Ergod. Th. & Dynam. Sys., page 1 of 42
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Strongly mixing systems are almost strongly mixing of all orders

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Abstract. We prove that any strongly mixing action of a countable abelian group on a probability space has higher-order mixing properties. This is achieved via the utilization of $\mathcal{R}$-limits, a notion of convergence which is based on the classical Ramsey theorem. $\mathcal{R}$-limits are intrinsically connected with a new combinatorial notion of largeness which is similar to but has stronger properties than the classical notions of uniform density one and IP*. While the main goal of this paper is to establish a *universal* property of strongly mixing actions of countable abelian groups, our results, when applied to $\mathbb{Z}$-actions, offer a new way of dealing with strongly mixing transformations. In particular, we obtain several new characterizations of strong mixing for $\mathbb{Z}$-actions, including a result which can be viewed as the analogue of the weak mixing of all orders property established by Furstenberg in the course of his proof of Szemerédi's theorem. We also demonstrate the versatility of $\mathcal{R}$-limits by obtaining new characterizations of higher-order weak and mild mixing for actions of countable abelian groups.

Key words: ergodic theory, mixing of higher orders, Ramsey theory
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Contents

1. *Introduction*

Let $G = (G, +)$ be a countable discrete abelian group and let $(T_g)_{g \in G}$ be a measure-preserving G-action on a separable probability space $(X, \mathcal{A}, \mu)$. We will call the quadruple $(X, \mathcal{A}, \mu, (T_g)_{g \in G})$ a measure-preserving system. A measure-preserving system $(X, \mathcal{A}, \mu, (T_g)_{g \in G})$ is strongly mixing (or 2-mixing) if for any $A_0, A_1 \in \mathcal{A}$, one has

$$\lim_{g \to \infty} \mu(A_0 \cap T_g A_1) = \mu(A_0)\mu(A_1). \tag{1.1}$$

The goal of this paper is to obtain new results about higher-order mixing properties of strongly mixing actions of abelian groups. These results are motivated by the following classical problem going back to Rohlin (who formulated it for $\mathbb{Z}$-actions; see [27]).

ROHLIN'S PROBLEM. *Assume that a measure-preserving system $(X, \mathcal{A}, \mu, (T_g)_{g \in G})$ is strongly mixing. Is it true that, given any $\ell \geq 2$, the system $(X, \mathcal{A}, \mu, (T_g)_{g \in G})$ is $(\ell + 1)$-mixing? This would mean that for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any sequences $(g_k^{(1)})_{k \in \mathbb{N}}, \ldots, (g_k^{(\ell)})_{k \in \mathbb{N}}$ in G satisfying that,*

(i) *for any $j \in \{1, \ldots, \ell\}$,*

$$\lim_{k \to \infty} g_k^{(j)} = \infty, \tag{1.2}$$

(ii) *and, for any distinct $i, j \in \{1, \ldots, \ell\}$,*

$$\lim_{k \to \infty} (g_k^{(j)} - g_k^{(i)}) = \infty, \tag{1.3}$$

one has

$$\lim_{k \to \infty} \mu(A_0 \cap T_{g_k^{(1)}} A_1 \cap \cdots \cap T_{g_k^{(\ell)}} A_\ell) = \prod_{j=0}^{\ell} \mu(A_j). \tag{1.4}$$

While for $\mathbb{Z}$-actions Rohlin's problem is still unsolved, an example for $\mathbb{Z}^2$-actions, due to Ledrappier, shows that, in general, mixing does not imply mixing of higher orders [22] (the reader is referred to [30] for more Ledrappier-type examples for $\mathbb{Z}^d$-actions). More precisely, Ledrappier provided an example of a pair S, T of commuting mixing automorphisms of a compact abelian group X such that, for some measurable set $A \subseteq X$,

$$\mu(A \cap T^{2^n} A \cap S^{2^n} A) \underset{n \to \infty}{\not\longrightarrow} \mu^3(A),$$

where μ is the normalized Haar measure on X. The analysis of Ledrappier's example undertaken in [1] reveals that Ledrappier's system is 'almost mixing of all orders' in the sense that, for any $\ell \in \mathbb{N}$, if the sequences $(g_k^{(1)})_{k \in \mathbb{N}}, \ldots, (g_k^{(\ell)})_{k \in \mathbb{N}}$ in $\mathbb{Z}^2$ satisfy (1.2) and (1.3) and, in addition, the ℓ-tuples $(g_k^{(1)}, \ldots, g_k^{(\ell)})$ avoid certain rather rarefied subsets of $\mathbb{Z}^{2\ell}$, equation (1.4) holds for any measurable $A_0, \ldots, A_\ell \subseteq X$ (see [1, Theorem 3.3]). The results obtained in [1] were extended in [2] to a rather large family of systems of

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algebraic origin. The notable classes of $\mathbb{Z}$-actions for which it is known that 2-mixing implies mixing of all orders include ergodic automorphisms of compact groups [27], mixing transformations with singular spectrum [18], and mixing actions of finite rank [19, 28]. It is also known that some natural actions of various locally compact groups possess the property of mixing of all orders (see, for example, [12, 24, 26, 29]).

In view of the results obtained in [1, 2], one might wonder if it could possibly be true that, similarly to the case of Ledrappier's system, *any* strongly mixing action $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ of an abelian group G is, in some sense, almost mixing of all orders. The goal of this paper is to establish a result that can be interpreted as a positive answer to this question.

At this point, we would like to mention that in the special case when $G = \mathbb{Z}$, our main theorem (Theorem 1.21 below) has corollaries (Theorem 1.4 and Corollary 1.12) which provide new non-trivial characterizations of the notion of strong mixing in terms of the largeness of sets of the form

$$R_\epsilon^{a_1,\ldots,a_\ell}(A_0, \ldots, A_\ell) = \left\{n \in \mathbb{Z} \;\middle|\; \left|\mu(A_0 \cap T^{a_1 n} A_1 \cap \cdots \cap T^{a_\ell n} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\right| < \epsilon\right\} \tag{1.5}$$

and

$$R_\epsilon(A_0, \ldots, A_\ell) = \left\{(n_1, \ldots, n_\ell) \in \mathbb{Z}^\ell \;\middle|\; \left|\mu(A_0 \cap T^{n_1} A_1 \cap \cdots \cap T^{n_\ell} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\right| < \epsilon\right\}. \tag{1.6}$$

So, if it turns out that sets of the form (1.5) and (1.6) are not always cofinite, our results still imply that these sets are *large in some natural sense*, thereby establishing the validity of the claim that strongly mixing $\mathbb{Z}$-actions are almost mixing of all orders.

Let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system. Let $\ell \in \mathbb{N}$ and $\epsilon > 0$. For any $A_0, \ldots, A_\ell \in \mathcal{A}$ consider the set

$$R_\epsilon(A_0, \ldots, A_\ell) = \left\{(g_1, \ldots, g_\ell) \in G^\ell \;\middle|\; \left|\mu(A_0 \cap T_{g_1} A_1 \cap \cdots \cap T_{g_\ell} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\right| < \epsilon\right\}. \tag{1.7}$$

Clearly, the higher is the degree of multiple mixing of the system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$, the more massive should the set $R_\epsilon(A_0, \ldots, A_\ell)$ be as a subset of G^ℓ. While, for $\ell = 1$, the strong mixing property of $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ implies that the set $R_\epsilon(A_0, A_1)$ is cofinite, this is no longer the case for $\ell \geq 2$ even if our system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ is mixing of all orders. For example, for any 3-mixing system and any $A \in \mathcal{A}$ with $\mu(A) \in (0, 1)$, one has that, if $\epsilon > 0$ is small enough, the set

$$R_\epsilon(A, A, A) = \{(g_1, g_2) \in G^2 \mid |\mu(A \cap T_{g_1} A \cap T_{g_2} A) - \mu^3(A)| < \epsilon\}$$

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can only have a finite intersection with any of the 'lines' $\{(g, g) \mid g \in G\}$, $\{(g, 0) \mid g \in G\}$ and $\{(0, g) \mid g \in G\}$.

In what follows we will show that, for *any* mixing system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$, the subsets of G^ℓ which are of the form $\mathcal{R}_\epsilon(A_0, \ldots, A_\ell)$ possess a strong ubiquity property which we will call $\tilde{\Sigma}_\ell^*$ and which is quite a bit stronger than the properties of largeness associated with weakly and mildly mixing systems. In other words, we will show that for any strongly mixing system the complement of any set of the form $R_\epsilon(A_0, \ldots, A_\ell)$ is very 'small', giving meaning to the claim that $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ is 'almost strongly mixing' of all orders. This will be achieved with the help of *$\mathcal{R}$-limits*, a notion of convergence which is based on a classical combinatorial result due to Ramsey and, as we will see, is adequate for dealing with strongly mixing systems. (In particular, we will show that the $\tilde{\Sigma}_\ell^*$ property of the sets $R_\epsilon(A_0, \ldots, A_\ell)$ implies the strong mixing of $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$.)

We would like to remark that while the results that we obtain are not as sharp as those obtained in [1, 2], they have the advantage of being applicable to *any* strongly mixing system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$, where G is a countable abelian group. Moreover, as will be demonstrated in §6, the versatility of $\mathcal{R}$-limits allows one to obtain new and recover some old results pertaining to multiple recurrence properties of weakly and mildly mixing actions of countable abelian groups. We would also like to mention that, as will be seen in §3, the utilization of $\mathcal{R}$-limits brings to life many new equivalent characterizations of strong mixing (some of which bear a strong analogy with the familiar characterizations of weak mixing via convergence in density and mild mixing via IP-convergence).

Before introducing the above-mentioned notion of largeness for subsets of G^ℓ, we define a related and somewhat simpler notion in G.

Definition 1.1. Let $m \in \mathbb{N}$, let $(G, +)$ be a countable abelian group, and let $E \subseteq G$.

