is a nonempty, compact, convex set in C^n . Let $h : C^n \to R^{2n}$ be the linear homeomorphism of C^n onto R^{2n} . Then $K' = h(K)$ is a nonempty, compact, convex set in R^{2n} . Since $G : K \times [0, r] \to K$ is continuous, there is a continuous map $G' : K' \times [0, r] \to K'$ defined by

$$G'(x, t) = hG\left(h^{-1}(z), t\right)$$

for $(z, t) \in K' \times [0, r]$. Therefore, by Browder's Theorem, there is a connected set $W' \subset K' \times [0, r]$ such that $W' \cap (K' \times \{0\}) \neq \emptyset$, $W' \cap (K' \times \{r\}) \neq \emptyset$, and $G'(z, t) = z$ for all $(z, t) \in W'$. Define a continuous map $\theta : K \times [0, r] \to K' \times [0, r]$ by the rule

$$\theta(x, t) = (hx, t) ,$$

and take $W = \theta^{-1}(W')$. Clearly W is connected, $W \subset K \times [0, r]$, and $G(x, t) = x$ for all $(x, t) \in W$. Since $W' \cap (K' \times \{0\}) \neq \emptyset$, there is a $w' = (z', 0) \in W' \cap (K' \times \{0\})$, and then

$$\theta^{-1}(w') = \left(h^{-1}(z'), 0\right) \in W \cap (K \times \{0\}) .$$

Thus $W \cap (K \times \{0\}) \neq \emptyset$ and similarly it can be shown that $W \cap (K \times \{r\}) \neq \emptyset$. This completes the proof.

LEMMA 2.4. *Let* $g : C^n \to C^n$ *be a continuous map on a closed convex cone* S *and let* $e \in S^*$. *If* x *is a stationary point of* $\left(D_r(e), g\right)$ *for some* $r \geq 0$, *then* $\operatorname{re}\left(g(x), x\right) \leq 0$. *In this case if* $x \in D_r^0(e)$ *then* $\operatorname{re}\left(g(x), x\right) = 0$.

Proof. Suppose that x is a stationary point of $\left(D_r(e), g\right)$. Then

$$\operatorname{re}\left(g(x), x\right) \leq \operatorname{re}\left(g(x), z\right) \quad \text{for all } z \in D_r(e) .$$

In particular,

https://doi.org/10.1017/S0004972700009102 Published online by Cambridge University Press

$$\text{re}\big(g(x),\ x\big) \leq \tfrac{1}{2}\ \text{re}\big(g(x),\ x\big)\ .$$

Therefore $\text{re}\big(g(x), x\big) \leq 0$. Now, if $x \in D_r^0(e)$, then there is some $\lambda > 1$ such that $\lambda x \in S_r(e)$. Then we have

$$\text{re}\big(g(x),\ x\big) \leq \lambda\ \text{re}\big(g(x),\ x\big)\ .$$

This is impossible unless $\text{re}\big(g(x), x\big) = 0$.

LEMMA 2.5. *Let* $g : C^n \to C^n$ *be a continuous map on a closed convex cone* S *and* $e \in S^*$. *If* x *is a stationary point of* $\big(D_r(e), g\big)$ *and* $x \in D_r^0(e)$, *then* x *is a stationary point of* (S, g) .

Proof. Let $x \in D_r^0(e)$. Then, from Lemma 2.4, it follows that $\text{re}\big(g(x), x\big) = 0$. Let $w \in S$; then $w = \lambda z$ for some $\lambda \geq 0$ and $z \in D_r(e)$. Since x is a stationary point of $\big(D_r(e), g\big)$ we have

$$\text{re}\big(g(x),\ x\big) \leq \text{re}\big(g(x),\ z\big) \quad \text{for all } z \in D_r(e)\ . \tag{2.1}$$

Since $\text{re}\big(g(x), x\big) = 0$ it follows from (2.1) that

$$\text{re}\big(g(x),\ x\big) \leq \text{re}\big(g(x),\ w\big)\ .$$

Thus x is a stationary point of (S, g) .

3. The main theorems

We are now ready to prove our main theorem.

THEOREM 3.1. *Let* $g : C^n \to C^n$ *be a continuous map on a closed convex cone* $S \subset C^n$ *and let* $e \in \text{int}(S^*)$. *Then there is a closed connected set* T *in* S *such that*

(i) *for every* $x \in T$ *there is an* $r \geq 0$ *such that* $x \in S_r(e)$ *and* x *is a stationary point of* $\big(D_r(e), g\big)$, *and*

(ii) *for each* $r \geq 0$ *there is an* $x \in T$ *such that* x *is a stationary point of* $\big(D_r(e), g\big)$.