(1) We say that E is a Σ_m set if it is of the form

$$\{g_{k_1}^{(1)} + \cdots + g_{k_m}^{(m)} \mid k_1 < \cdots < k_m\}$$

where, for each $j \in \{1, \ldots, m\}$, $(g_k^{(j)})_{k\in\mathbb{N}}$ is a sequence in G which satisfies $\lim_{k\to\infty} g_k^{(j)} = \infty$.

(2) We say that E is a Σ_m^* set if it has a non-trivial intersection with every Σ_m set.

Remark 1.2

(a) Note that a subset of G is Σ_1^* if and only if it is cofinite. On the other hand, for any $m \geq 2$, a Σ_m^* set does not need to be cofinite. Moreover, one can show that for each $m \geq 2$, there exists a Σ_m^* set which fails to be a Σ_n^* set for each $n < m$ [8].

(b) The notion of Σ_m^* is similar to (but much stronger than) the notion of IP* which has an intrinsic connection to *mild* mixing and which plays an instrumental role in IP ergodic theory and in Ramsey theory (see, for example, [5, 14, 15]). The connection between these two notions will be discussed in detail in §5.

Since the sets $R_\epsilon(A_0, \ldots, A_\ell)$ are, by definition, subsets of G^ℓ, the above-defined notion of Σ_m^* has to be 'upgraded' to the subsets of the Cartesian power G^ℓ in order to

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be useful in the study of the asymptotic behavior of *multiparameter* expressions of the form

$$\mu(A_0 \cap T_{g_1} A_1 \cap \cdots \cap T_{g_\ell} A_\ell), \quad g_1, \ldots, g_\ell \in G. \tag{1.8}$$

However, it is worth noting that the family of Σ_m^* sets is quite adequate for dealing with 'diagonal' multicorrelation sequences. In the case $G = \mathbb{Z}$, such diagonal sequences have the form

$$\mu(A_0 \cap T^{a_1 n} A_1 \cap \cdots \cap T^{a_\ell n} A_\ell), \tag{1.9}$$

where $a_1, \ldots, a_\ell \in \mathbb{Z}$, and play an instrumental role in Furstenberg's ergodic approach to Szemerédi's theorem [13, 14]. For example, our main result (Theorem 1.21), while dealing with the multiparameter expressions (1.8), has strong corollaries of a 'diagonal' nature. The following theorem (which is a version of Theorem 4.4 below) is an example of a new result of this kind. Note the appearance of Σ_ℓ^* sets in the formulation.

THEOREM 1.3. *Let $(G, +)$ be a countable abelian group, let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a strongly mixing system, and let the homomorphisms $\phi_1, \ldots, \phi_\ell : G \to G$ be such that, for any $j \in \{1, \ldots, \ell\}$, $\ker(\phi_j)$ is finite and, for any $i \neq j$, $\ker(\phi_j - \phi_i)$ is also finite. Then, for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set*

$$R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A_0, \ldots, A_\ell) = \left\{ g \in G \;\middle|\; \left| \mu(A_0 \cap T_{\phi_1(g)} A_1 \cap \cdots \cap T_{\phi_\ell(g)} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j) \right| < \epsilon \right\} \tag{1.10}$$

is Σ_ℓ^*.

When G is finitely generated, Theorem 1.3 has a stronger version (Theorem 4.2), which in the case $G = \mathbb{Z}$ can be formulated as follows.

THEOREM 1.4. *Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system, let $\ell \in \mathbb{N}$, and let $a_1, \ldots, a_\ell$ be distinct non-zero integers. Then T is strongly mixing if and only if, for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set*

$$R_\epsilon^{a_1,\ldots,a_\ell}(A_0, \ldots, A_\ell) = \left\{ n \in \mathbb{Z} \;\middle|\; \left| \mu(A_0 \cap T^{a_1 n} A_1 \cap \cdots \cap T^{a_\ell n} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j) \right| < \epsilon \right\} \tag{1.11}$$

is Σ_ℓ^*.

For a related result see [7, Theorem 1.11]. See also [20].

Remark 1.5. One can view Theorem 1.4 as a strongly mixing analogue of two theorems due to Furstenberg which pertain to weak and mild mixing (see Theorems 4.11 and 9.27 in [14], respectively). The first of these two theorems states that the assumption that $(X, \mathcal{A}, \mu, T)$ is weakly mixing implies (and is implied by the fact) that the sets $R_\epsilon^{a_1,\ldots,a_\ell}(A_0, \ldots, A_\ell)$ defined in (1.11) have uniform density one. The second one states that the assumption that $(X, \mathcal{A}, \mu, T)$ is mildly mixing implies (and is implied by) the IP*

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property of the sets $R_\epsilon^{a_1,\ldots,a_\ell}(A_0,\ldots,A_\ell)$. These theorems are instrumental for the proofs of the ergodic Szemerédi [13] and IP-Szemerédi [15] theorems.

Note that, for $\ell = 1$, both diagonal (see (1.9)) and multiparameter (see (1.8)) multicorrelation sequences reduce to the classical expression $\mu(A_0 \cap T_g A_1)$. The following theorem (which is a very special case of stronger results to be established in this paper) shows that, even in the rather degenerated case $\ell = 1$, Σ_m^* sets provide a new characterization for the notion of strong mixing for actions of abelian groups.

THEOREM 1.6. *Let $(G,+)$ be a countable abelian group and let $(X,\mathcal{A},\mu,(T_g)_{g\in G})$ be a measure-preserving system. The following statements are equivalent.*

(i) *$(T_g)_{g\in G}$ is strongly mixing. In other words, for any $\epsilon > 0$ and any $A_0, A_1 \in \mathcal{A}$, the set*

$$R_\epsilon(A_0,A_1) = \{g \in G \mid |\mu(A_0 \cap T_g A_1) - \mu(A_0)\mu(A_1)| < \epsilon\}$$

is cofinite (that is, it is Σ_1^* in G).

(ii) *For any $m \in \mathbb{N}$, any $\epsilon > 0$ and any $A_0, A_1 \in \mathcal{A}$, the set $R_\epsilon(A_0,A_1)$ is Σ_m^* in G.*

(iii) *There exists an $m \in \mathbb{N}$ such that, for any $\epsilon > 0$ and any $A_0, A_1 \in \mathcal{A}$, the set $R_\epsilon(A_0,A_1)$ is Σ_m^* in G.*

We next define the modified versions of Σ_m and Σ_m^* sets which will be instrumental in dealing with the multiple mixing properties of strongly mixing systems.

Definition 1.7. Let $(G,+)$ be a countable abelian group and let $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ be two sequences in G. We say that $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ *grow apart* if $\lim_{k\to\infty}(g_k - h_k) = \infty$.

Definition 1.8. Let $(G,+)$ be a countable abelian group, let $d \in \mathbb{N}$ and let $(\mathbf{g}_k)_{k\in\mathbb{N}} = (g_{k,1},\ldots,g_{k,d})_{k\in\mathbb{N}}$ be a sequence in G^d. We say that $(\mathbf{g}_k)_{k\in\mathbb{N}}$ is *non-degenerated* if, for each $j \in \{1,\ldots,d\}$,

$$\lim_{k\to\infty} g_{k,j} = \infty.$$

Definition 1.9. Let $d, m \in \mathbb{N}$ and let $(G,+)$ be a countable abelian group.

(1) We say that $E \subseteq G^d$ is a $\tilde{\Sigma}_m$ set if it is of the form

$$\{\mathbf{g}_{k_1}^{(1)} + \cdots + \mathbf{g}_{k_m}^{(m)} \mid k_1 < \cdots < k_m\}$$

where, for each $j \in \{1,\ldots,m\}$, $(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}} = (g_{k,1}^{(j)},\ldots,g_{k,d}^{(j)})_{k\in\mathbb{N}}$ is a non- degenerated sequence in G^d and for any distinct $t, t' \in \{1,\ldots,d\}$ the sequences $(g_{k,t}^{(j)})_{k\in\mathbb{N}}$ and $(g_{k,t'}^{(j)})_{k\in\mathbb{N}}$ grow apart. (Note that if $d = 1$, then $E \subseteq G$ is a Σ_m set if and only if it is a $\tilde{\Sigma}_m$ set.)

(2) We say that $E \subseteq G^d$ is a $\tilde{\Sigma}_m^*$ set if it has a non-trivial intersection with every $\tilde{\Sigma}_m$ set in G^d.

Remark 1.10. The main difference between $\tilde{\Sigma}_m$ sets and Σ_m sets is that $\tilde{\Sigma}_m$ sets are subsets of Cartesian powers of G and have a built-in feature which guarantees that, asymptotically,

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the elements of $\tilde{\Sigma}_m$ sets stay away from 'degenerated' subsets such as the following subsets of G^3: $\{(g, g, g) \mid g \in G\}$, $\{(g, 2g, 0) \mid g \in G\}$ and $\{(g, g, h) \mid g, h \in G\}$.

The following theorem, which is a corollary of Theorem 1.21 below, demonstrates the relevance of $\tilde{\Sigma}_m$ sets for dealing with mixing of higher orders.

THEOREM 1.11. *Let $(G, +)$ be a countable abelian group and let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system. The following statements are equivalent.*

(i) *$(T_g)_{g\in G}$ is strongly mixing.*

(ii) *For any $\ell \in \mathbb{N}$, any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set*

$$R_\epsilon(A_0, \ldots, A_\ell) = \bigg\{(g_1, \ldots, g_\ell) \in G^\ell \;\bigg|\; \bigg|\mu(A_0 \cap T_{g_1} A_1 \cap \cdots \cap T_{g_\ell} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\bigg| < \epsilon\bigg\}$$

is $\tilde{\Sigma}_\ell^*$ in G^ℓ.

(iii) *There exists an $\ell \in \mathbb{N}$ such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set $R_\epsilon(A_0, \ldots, A_\ell)$ is $\tilde{\Sigma}_\ell^*$ in G^ℓ.*

We take the liberty of stating explicitly the following special case of Theorem 1.11 to stress the applicability of the apparatus developed in this paper to $\mathbb{Z}$-actions.

COROLLARY 1.12. *Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system. The following statements are equivalent.*

(i) *T is strongly mixing.*

(ii) *For any $\ell \in \mathbb{N}$, any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set*

$$R_\epsilon(A_0, \ldots, A_\ell) = \bigg\{(n_1, \ldots, n_\ell) \in \mathbb{Z}^\ell \;\bigg|\; \bigg|\mu(A_0 \cap T^{n_1} A_1 \cap \cdots \cap T^{n_\ell} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\bigg| < \epsilon\bigg\}$$

is $\tilde{\Sigma}_\ell^*$ in $\mathbb{Z}^\ell$.

(iii) *There exists an $\ell \in \mathbb{N}$ such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set $R_\epsilon(A_0, \ldots, A_\ell)$ is $\tilde{\Sigma}_\ell^*$ in $\mathbb{Z}^\ell$.*

We introduce now the notion of convergence that is utilized in the proof of Theorem 1.11 and is based on the classical Ramsey theorem (which, for convenience of the reader, we state below). We remark that variants of this notion of convergence can also be found in [**10**, **11**, **21**, **23**, **25**, **31**]. Given $m \in \mathbb{N}$ and an infinite set $S \subseteq \mathbb{N}$, we denote by $S^{(m)}$ the family of all m-element subsets of S. When writing $\{k_1, \ldots, k_m\} \in S^{(m)}$, we will always assume that $k_1 < \cdots < k_m$.

THEOREM 1.13. (Ramsey's theorem) *Let $r, m \in \mathbb{N}$ and let $C_1, \ldots, C_r \subseteq \mathbb{N}^{(m)}$ be such that*

$$\mathbb{N}^{(m)} = \bigcup_{j=1}^{r} C_j. \tag{1.12}$$

Then there exist $j_0 \in \{1, \ldots, r\}$ *and an infinite subset* $S \subseteq \mathbb{N}$ *satisfying* $S^{(m)} \subseteq C_{j_0}$.

Remark 1.14. It is easy to see that Theorem 1.13 can be formulated in the following equivalent form that will be frequently used in the sequel.

Let $r, m \in \mathbb{N}$, *let* P *be an infinite subset of* $\mathbb{N}$ *and let* $C_1, \ldots, C_r \subseteq \mathbb{N}^{(m)}$ *be such that*

$$P^{(m)} \subseteq \bigcup_{j=1}^{r} C_j. \tag{1.13}$$

Then there exist $j_0 \in \{1, \ldots, r\}$ *and an infinite subset* $S \subseteq P$ *satisfying* $S^{(m)} \subseteq C_{j_0}$.

Definition 1.15. Let $m \in \mathbb{N}$, let (X, d) be a compact metric space, let $x \in X$, let $(x_\alpha)_{\alpha \in \mathbb{N}^{(m)}}$ be an $\mathbb{N}^{(m)}$-sequence in X, and let S be an infinite subset of $\mathbb{N}$. We write

$$\mathcal{R}\text{-}\lim_{\alpha \in S^{(m)}} x_\alpha = x \tag{1.14}$$

if, for every $\epsilon > 0$, there exists $\alpha_0 \in \mathbb{N}^{(m)}$ such that, for any $\alpha \in S^{(m)}$ satisfying $\min \alpha > \max \alpha_0$, one has

$$d(x_\alpha, x) < \epsilon.$$

The following theorem can be viewed as a version of Bolzano–Weierstrass theorem for $\mathcal{R}$-convergence. It follows from Theorem 1.13 with the help of a diagonalization argument.

THEOREM 1.16. *Let* $m \in \mathbb{N}$, *let* (X, d) *be a compact metric space and let* $(x_\alpha)_{\alpha \in \mathbb{N}^{(m)}}$ *be an* $\mathbb{N}^{(m)}$*-sequence in X. Then, for any infinite set* $S_1 \subseteq \mathbb{N}$, *there exist an* $x \in X$ *and an infinite set* $S \subseteq S_1$ *such that*

$$\mathcal{R}\text{-}\lim_{\alpha \in S^{(m)}} x_\alpha = x. \tag{1.15}$$

Remark 1.17. Let $(x_\alpha)_{\alpha \in \mathbb{N}^{(m)}}$ be an $\mathbb{N}^{(m)}$-sequence in a compact metric space (X, d). The above-introduced $\mathcal{R}$-limits have an intrinsic connection with iterated limits of the form

$$\lim_{j_1 \to \infty} \cdots \lim_{j_m \to \infty} x_{\{k_{j_1}, \ldots, k_{j_m}\}}. \tag{1.16}$$

The goal of this extended remark is to clarify this connection.

(a) Using the compactness of X, one can show with the help of a diagonalization argument that for any increasing sequence $(k_j)_{j \in \mathbb{N}}$, there exists a subsequence $(k'_j)_{j \in \mathbb{N}}$ for which all the limits in (1.16) exist.

(b) By Theorem 1.16, there exists an increasing sequence of natural numbers $(k_j)_{j \in \mathbb{N}}$ so that, for $S = \{k_j \mid j \in \mathbb{N}\}$, $\mathcal{R}\text{-}\lim_{\alpha \in S^{(m)}} x_\alpha$ exists. Let $(k'_j)_{j \in \mathbb{N}}$ be the subsequence of $(k_j)_{j \in \mathbb{N}}$ which is guaranteed to exist by (a). Letting $S_1 = \{k'_j \mid j \in \mathbb{N}\}$, we have

$$\mathcal{R}\text{-}\lim_{\alpha \in S_1^{(m)}} x_\alpha = \lim_{j_1 \to \infty} \cdots \lim_{j_m \to \infty} x_{\{k'_{j_1}, \ldots, k'_{j_m}\}}. \tag{1.17}$$

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(c) When $X = \{1, \ldots, r\}$, one can use (a) to prove Theorem 1.13. Let $r, m \in \mathbb{N}$ and consider a partition $\mathbb{N}^{(m)} = \bigcup_{j=1}^{r} C_j$. Let $(x_\alpha)_{\alpha \in \mathbb{N}^{(m)}}$ be defined by $x_\alpha = j$ if $\alpha \in C_j$. For some increasing sequence $(k_j)_{j \in \mathbb{N}}$ in $\mathbb{N}$ there exists a $j_0 \in \{1, \ldots, r\}$ such that

$$\lim_{j_1 \to \infty} \cdots \lim_{j_m \to \infty} x_{\{k_{j_1}, \ldots, k_{j_m}\}} = j_0.$$

By using a diagonalization argument, we obtain a subsequence $(k'_j)_{j \in \mathbb{N}}$ of $(k_j)_{j \in \mathbb{N}}$ with the property that $x_{\{k'_{j_1}, \ldots, k'_{j_m}\}} = j_0$ for any $j_1 < \cdots < j_m$. Now let $S = \{k'_j \mid j \in \mathbb{N}\}$. It follows that $S^{(m)} \subseteq C_{j_0}$.

Before formulating our main result, we need two more definitions.

Definition 1.18. Let $m \in \mathbb{N}$ and let $(G, +)$ be a countable abelian group. For any sequence $(\mathbf{g}_k)_{k \in \mathbb{N}} = (g_{k,1}, \ldots, g_{k,m})_{k \in \mathbb{N}}$ and any $\alpha = \{k_1, \ldots, k_m\} \in \mathbb{N}^{(m)}$ we let

$$g_\alpha = \sum_{j=1}^{m} g_{k_j, j} = g_{k_1,1} + g_{k_2,2} + \cdots + g_{k_m,m}, \tag{1.18}$$

where $k_1 < \cdots < k_m$.

Definition 1.19. Let $m \in \mathbb{N}$, let $(G, +)$ be a countable abelian group and let

$$(\mathbf{g}_k)_{k \in \mathbb{N}} = (g_{k,1}, \ldots, g_{k,m})_{k \in \mathbb{N}} \quad \text{and} \quad (\mathbf{h}_k)_{k \in \mathbb{N}} = (h_{k,1}, \ldots, h_{k,m})_{k \in \mathbb{N}}$$

be sequences in G^m. We say that $(\mathbf{g}_k)_{k \in \mathbb{N}}$ and $(\mathbf{h}_k)_{k \in \mathbb{N}}$ are *essentially distinct* if, for each $t \in \{1, \ldots, m\}$, $(g_{k,t})_{k \in \mathbb{N}}$ and $(h_{k,t})_{k \in \mathbb{N}}$ grow apart (that is, $\lim_{k \to \infty}(g_{k,t} - h_{k,t}) = \infty$).

Remark 1.20. The following observation indicates the natural connection between non-degenerated, essentially distinct sequences in G^m and $\tilde{\Sigma}_m$ sets. Let $d, m \in \mathbb{N}$ and let $(G, +)$ be a countable abelian group. Then for any non-degenerated and essentially distinct sequences

$$(\mathbf{g}_k^{(j)})_{k \in \mathbb{N}} = (g_{k,1}^{(j)}, \ldots, g_{k,m}^{(j)})_{k \in \mathbb{N}}, \quad j \in \{1, \ldots, d\},$$

in G^m, the set

$$\begin{aligned}
&\{(g_\alpha^{(1)}, \ldots, g_\alpha^{(d)}) \mid \alpha \in \mathbb{N}^{(m)}\} \\
&\quad = \{(g_{k_1,1}^{(1)} + \cdots + g_{k_m,m}^{(1)}, \ldots, g_{k_1,1}^{(d)} + \cdots + g_{k_m,m}^{(d)}) \mid k_1 < \cdots < k_m\} \\
&\quad = \{(g_{k_1,1}^{(1)}, \ldots, g_{k_1,1}^{(d)}) + \cdots + (g_{k_m,m}^{(1)}, \ldots, g_{k_m,m}^{(d)}) \mid k_1 < \cdots < k_m\}
\end{aligned}$$

is a $\tilde{\Sigma}_m$ set in G^d.