Proof. It follows from Proposition 2.1 and Lemma 2.2 that $\big(D_r(e), g\big)$

https://doi.org/10.1017/S0004972700009102 Published online by Cambridge University Press

has a stationary point for each $r \geq 0$. Let T_r be the set of all stationary points of $(D_r(e), g)$ and let T be the connected component of $\bigcup_{r\geq 0} T_r$ which contains $0 \in C^n$. Clearly T is closed and is the maximal connected set containing 0 and satisfying *(i)*. We now show that T satisfies *(ii)*. Let $G : D_r(e) \times [0, r] \to D_r(e)$ be defined by

$$G(x, t) = \arg \min_{y \in D_t(e)} \|y-x+g(x)\| .$$

Clearly G is continuous. By Proposition 2.3 it follows that there is a connected set W in $D_r(e) \times [0, r]$ which contains $(0, 0)$ and (x, r) for some $x \in T_r$. But T contains the connected set $\{y : (y, t) \in W\}$, and hence $x \in T$.

THEOREM 3.2. *Let $g : C^n \to C^n$ be continuous and strictly monotone on a closed convex cone S and let $e \in \operatorname{int}(S^*)$. Then the set T of Theorem* 3.1 *is unique; it is also arcwise connected.*

Proof. Assume that x_1 and x_2 are two stationary points of $(D_r(e), g)$ for some $r > 0$. Then we have

$$\operatorname{re}\left(g(x_1), x_1-x_2\right) \leq 0 \tag{3.1}$$

and

$$\operatorname{re}\left(g(x_2), x_2-x_1\right) \leq 0 . \tag{3.2}$$

By adding (3.1) and (3.2) we get

$$\operatorname{re}\left(g(x_1)-g(x_2), x_1-x_2\right) \leq 0 .$$

Since g is strictly monotone, this is impossible unless $x_1 = x_2$. This proves the uniqueness of T.

To show that T is arcwise connected, it is enough to show that the correspondence $r \mapsto x_r$ is a continuous map, where x_r is the unique stationary point of $D_r(e)$. It will therefore suffice to show that if $r_n \to r$ (in R_+^1), then $x_{r_n} \to x_r$ in S. Let $r_n \to r^-$ and consider the sequence $\{x_{r_n}\}$ in S. Notice first of all that x_{r_n} is a stationary point of $D_{r_n}(e)$; moreover, if $r_n < r_{n+1}$, then $D_{r_n}(e) \subset D_{r_{n+1}}(e)$. Thus it follows that $\{x_{r_n}\} \subset D_r(e)$. Since $D_r(e)$ is compact, there is a subsequence $\{x_{r_m}\}$ which converges to x in $D_r(e)$. Suppose that $x \in D_r^0(e)$; then we can find an $\varepsilon > 0$ such that $x \in D_{r-\varepsilon}(e)$. Since $r_n \to r$, it follows that all except a finite number of x_{r_m}'s will lie outside $D_{r-\varepsilon}(e)$; thus x cannot be a point of accumulation. This contradiction shows that $x \in S_r(e)$. Since x_r is the unique stationary point of $D_r(e)$ we have

$$\text{re}\big(g(x_r),\ x_r-y\big) \le 0 \quad \text{for all} \quad y \in D_r(e) . \tag{3.3}$$

We shall now show that

$$\text{re}\big(g(x),\ x-y\big) \le 0 \quad \text{for all} \quad y \in D_r(e) . \tag{3.4}$$

If $y \in D_r^0(e)$ then we have

$$\text{re}\big(g(x_{r_m}),\ x_{r_m}-y\big) \le 0 .$$

Since g is continuous, taking the limits as $r_m \to r$ we get

$$\text{re}\big(g(x),\ x-y\big) \le 0 .$$

If, however, $y \in S_r(e)$, then given x_{r_m} we can choose $\lambda_m \in [0, 1]$ such that $\lambda_m y \in D_{r_m}(e)$; thus

$$\text{re}\big(g(x),\ x-\lambda_m y\big) \le 0 .$$

Notice that as $r_m \to r$, $\lambda_m \to 1$, and thus taking the limit as $\lambda_m \to 1$, we get

$$\text{re}\big(g(x),\ x-y\big) \le 0 .$$

Therefore (3.4) holds, and combining (3.3) and (3.4) we have that $x = x_r$.

https://doi.org/10.1017/S0004972700009102 Published online by Cambridge University Press

This shows that all convergent subsequences of the sequence $\{x_{r_n}\}$ will have the same limit x_r , and therefore the sequence $\{x_{r_n}\}$ converges to x_r .