We are now ready to formulate our main result (it appears as Theorem 3.1 in §3). It incorporates some of the characterizations of strongly mixing systems which were mentioned above.

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THEOREM 1.21. *Let $\ell \in \mathbb{N}$, let $(G, +)$ be a countable abelian group and let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system. The following statements are equivalent.*

(i) $(T_g)_{g\in G}$ *is strongly mixing.*

(ii) *For any non-degenerated and essentially distinct sequences*

$$(\boldsymbol{g}_k^{(j)})_{k\in\mathbb{N}}, \quad j \in \{1, \ldots, \ell\},$$

in $G^{(\ell)}$, there exists an infinite $S \subseteq \mathbb{N}$ such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$,

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \mu(A_0 \cap T_{g_\alpha^{(1)}} A_1 \cap \cdots \cap T_{g_\alpha^{(\ell)}} A_\ell) = \prod_{j=0}^{\ell} \mu(A_j). \tag{1.19}$$

More explicitly, if

$$(\boldsymbol{g}_k^{(j)})_{k\in\mathbb{N}} = (g_{k,1}^{(j)}, \ldots, g_{k,\ell}^{(j)})_{k\in\mathbb{N}},$$

for each $j \in \{1, \ldots, \ell\}$, then

$$\mathcal{R}\text{-}\lim_{\{k_1,\ldots,k_\ell\}\in S^{(\ell)}} \mu(A_0 \cap T_{g_{k_1,1}^{(1)}+\cdots+g_{k_\ell,\ell}^{(1)}} A_1 \cap \cdots \cap T_{g_{k_1,1}^{(\ell)}+\cdots+g_{k_\ell,\ell}^{(\ell)}} A_\ell) = \prod_{j=0}^{\ell} \mu(A_j).$$

(iii) *For any $\epsilon > 0$ and any $A_0, \ldots, A_\ell \in \mathcal{A}$, the set*

$$R_\epsilon(A_0, \ldots, A_\ell) = \bigg\{(g_1, \ldots, g_\ell) \in G^\ell \;\bigg|\; \bigg|\mu(A_0 \cap T_{g_1} A_1 \cap \cdots \cap T_{g_\ell} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\bigg| < \epsilon\bigg\}$$

is $\tilde{\Sigma}_\ell^*$ in G^ℓ.

(iv) *For any $\epsilon > 0$ and any $A_0, A_1 \in \mathcal{A}$, the set $R_\epsilon(A_0, A_1)$ is Σ_ℓ^* in G.*

The structure of this paper is as follows. In §2 we review some basic facts about couplings of probability spaces and establish some auxiliary results which will be needed in §3 and §6. In §3 we prove our main result, Theorem 1.21 (=Theorem 3.1). In §4 we derive some diagonal results for strongly mixing systems. In §5 we describe the largeness properties of $\tilde{\Sigma}_m^*$ sets and, more specifically, of the sets $R_\epsilon(A_0, \ldots, A_\ell)$. We also juxtapose the properties of $\tilde{\Sigma}_m^*$ sets with those of $\widetilde{\mathrm{IP}}^*$ sets and sets of uniform density one which are characteristic, correspondingly, of mild and weak mixing. In §6 we utilize the methods developed in §2 and §5 to obtain analogues of Theorem 1.21 for mildly and weakly mixing systems.

Remark 1.22. Throughout this paper, we will be tacitly assuming that the measure-preserving systems $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ that we are working with are *regular*, meaning that the underlying probability space $(X, \mathcal{A}, \mu)$ is regular (that is, X is a compact metric space and $\mathcal{A} = \text{Borel}(X)$). Note that this assumption can be made without loss of generality since every separable measure-preserving system is equivalent to a regular one (see, for instance, [**14**, Proposition 5.3]).

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2. *Some auxiliary facts involving couplings and $\mathcal{R}$-limits*

In this section we review some basic facts about couplings of probability spaces and establish some auxiliary results which will be needed in §3 and §6.

Definition 2.1. Let $N \in \mathbb{N}$. Given regular probability spaces $\mathbf{X}_j = (X_j, \mathcal{A}_j, \mu_j)$, $j \in \{1, \ldots, N\}$, a *coupling* of $\mathbf{X}_1, \ldots, \mathbf{X}_N$ is a Borel probability measure λ defined on the measurable space

$$\bigg(\prod_{j=1}^{N} X_j, \bigotimes_{j=1}^{N} \mathcal{A}_j\bigg)$$

having the property that, for any $j \in \{1, \ldots, N\}$ and any $A \in \mathcal{A}_j$, $\lambda(\pi_j^{-1}(A)) = \mu_j(A)$, where $\pi_j : \prod_{i=1}^{N} X_i \to X_j$ is the projection map onto the *j*th coordinate of $\prod_{j=1}^{N} X_j$. (A coupling is just a *joining* of the trivial measure-preserving systems $(X_j, \mathcal{A}_j, \mu_j, \mathrm{Id}_j)$, $j \in \{1, \ldots, N\}$, where $\mathrm{Id}_j : X_j \to X_j$ denotes the identity map on X_j.)

We will let $\mathcal{C}(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ denote the set of all couplings of $\mathbf{X}_1, \ldots, \mathbf{X}_N$. $\mathcal{C}(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ is a closed subspace of the set of all probability Borel measures on $\prod_{j=1}^{N} X_j$ endowed with the weak-* topology. With this topology, $\mathcal{C}(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ is a compact metrizable space. Given a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $\mathcal{C}(\mathbf{X}_1, \ldots, \mathbf{X}_N)$,

$$\lambda_k \xrightarrow[k\to\infty]{} \lambda$$

if and only if, for any $A_1 \in \mathcal{A}_1, \ldots, A_N \in \mathcal{A}_N$,

$$\lambda_k(A_1 \times \cdots \times A_N) \xrightarrow[k\to\infty]{} \lambda(A_1 \times \cdots \times A_N).$$

The following proposition follows immediately from the compactness of $\mathcal{C}(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ and Theorem 1.16.

PROPOSITION 2.2. *Let $\boldsymbol{X}_j = (X_j, \mathcal{A}_j, \mu_j)$, $j \in \{1, \ldots, N\}$, be regular probability spaces. For any $m \in \mathbb{N}$, any infinite $S \subseteq \mathbb{N}$ and any $\mathbb{N}^{(m)}$-sequence $(\lambda_\alpha)_{\alpha\in\mathbb{N}^{(m)}}$ in $\mathcal{C}(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N)$,*

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(m)}} \lambda_\alpha = \lambda$$

if and only if, for any $A_1 \in \mathcal{A}_1, \ldots, A_N \in \mathcal{A}_N$,

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(m)}} \lambda_\alpha(A_1 \times \cdots \times A_N) = \lambda(A_1 \times \cdots \times A_N).$$

Our next goal is to establish a useful criterion for mixing of higher orders (Proposition 2.9). First, we need a definition and two lemmas.

Definition 2.3. Let $(Z, \mathcal{D}, \lambda)$ be a regular probability space and let, for each $k \in \mathbb{N}$, $T_k : Z \to Z$ be a measure-preserving transformation. The sequence $(T_k)_{k\in\mathbb{N}}$ has the mixing property if, for every $A_0, A_1 \in \mathcal{D}$,

$$\lim_{k\to\infty} \lambda(A_0 \cap T_k^{-1} A_1) = \lambda(A_0)\lambda(A_1).$$

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Remark 2.4

(a) If each of the transformations T_k, $k \in \mathbb{N}$, is invertible, $(T_k)_{k\in\mathbb{N}}$ has the mixing property if and only if $(T_k^{-1})_{k\in\mathbb{N}}$ has the mixing property.

(b) $(T_k)_{k\in\mathbb{N}}$ has the mixing property if and only if, for any $f, g \in L^2(\mu)$,

$$\lim_{k\to\infty} \int_X f T_k g \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu \int_X g \, \mathrm{d}\mu.$$

LEMMA 2.5. *Let $\boldsymbol{X} = (X, \mathcal{A}, \mu)$ and $\boldsymbol{Y} = (Y, \mathcal{B}, \nu)$ be regular probability spaces. For each $k \in \mathbb{N}$, let $T_k : Y \to Y$ be a measure-preserving transformation, and assume that the sequence $(T_k)_{k\in\mathbb{N}}$ has the mixing property. Let λ_0 be a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Assume that λ is a probability measure on $\mathcal{A} \otimes \mathcal{B}$ such that, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, one has*

$$\lim_{k\to\infty} \lambda_0((\mathrm{Id} \times T_k^{-1})(A \times B)) = \lambda(A \times B). \tag{2.1}$$

Then $\lambda = \mu \otimes \nu$.

Proof. Note that it suffices to show that, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\lambda(A \times B) = \mu(A)\nu(B). \tag{2.2}$$

Fix $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Since $\mathbb{1}_A \otimes \mathbb{1}_B = (\mathbb{1}_A \otimes \mathbb{1}_Y)(\mathbb{1}_X \otimes \mathbb{1}_B)$, we have by (2.1) that

$$\begin{aligned}\int_{X\times Y} (\mathbb{1}_A \otimes \mathbb{1}_Y)(\mathbb{1}_X \otimes \mathbb{1}_B) \, \mathrm{d}\lambda &= \lambda(A \times B)\\ &= \lim_{k\to\infty} \lambda_0((\mathrm{Id} \times T_k^{-1})(A \times B))\\ &= \lim_{k\to\infty} \int_{X\times Y} (\mathrm{Id} \times T_k)(\mathbb{1}_A \otimes \mathbb{1}_Y)(\mathrm{Id} \times T_k)(\mathbb{1}_X \otimes \mathbb{1}_B) \, \mathrm{d}\lambda_0.\end{aligned} \tag{2.3}$$

Note that $(\mathrm{Id} \times T_k)(\mathbb{1}_A \otimes \mathbb{1}_Y) = \mathbb{1}_A \otimes \mathbb{1}_Y$ and, if we regard $\mathcal{B}$ as a sub-σ-algebra of $\mathcal{A} \otimes \mathcal{B}$, $\lambda_0|_{\mathcal{B}} = \nu$. The rightmost expression in (2.3) equals

$$\begin{aligned}&\lim_{k\to\infty} \int_{X\times Y} (\mathbb{1}_A \otimes \mathbb{1}_Y)(\mathbb{1}_X \otimes T_k \mathbb{1}_B) \, \mathrm{d}\lambda_0\\ &\quad= \lim_{k\to\infty} \int_{X\times Y} \mathbb{E}(\mathbb{1}_A \otimes \mathbb{1}_Y \mid \mathcal{B})(\mathbb{1}_X \otimes T_k \mathbb{1}_B) \, \mathrm{d}\lambda_0\\ &\quad= \lim_{k\to\infty} \int_Y \mathbb{E}(\mathbb{1}_A \otimes \mathbb{1}_Y \mid \mathcal{B}) T_k \mathbb{1}_B \, \mathrm{d}\nu,\end{aligned} \tag{2.4}$$

where $\mathbb{E}(\mathbb{1}_A \otimes \mathbb{1}_Y \mid \mathcal{B})$ denotes the conditional expectation of $\mathbb{1}_A \otimes \mathbb{1}_Y$ with respect to $\mathcal{B}$.

But $(T_k)_{k\in\mathbb{N}}$ has the mixing property, so the rightmost expression in (2.4) equals

$$\int_Y \mathbb{E}(\mathbb{1}_A \otimes \mathbb{1}_Y \mid \mathcal{B}) \, \mathrm{d}\nu \int_Y \mathbb{1}_B \, \mathrm{d}\nu = \lambda(A \times B). \tag{2.5}$$

By noting that

$$\int_Y \mathbb{E}(\mathbb{1}_A \otimes \mathbb{1}_Y \mid \mathcal{B}) \, \mathrm{d}\nu = \int_{X\times Y} (\mathbb{1}_A \otimes \mathbb{1}_Y) \, \mathrm{d}\lambda_0 = \int_X \mathbb{1}_A \, \mathrm{d}\mu,$$

we have that (2.5) equals $\mu(A)\nu(B)$. □

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LEMMA 2.6. *Let $m \in \mathbb{N}$, let (X, d) be a compact metric space, and let $(x_\alpha)_{\alpha\in\mathbb{N}^{(m+1)}}$ be an $\mathbb{N}^{(m+1)}$-sequence in X. Assume that there exists an infinite $S \subseteq \mathbb{N}$ with the following properties: (a) for some $x \in X$, $\mathcal{R}\text{-}\lim_{\alpha\in S^{(m+1)}} x_\alpha = x$; (b) for each $k \in S$, there exists $y_k \in X$ such that*

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(m)},\, k<\min\alpha} x_{\{k\}\cup\alpha} = y_k.$$

Then

$$\lim_{k\to\infty,\, k\in S} \mathcal{R}\text{-}\lim_{\alpha\in S^{(m)},\, k<\min\alpha} x_{\{k\}\cup\alpha} = \lim_{k\to\infty,\, k\in S} y_k = \mathcal{R}\text{-}\lim_{\alpha\in S^{(m+1)}} x_\alpha.$$

Proof. Let $\epsilon > 0$. Note that (1) there exists $k_0 \in S$ such that, for any $\alpha \in S^{(m+1)}$ with $k_0 \leq \min\alpha$, $d(x_\alpha, x) < \epsilon/2$ and (2) for any $k \in S$, there exists an $\alpha_k \in S^{(m)}$ such that, for any $\alpha \in S^{(m)}$ with $\min\alpha > \max(\alpha_k \cup \{k\})$, $d(x_{\{k\}\cup\alpha}, y_k) < \epsilon/2$. It follows that, for any $k \in S$ with $k \geq k_0$ and any $\alpha \in S^{(m)}$ with $\min\alpha > \max(\alpha_k \cup \{k\})$, $d(y_k, x) < d(x_{\{k\}\cup\alpha}, y_k) + d(x_{\{k\}\cup\alpha}, x) < \epsilon$. Since $\epsilon > 0$ was arbitrary,

$$\lim_{k\to\infty,\, k\in S} y_k = x = \mathcal{R}\text{-}\lim_{\alpha\in S^{(m+1)}} x_\alpha. \qquad \square$$

Remark 2.7. Let $m \in \mathbb{N}$ and let $(x_\alpha)_{\alpha\in\mathbb{N}^{(m+1)}}$ be an $\mathbb{N}^{(m+1)}$-sequence in a compact metric space X. By applying Theorem 1.16 first to the $\mathbb{N}^{(m)}$-sequence $(\omega_\alpha)_{\alpha\in\mathbb{N}^{(m)}} = ((x_{\{k\}\cup\alpha})_{k\in\mathbb{N}})_{\alpha\in\mathbb{N}^{(m)}}$ in $X^{\mathbb{N}}$ (here $x_{\{k\}\cup\alpha} = x_0$ for some fixed $x_0 \in X$, whenever $k \geq \min\alpha$), and then to the $\mathbb{N}^{(m+1)}$-sequence $(x_\alpha)_{\alpha\in\mathbb{N}^{(m+1)}}$, we obtain an infinite set $S \subseteq \mathbb{N}$ for which (a) and (b) in the statement of Lemma 2.6 hold. A similar reasoning shows that one can pick S to be a subset of any prescribed in advance infinite set $S_1 \subseteq \mathbb{N}$.

Remark 2.8. In Remark 1.17(c), we indicated how the utilization of iterated limits

$$\lim_{j_1\to\infty} \cdots \lim_{j_m\to\infty} x_{\{k_{j_1},\ldots,k_{j_m}\}}$$

leads to a proof of Ramsey's theorem (Theorem 1.13). In this remark, we show that Lemma 2.6 and Remark 2.7 (which are corollaries of Ramsey's Theorem) imply that, for any infinite set $S_1 \subseteq \mathbb{N}$ and any $\mathbb{N}^{(m)}$-sequence $(x_\alpha)_{\alpha\in\mathbb{N}^{(m)}}$ in a compact metric space X, there exists an increasing sequence $(k_j)_{j\in\mathbb{N}}$ in S_1 such that, for $S = \{k_j \mid j \in \mathbb{N}\}$, each of the limits in the formula

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(m)}} x_\alpha = \lim_{j_1\to\infty} \cdots \lim_{j_m\to\infty} x_{\{k_{j_1},\ldots,k_{j_m}\}}$$

exists. The proof is by induction on $m \in \mathbb{N}$. When $m = 1$, the result follows from the compactness of X. Now let $m > 1$ and let S_1 be an infinite subset of $\mathbb{N}$. By Remark 2.7 and Lemma 2.6, there exists an increasing sequence $(k_j)_{j\in\mathbb{N}}$ in S_1 such that, for $S = \{k_j \mid j \in \mathbb{N}\}$,

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(m)}} x_\alpha = \lim_{j\to\infty} \mathcal{R}\text{-}\lim_{\alpha\in S^{(m-1)}} x_{\{k_j\}\cup\alpha}.$$

The result now follows from the inductive hypothesis applied to the infinite set $S \subseteq \mathbb{N}$ and the $\mathbb{N}^{(m-1)}$-sequence $((x_{\{k\}\cup\alpha})_{k\in\mathbb{N}})_{\alpha\in\mathbb{N}^{(m-1)}}$ in the compact metric space $X^{\mathbb{N}}$.

The following proposition provides a useful technical tool for establishing higher-order mixing properties of measure-preserving systems. It will be instrumental in §3 for dealing with strongly mixing systems and in §6 where we will focus on mildly and weakly mixing systems.

PROPOSITION 2.9. *Let $(G, +)$ be a countable abelian group, let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system, let $\ell \in \mathbb{N}$ and, for each $j \in \{1, \ldots, \ell\}$, let*

$$(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}} = (g_{k,1}^{(j)}, \ldots, g_{k,\ell}^{(j)})_{k\in\mathbb{N}}$$

be a sequence in G^ℓ. Suppose that, for any $t \in \{1, \ldots, \ell\}$ and any $j \in \{1, \ldots, \ell\}$, $(T_{g_{k,t}^{(j)}})_{k\in\mathbb{N}}$ has the mixing property and that, for any t and any $i \neq j$, $(T_{(g_{k,t}^{(j)}-g_{k,t}^{(i)})})_{k\in\mathbb{N}}$ also has the mixing property. Then, there exists an infinite set $S \subseteq \mathbb{N}$ such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$,

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \mu(A_0 \cap T_{g_\alpha^{(1)}} A_1 \cap \cdots \cap T_{g_\alpha^{(\ell)}} A_\ell) = \prod_{j=0}^{\ell} \mu(A_j).$$

Proof. The proof is by induction on ℓ. When $\ell = 1$, it follows from our hypothesis that, for any $A_0, A_1 \in \mathcal{A}$,

$$\mathcal{R}\text{-}\lim_{\alpha\in\mathbb{N}^{(1)}} \mu(A_0 \cap T_{g_\alpha^{(1)}} A_1) = \lim_{k\to\infty} \mu(A_0 \cap T_{g_{k,1}^{(1)}} A_1) = \mu(A_0)\mu(A_1).$$