Now if $r_n \to r^+$, we can choose an $s > 0$ such that $s > r_n$ for all n . Now since $D_s(e)$ is compact we can go through the same argument as in the case above to show that $x_{r_n} \to x_r$. This completes the proof that $r \mapsto x_r$ is continuous.

In order to show the existence of a stationary point of (S, g) , we introduce the following definition.

DEFINITION. Let D be a subset of S . We say that a bounded set $U \subset S \cap D^c$ separates D from ∞ if each unbounded closed connected set in S that meets D also meets U .

THEOREM 3.3. *Let* $g : C^n \to C^n$ *be a continuous map on* S *and let* $e \in \operatorname{int}(S^*)$. *Suppose that* U *separates* $D_r(e)$ *from* ∞ *and that for each* $x \in U$ *there is a* $w \in D_r(e)$ *for which* $\operatorname{re}(g(x), w) \le \operatorname{re}(g(x), x)$. *Then* (S, g) *has a stationary point.*

Proof. Let T be as in Theorem 3.1. If $T \cap U = \emptyset$, then T is bounded and therefore the result holds. Assume that $x \in T \cap U$. Since $x \in T$, by Theorem 3.1, there is a $k > 0$ such that $x \in S_k(e)$ and x is a stationary point of $D_k(e)$. If $k < r$, then clearly $x \in D_r^0(e)$ and by Lemma 2.4, $\operatorname{re}(g(x), x) = 0$. Now assume that $r \le k$. Then we have

$$\operatorname{re}(g(x), x) \le \operatorname{re}(g(x), z) \tag{3.5}$$

for all $z \in D_k(e)$. Since $x \in U$ by the hypothesis, there is a $w \in D_r(e)$ for which

$$\operatorname{re}(g(x), w) \le \operatorname{re}(g(x), x) . \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$\operatorname{re}(g(x), w) \le \operatorname{re}(g(x), z) \tag{3.7}$$

https://doi.org/10.1017/S0004972700009102 Published online by Cambridge University Press

for all $z \in D_k(e)$. Since $r \le k$, $D_r(e) \subset D_k(e)$, and therefore $w \in D_k(e)$. If we take $z = x/2 + w$ in (3.7), we get

$$\text{re}\big(g(x),\, x\big) \ge 0 \; . \tag{3.8}$$

Now, from (3.8) and Lemma 2.4, it follows that $\text{re}\big(g(x),\, x\big) = 0$. The required result then follows from Lemma 2.5.

References

[1] M.S. Bazaraa, J.J. Goode and M.Z. Nashed, "A nonlinear complementarity problem in mathematical programming in Banach space", *Proc. Amer. Math. Soc.* **35** (1972), 165-170.

[2] Felix E. Browder, "On continuity of fixed points under deformations of continuous mappings", *Summa Brasil. Math.* **4** (1960), 183-191.

[3] B.C. Eaves, "On the basic theorem of complementarity", *Math. Programming* **1** (1971), 68-75.

[4] G.J. Habetler and A.L. Price, "Existence theory for generalized non-linear complementarity problems", *J. Optimization Theory Appl.* **7** (1971), 223-239.

[5] Philip Hartman and Guido Stampacchia, "On some non-linear elliptic differential-functional equations", *Acta Math.* **115** (1966), 271-310.

[6] S. Karamardian, "Generalized complementarity problem", *J. Optimization Theory Appl.* **8** (1971), 161-168.

[7] C.E. Lemke, "Bimatrix equilibrium points and mathematical programming", *Management Sci. Ser. A* **11** (1965), 681-689.

[8] Bertram Mond, "On the complex complementarity problem", *Bull. Austral. Math. Soc.* **9** (1973), 249-257.

[9] Sribatsa Nanda and Sudarsan Nanda, "A complex nonlinear complementarity problem", *Bull. Austral. Math. Soc.* **19** (1978), 437-444 (1979).

[10] J. Parida and B. Sahoo, "On the complex nonlinear complementary problem", *Bull. Austral. Math. Soc.* 14 (1976), 129-136.

Department of Mathematics,
Regional Engineering College,
Rourkela,
Orissa,
India.

https://doi.org/10.1017/S0004972700009102 Published online by Cambridge University Press