Now fix $\ell \in \mathbb{N}$ and suppose that Proposition 2.9 holds for any $\ell' \leq \ell$. Let $\mathbf{X} = (X, \mathcal{A}, \mu)$ and let $\mu_\Delta \in \mathcal{C} = \mathcal{C}(\underbrace{\mathbf{X}, \ldots, \mathbf{X}}_{\ell+2 \text{ times}})$ be defined by $\mu(A_0 \times \cdots \times A_{\ell+1}) = \mu(A_0 \cap \cdots \cap A_{\ell+1})$. By the inductive hypothesis, there exists an infinite $S \subseteq \mathbb{N}$ such that, for any $A_1, \ldots, A_{\ell+1} \in \mathcal{A}$,

$$\begin{aligned}
&\mathcal{R}\text{-}\lim_{\{j_1,\ldots,j_\ell\}\in S^{(\ell)}} \mu_\Delta(X \times T_{g_{j_1,2}^{(1)}+\cdots+g_{j_\ell,\ell+1}^{(1)}} A_1 \times \cdots \times T_{g_{j_1,2}^{(\ell+1)}+\cdots+g_{j_\ell,\ell+1}^{(\ell+1)}} A_{\ell+1}) \\
&\quad = \mathcal{R}\text{-}\lim_{\{j_1,\ldots,j_\ell\}\in S^{(\ell)}} \mu(X \cap T_{g_{j_1,2}^{(1)}+\cdots+g_{j_\ell,\ell+1}^{(1)}} A_1 \cap \cdots \cap T_{g_{j_1,2}^{(\ell+1)}+\cdots+g_{j_\ell,\ell+1}^{(\ell+1)}} A_{\ell+1}) \\
&\quad = \mathcal{R}\text{-}\lim_{\{j_1,\ldots,j_\ell\}\in S^{(\ell)}} \mu(T_{g_{j_1,2}^{(1)}+\cdots+g_{j_\ell,\ell+1}^{(1)}} A_1 \cap \cdots \cap T_{g_{j_1,2}^{(\ell+1)}+\cdots+g_{j_\ell,\ell+1}^{(\ell+1)}} A_{\ell+1}) \\
&\quad = \mathcal{R}\text{-}\lim_{\{j_1,\ldots,j_\ell\}\in S^{(\ell)}} \mu(A_1 \cap T_{(g_{j_1,2}^{(2)}-g_{j_1,2}^{(1)})+\cdots+(g_{j_\ell,\ell+1}^{(2)}-g_{j_\ell,\ell+1}^{(1)})} \\
&\qquad A_2 \cap \cdots \cap T_{(g_{j_1,2}^{(\ell+1)}-g_{j_1,2}^{(1)})+\cdots+(g_{j_\ell,\ell+1}^{(\ell+1)}-g_{j_\ell,\ell+1}^{(1)})} A_{\ell+1}) \\
&\quad = \prod_{j=1}^{\ell+1} \mu(A_j).
\end{aligned} \tag{2.6}$$

By Theorem 1.16 and the compactness of $\mathcal{C}$, there exist an infinite set $S_0 \subseteq S$ and $\lambda_0 \in \mathcal{C}$ such that, for any $A_0, \ldots, A_{\ell+1} \in \mathcal{A}$,

$$\mathcal{R}\text{-}\lim_{\{j_1,\ldots,j_\ell\}\in S_0^{(\ell)}} \mu_\Delta(A_0 \times T_{g_{j_1,2}^{(1)}+\cdots+g_{j_\ell,\ell+1}^{(1)}} A_1 \times \cdots \times T_{g_{j_1,2}^{(\ell+1)}+\cdots+g_{j_\ell,\ell+1}^{(\ell+1)}} A_{\ell+1}) = \lambda_0\Big(\prod_{j=0}^{\ell+1} A_j\Big). \tag{2.7}$$

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Likewise, there exist an infinite set $S_1 \subseteq S_0$ and $\lambda \in \mathcal{C}$ such that, for any $A_0, \ldots, A_{\ell+1} \in \mathcal{A}$,

$$\mathcal{R}\text{-}\lim_{\{j_1,\ldots,j_{\ell+1}\}\in S_1^{(\ell+1)}} \mu_\Delta(A_0 \times T_{g^{(1)}_{j_1,1}+\cdots+g^{(1)}_{j_{\ell+1},\ell+1}} A_1 \times \cdots \times T_{g^{(\ell+1)}_{j_1,1}+\cdots+g^{(\ell+1)}_{j_{\ell+1},\ell+1}} A_{\ell+1}) = \lambda\bigg(\prod_{j=0}^{\ell+1} A_j\bigg). \tag{2.8}$$

Let $\mathbf{Y} = (\prod_{j=1}^{\ell+1} X, \bigotimes_{j=1}^{\ell+1} \mathcal{A}, \bigotimes_{j=1}^{\ell+1} \mu)$. Note that (2.6) holds if we substitute S_1 for S and (2.7) holds when we substitute S_1 for S_0. Performing this substitution and applying first (2.7) and then (2.6) to $A_1, \ldots, A_{\ell+1} \in \mathcal{A}$, we have

$$\lambda_0(X \times A_1 \times \cdots \times A_{\ell+1}) = \prod_{j=1}^{\ell+1} \mu(A_j).$$

Also, trivially, for any $A_0 \in \mathcal{A}$,

$$\lambda_0(A_0 \times X \times \cdots \times X) = \mu(A_0).$$

Thus, λ_0 is a coupling of $\mathbf{X}$ and $\mathbf{Y}$.

Using formula (2.7), Lemma 2.6 and applying (2.8) to the set $S_1 = \{k_j \mid j \in \mathbb{N}\}$ (where we assume that $(k_j)_{j\in\mathbb{N}}$ is an increasing sequence), we have

$$\begin{aligned}
&\lim_{t\to\infty} \lambda_0(A_0 \times T_{g^{(1)}_{k_t,1}} A_1 \times \cdots \times T_{g^{(\ell+1)}_{k_t,1}} A_{\ell+1}) \\
&\quad = \lim_{t\to\infty} \mathcal{R}\text{-}\lim_{\{j_2,\ldots,j_{\ell+1}\}\in S_1^{(\ell)}} \mu_\Delta(A_0 \times T_{g^{(1)}_{j_2,2}+\cdots+g^{(1)}_{j_{\ell+1},\ell+1}} (T_{g^{(1)}_{k_t,1}} A_1) \\
&\qquad\quad \times \cdots \times T_{g^{(\ell+1)}_{j_2,2}+\cdots+g^{(\ell+1)}_{j_{\ell+1},\ell+1}} (T_{g^{(\ell+1)}_{k_t,1}} A_{\ell+1})) \\
&\quad = \lim_{t\to\infty} \mathcal{R}\text{-}\lim_{\{j_2,\ldots,j_{\ell+1}\}\in S_1^{(\ell)},\, k_t<j_2} \mu_\Delta(A_0 \times T_{g^{(1)}_{k_t,1}+g^{(1)}_{j_2,2}+\cdots+g^{(1)}_{j_{\ell+1},\ell+1}} A_1 \\
&\qquad\quad \times \cdots \times T_{g^{(\ell+1)}_{k_t,1}+g^{(\ell+1)}_{j_2,2}+\cdots+g^{(\ell+1)}_{j_{\ell+1},\ell+1}} A_{\ell+1}) \\
&\quad = \mathcal{R}\text{-}\lim_{\{j_1,\ldots,j_{\ell+1}\}\in S_1^{(\ell+1)}} \mu_\Delta(A_0 \times T_{g^{(1)}_{j_1,1}+\cdots+g^{(1)}_{j_{\ell+1},\ell+1}} A_1 \times \cdots \times T_{g^{(\ell+1)}_{j_1,1}+\cdots+g^{(\ell+1)}_{j_{\ell+1},\ell+1}} A_{\ell+1}) \\
&\quad = \lambda\bigg(\prod_{j=0}^{\ell+1} A_j\bigg),
\end{aligned} \tag{2.9}$$

For each $j \in \mathbb{N}$, let $\mathbf{T}_j = T_{g^{(1)}_{k_j,1}} \times \cdots \times T_{g^{(\ell+1)}_{k_j,1}}$. Note that, for any increasing sequence $(t_s)_{s\in\mathbb{N}}$ in $\mathbb{N}$, there exist a subsequence $(t'_s)_{s\in\mathbb{N}}$ and a measure $\lambda' \in \mathcal{C}(\mathbf{X}, \mathbf{Y})$, such that, for any $A \in \mathcal{A}$ and any $B \in \bigotimes_{j=1}^{\ell+1} \mathcal{A}$, $\lim_{s\to\infty} \lambda_0(A \times \mathbf{T}_{t'_s} B) = \lambda'(A \times B)$. By (2.9), $\lambda' = \lambda$ and hence, for any $A \in \mathcal{A}$ and any $B \in \bigotimes_{j=1}^{\ell+1} \mathcal{A}$, $\lim_{j\to\infty} \lambda_0(A \times \mathbf{T}_j B) = \lambda(A \times B)$.

By Lemma 2.5 applied to $\mathbf{X} = (X, \mathcal{A}, \mu)$, $\mathbf{Y} = (\prod_{j=1}^{\ell+1} X, \bigotimes_{j=1}^{\ell+1} \mathcal{A}, \bigotimes_{j=1}^{\ell+1} \mu)$ and the sequence of measure-preserving transformations $(T^{-1}_{g^{(1)}_{k_j,1}} \times \cdots \times T^{-1}_{g^{(\ell+1)}_{k_j,1}})_{j\in\mathbb{N}}$, we have that $\lambda = \bigotimes_{j=0}^{\ell+1} \mu$. It follows that, for any $A_0, \ldots, A_{\ell+1} \in \mathcal{A}$,

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$$\begin{aligned}&\mathcal{R}\text{-}\lim_{\alpha\in S_1^{(\ell+1)}} \mu(A_0\cap T_{g_\alpha^{(1)}}A_1\cap\cdots\cap T_{g_\alpha^{(\ell+1)}}A_{\ell+1})\\ &\quad= \mathcal{R}\text{-}\lim_{\alpha\in S_1^{(\ell+1)}} \mu_\Delta(A_0\times T_{g_\alpha^{(1)}}A_1\times\cdots\times T_{g_\alpha^{(\ell+1)}}A_{\ell+1}) = \prod_{j=0}^{\ell+1}\mu(A_j),\end{aligned}$$

completing the proof. □

3. *Strongly mixing systems are 'almost' strongly mixing of all orders*

In this section we will prove the following theorem (Theorem 1.21 from the Introduction) which is the main result of this paper.

THEOREM 3.1. *Let $\ell\in\mathbb{N}$ and let $(X,\mathcal{A},\mu,(T_g)_{g\in G})$ be a measure-preserving system. The following statements are equivalent.*

(i) *$(T_g)_{g\in G}$ is strongly mixing.*

(ii) *For any ℓ non-degenerated and essentially distinct sequences*

$$(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}} = (g_{k,1}^{(j)},\ldots,g_{k,\ell}^{(j)})_{k\in\mathbb{N}},\ j\in\{1,\ldots,\ell\},$$

in G^ℓ, there exists an infinite $S\subseteq\mathbb{N}$ such that, for any $A_0,\ldots,A_\ell\in\mathcal{A}$,

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \mu(A_0\cap T_{g_\alpha^{(1)}}A_1\cap\cdots\cap T_{g_\alpha^{(\ell)}}A_\ell) = \prod_{j=0}^{\ell}\mu(A_j). \tag{3.1}$$

(iii) *For any $\epsilon>0$ and any $A_0,\ldots,A_\ell\in\mathcal{A}$, the set*

$$\begin{aligned}&R_\epsilon(A_0,\ldots,A_\ell)\\ &\quad=\left\{(g_1,\ldots,g_\ell)\in G^\ell \;\middle|\; \left|\mu(A_0\cap T_{g_1}A_1\cap\cdots\cap T_{g_\ell}A_\ell)-\prod_{j=0}^{\ell}\mu(A_j)\right|<\epsilon\right\}\end{aligned}$$

is $\tilde{\Sigma}_\ell^*$ in G^ℓ.

(iv) *For any $\epsilon>0$ and any $A_0,A_1\in\mathcal{A}$, the set $R_\epsilon(A_0,A_1)$ is Σ_ℓ^* in G.*

Proof. (i) $\Longrightarrow$ (ii): Note that since $(T_g)_{g\in G}$ is strongly mixing, for any $t\in\{1,\ldots,\ell\}$ and any $j\in\{1,\ldots,\ell\}$, $(T_{g_{k,t}^{(j)}})_{k\in\mathbb{N}}$ has the mixing property and that for any t and any $i\neq j$, $(T_{(g_{k,t}^{(j)}-g_{k,t}^{(i)})})_{k\in\mathbb{N}}$ also has the mixing property. Thus (ii) follows from Proposition 2.9.

(ii) $\Longrightarrow$ (iii): By (ii), we have that, for any $\epsilon>0$, any $A_0,\ldots,A_\ell\in\mathcal{A}$ and any ℓ non-degenerated and essentially distinct sequences

$$(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}} = (g_{k,1}^{(j)},\ldots,g_{k,\ell}^{(j)})_{k\in\mathbb{N}},\quad j\in\{1,\ldots,\ell\},$$

in G^ℓ, there exists an $\alpha\in\mathbb{N}^{(\ell)}$ such that

$$(g_\alpha^{(1)},\ldots,g_\alpha^{(\ell)})\in R_\epsilon(A_0,\ldots,A_\ell),$$

which implies that $R_\epsilon(A_0,\ldots,A_\ell)$ is $\tilde{\Sigma}_\ell^*$.

(iii) $\Longrightarrow$ (iv): Let $\epsilon > 0$, let $A_0, A_1 \in \mathcal{A}$ and let $(\mathbf{g}_k^{(1)})_{k\in\mathbb{N}} = (g_{k,1}^{(1)}, \ldots, g_{k,\ell}^{(1)})_{k\in\mathbb{N}}$ be a non-degenerated sequence in G^ℓ. In order to prove that $\mathcal{R}_\epsilon(A_0, A_1)$ is Σ_ℓ^*, it suffices to show that for some $\alpha \in \mathbb{N}^{(\ell)}$, $g_\alpha^{(1)} \in \mathcal{R}_\epsilon(A_0, A_1)$.

Note that, for any sequence $(h_k^{(1)})_{k\in\mathbb{N}}$ in G with $\lim_{k\to\infty} h_k^{(1)} = \infty$, one can pick sequences $(h_k^{(2)})_{k\in\mathbb{N}}, \ldots, (h_k^{(\ell)})_{k\in\mathbb{N}}$ in G with the property that, for any distinct $i, j \in \{1, \ldots, \ell\}$,

$$\lim_{k\to\infty} h_k^{(j)} = \infty \quad \text{and} \quad \lim_{k\to\infty} (h_k^{(j)} - h_k^{(i)}) = \infty.$$

Hence, one can find non-degenerated sequences $(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}}$ in G^ℓ, $j \in \{2, \ldots, \ell\}$, such that $(\mathbf{g}_k^{(1)})_{k\in\mathbb{N}}, \ldots, (\mathbf{g}_k^{(\ell)})_{k\in\mathbb{N}}$ are essentially distinct. By (iii), there exists an $\alpha \in \mathbb{N}^{(\ell)}$ for which

$$(g_\alpha^{(1)}, \ldots, g_\alpha^{(\ell)}) \in \mathcal{R}_\epsilon(A_0, A_1, \underbrace{X, \ldots, X}_{\ell-1 \text{ times}}).$$

This implies that $g_\alpha^{(1)} \in R_\epsilon(A_0, A_1)$.

(iv) $\Longrightarrow$ (i): We will show that, for any $\xi, \eta \in L_0^2(\mu) = \{f \in L^2(\mu) \mid \int_X f \, d\mu = 0\}$, $\lim_{g\to\infty} \langle T_g \xi, \eta\rangle = 0$. To do this, it suffices to prove that for any sequence $(g_k)_{k\in\mathbb{N}}$ in G with $\lim_{k\to\infty} g_k = \infty$, there exists an increasing sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that, for any $\xi, \eta \in L_0^2(\mu)$,

$$\lim_{j\to\infty} \langle T_{g_{k_j}} \xi, \eta\rangle = 0. \tag{3.2}$$

Let $(g_k)_{k\in\mathbb{N}} \subseteq G$ with $\lim_{k\to\infty} g_k = \infty$. Let $(\mathbf{g}_k)_{k\in\mathbb{N}} = (\underbrace{g_k, \ldots, g_k}_{\ell \text{ times}})_{k\in\mathbb{N}}$ (note that $(\mathbf{g}_k)_{k\in\mathbb{N}}$ is a non-degenerated sequence in G^ℓ). We claim that there exist an increasing sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ and a bounded linear operator $V : L_0^2(\mu) \to L_0^2(\mu)$ such that, if we set $S = \{k_j \mid j \in \mathbb{N}\}$, the following assertions hold.

(1) For any $\xi, \eta \in L_0^2(\mu)$,

$$\langle V\xi, \eta\rangle = \lim_{j\to\infty} \langle T_{g_{k_j}} \xi, \eta\rangle. \tag{3.3}$$

(2) For any $A_0, A_1 \in \mathcal{A}$, there exists a real number r_{A_0,A_1} such that

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \mu(A_0 \cap T_{-g_\alpha} A_1) = r_{A_0,A_1}. \tag{3.4}$$

Let $\mathcal{D}$ be a countable dense subset of $L_0^2(\mu)$. By a diagonalization argument, one obtains an increasing sequence $(k'_j)_{j\in\mathbb{N}}$ for which the limit in (3.3) exists for any $\xi, \eta \in \mathcal{D}$. Diagonalizing once more, we can pick a subsequence $(k_j)_{j\in\mathbb{N}}$ of $(k'_j)_{j\in\mathbb{N}}$ for which (3.4) holds for any A_0, A_1 from a countable dense subset of $\mathcal{A}$. It follows (by a standard approximation argument) that all the limits appearing in (3.3) and (3.4) exist for any $\xi, \eta \in L_0^2(\mu)$ and any $A_0, A_1 \in \mathcal{A}$. Notice that (3.3) holds for a unique linear operator V. Since

$$\sup_{\|\xi\|\le 1} \|V\xi\| \le \sup_{g\in G} \sup_{\|\xi\|\le 1} \|T_g\xi\| = 1,$$

we have that V is norm-bounded.

We claim that $V^\ell = 0$. To see this, note that, by (iv), for every $A_0, A_1 \in \mathcal{A}$, $r_{A_0,A_1} = \mu(A_0)\mu(A_1)$ (otherwise we would be able to find an $\epsilon > 0$ for which the set $\mathcal{R}_\epsilon(A_0, A_1)$ is not Σ_ℓ^*). Since the linear combinations of indicator functions are dense in $L^2(\mu)$, it follows that, for any $f_1, f_2 \in L^2(\mu)$,

$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \int_X f_1 T_{g_\alpha} f_2 \, \mathrm{d}\mu = \int_X f_1 \, \mathrm{d}\mu \int_X f_2 \, \mathrm{d}\mu. \tag{3.5}$$

Observe that, by (3.3), $T_g V = V T_g$ for all $g \in G$. Thus, all the limits appearing in the expression

$$\lim_{j_1\to\infty} \cdots \lim_{j_\ell\to\infty} \langle T_{(g_{k_{j_1}}+\cdots+g_{k_{j_\ell}})}\xi, \eta\rangle$$

exist for any $\xi, \eta \in L_0^2(\mu)$. Combining (3.3) and (3.5), we obtain that, for any $\xi, \eta \in L_0^2(\mu)$,

$$\begin{aligned} 0 &= \mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \int_X \overline{\eta} T_{g_\alpha}\xi \, \mathrm{d}\mu = \mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \langle T_{g_\alpha}\xi, \eta\rangle \\ &= \lim_{j_1\to\infty} \cdots \lim_{j_\ell\to\infty} \langle T_{(g_{k_{j_1}}+\cdots+g_{k_{j_\ell}})}\xi, \eta\rangle = \langle V^\ell \xi, \eta\rangle, \end{aligned}$$

proving our claim.

It follows that in order to prove that (3.2) holds, it is enough to show that $L_0^2(\mu) = \mathrm{Ker}(V^\ell) \subseteq \mathrm{Ker}(V)$. To do this, we will first show that V is a normal operator. Indeed, for any $\xi, \eta \in L_0^2(\mu)$,

$$\langle V^*\xi, \eta\rangle = \overline{\langle V\eta, \xi\rangle} = \lim_{j\to\infty} \overline{\langle T_{g_{k_j}}\eta, \xi\rangle} = \lim_{j\to\infty} \langle T_{-g_{k_j}}\xi, \eta\rangle$$

and, hence,

$$V^*V\xi = \lim_{j\to\infty} T_{-g_{k_j}} V\xi = \lim_{j\to\infty} VT_{-g_{k_j}}\xi = VV^*\xi.$$

So, for any $\xi \in L_0^2(\mu)$,

$$\|V\xi\|^2 = \langle V\xi, V\xi\rangle = \langle V^*V\xi, \xi\rangle = \langle V^*\xi, V^*\xi\rangle = \|V^*\xi\|^2. \tag{3.6}$$

Now take $t \in \mathbb{N}$, $\eta \in L_0^2(\mu)$, and set $\xi = V^{t-1}\eta$. Suppose that $\eta \notin \mathrm{Ker}(V^t)$. Then $\xi \notin \mathrm{Ker}(V)$ and, by (3.6), $\langle V^*V\xi, \xi\rangle \neq 0$. Applying (3.6) to $V\xi$, we obtain $\|V^2\xi\|^2 = \|V^*V\xi\|^2$. So, since $\langle V^*V\xi, \xi\rangle \neq 0$, $V^{t+1}\eta = V^2\xi \neq 0$. This proves that, for each $t \in \mathbb{N}$, if $\eta \notin \mathrm{Ker}(V^t)$, then $\eta \notin \mathrm{Ker}(V^{t+1})$. So, $L_0^2(\mu) = \mathrm{Ker}(V^\ell) \subseteq \mathrm{Ker}(V)$ and, hence, for any $\xi, \eta \in L_0^2(\mu)$,

$$0 = \langle V\xi, \eta\rangle = \lim_{j\to\infty} \langle T_{g_{k_j}}\xi, \eta\rangle. \qquad \square$$

4. *Some 'diagonal' results for strongly mixing systems*

In order to give the reader the flavor of the main theme of this section, we start by formulating a slightly enhanced form of Theorem 1.4. (This theorem is a rather special case of the results of 'diagonal' nature to be proved in this section.)

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PROPOSITION 4.1. *Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system and let $a_1, \ldots, a_\ell$ be non-zero distinct integers. Then T is strongly mixing if and only if, for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set*

$$\left\{n \in \mathbb{Z} \;\middle|\; \left|\mu(A_0 \cap T^{a_1 n} A_1 \cap \cdots \cap T^{a_\ell n} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\right| < \epsilon\right\}$$

is Σ_ℓ^*.

We move now to formulations of more general 'diagonal' results.

Let $(G, +)$ be a countable abelian group, let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system, let $\ell \in \mathbb{N}$ and let $\phi_1, \ldots, \phi_\ell : G \to G$ be homomorphisms. For any $\epsilon > 0$ and any $A_0, \ldots, A_\ell \in \mathcal{A}$, define

$$R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A_0, \ldots, A_\ell) = \left\{g \in G \;\middle|\; \left|\mu(A_0 \cap T_{\phi_1(g)} A_1 \cap \cdots \cap T_{\phi_\ell(g)} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\right| < \epsilon\right\}.$$

We first give two equivalent formulations of a general result which deals with finitely generated groups.

THEOREM 4.2. *Let $(G, +)$ be a finitely generated abelian group, let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system and let the homomorphisms $\phi_1, \ldots, \phi_\ell : G \to G$ be such that, for any $j \in \{1, \ldots, \ell\}$, $\ker(\phi_j)$ is finite and, for any $i \neq j$, $\ker(\phi_j - \phi_i)$ is also finite. Then $(T_g)_{g\in G}$ is strongly mixing if and only if, for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set $R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A_0, \ldots, A_\ell)$ is Σ_ℓ^*.*

Note that if G is a finitely generated abelian group and $\phi : G \to G$ is a homomorphism, $\ker(\phi)$ is finite if and only if the index of $\phi(G)$ in G is finite. It follows that Theorem 4.2 can be formulated in the following equivalent form.

THEOREM 4.3. *Let $(G, +)$ be a finitely generated abelian group, let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system and let the homomorphisms $\phi_1, \ldots, \phi_\ell : G \to G$ be such that, for any $j \in \{1, \ldots, \ell\}$, the index of $\phi_j(G)$ in G is finite and, for any $i \neq j$, the index of $(\phi_j - \phi_i)$ in G is also finite. Then $(T_g)_{g\in G}$ is strongly mixing if and only if, for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set $R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A_0, \ldots, A_\ell)$ is Σ_ℓ^*.*

We now formulate and prove variants of Theorems 4.2 and 4.3 which pertain to mixing actions of general (not necessarily finitely generated) countable abelian groups. Unlike Theorems 4.2 and 4.3, the following two theorems are not equivalent. We will provide the relevant counterexamples at the end of this section.

THEOREM 4.4. *Let $(G, +)$ be a countable abelian group, let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a strongly mixing system and let the homomorphisms $\phi_1, \ldots, \phi_\ell : G \to G$ be such that, for any $j \in \{1, \ldots, \ell\}$, $\ker(\phi_j)$ is finite and, for any $i \neq j$, $\ker(\phi_j - \phi_i)$ is also finite. For any non-degenerated sequence $(\boldsymbol{g}_k)_{k\in\mathbb{N}} = (g_{k,1}, \ldots, g_{k,\ell})_{k\in\mathbb{N}}$ in G^ℓ, there exists an infinite set $S \subseteq \mathbb{N}$ such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$,*

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$$\mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \mu(A_0 \cap T_{\phi_1(g_\alpha)}A_1 \cap \cdots \cap T_{\phi_\ell(g_\alpha)}A_\ell) = \prod_{j=0}^{\ell} \mu(A_j).$$

Equivalently, for any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set $R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A_0, \ldots, A_\ell)$ is Σ_ℓ^*.

Proof. Since, for any distinct $i, j \in \{1, \ldots, \ell\}$, $\ker(\phi_j)$ and $\ker(\phi_j - \phi_i)$ are both finite, we have for each $t \in \{1, \ldots, \ell\}$,

$$\lim_{k\to\infty} \phi_j(g_{k,t}) = \infty \quad \text{and} \quad \lim_{k\to\infty} (\phi_j(g_{k,t}) - \phi_i(g_{k,t})) = \infty.$$

For each $j \in \{1, \ldots, \ell\}$, let

$$(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}} = (\phi_j(g_{k,1}), \ldots, \phi_j(g_{k,\ell}))_{k\in\mathbb{N}}.$$

Then the sequences $(\mathbf{g}_k^{(1)})_{k\in\mathbb{N}}, \ldots, (\mathbf{g}_k^{(\ell)})_{k\in\mathbb{N}}$ are non-degenerated and essentially distinct. By Theorem 3.1(ii), there exists an infinite set $S \subseteq \mathbb{N}$ such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$,

$$\begin{aligned}\mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} &\mu(A_0 \cap T_{\phi_1(g_\alpha)}A_1 \cap \cdots \cap T_{\phi_\ell(g_\alpha)}A_\ell) \\ &= \mathcal{R}\text{-}\lim_{\alpha\in S^{(\ell)}} \mu(A_0 \cap T_{g_\alpha^{(1)}}A_1 \cap \cdots \cap T_{g_\alpha^{(\ell)}}A_\ell) = \mu\bigg(\prod_{j=0}^{\ell} A_j\bigg).\end{aligned}$$ □

Remark 4.5. The goal of this remark is to indicate an alternative way of proving Theorem 4.4. Let G and $\phi_1, \ldots, \phi_\ell$ be as in the hypothesis of Theorem 4.4. In §5 we will show that if E is a $\tilde{\Sigma}_\ell^*$ set in G^ℓ, then $\{g \in G \mid (\phi_1(g), \ldots, \phi_\ell(g)) \in E\}$ is a Σ_ℓ^* set in G (see Proposition 5.22). Thus, for any measure-preserving system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$, any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, if $R_\epsilon(A_0, \ldots, A_\ell)$ is a $\tilde{\Sigma}_\ell^*$ set, then $R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A_0, \ldots, A_\ell)$ is a Σ_ℓ^* set. One can now invoke Theorem 3.1(iii).

The next result complements Theorem 4.4. Note that it provides a somewhat stronger version of one of the directions in Theorem 4.3.

THEOREM 4.6. *Let $(G, +)$ be a countable abelian group, let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system and let the homomorphisms $\phi_1, \ldots, \phi_\ell : G \to G$ be such that at least one of $\phi_1(G)$, $\phi_2(G)$, $(\phi_2 - \phi_1)(G)$ has finite index in G. If, for all $A_0, \ldots, A_\ell \in \mathcal{A}$ and all $\epsilon > 0$, the set $R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A_0, \ldots, A_\ell)$ is Σ_ℓ^*, then $(T_g)_{g\in G}$ is strongly mixing.*

Proof. We will assume that $(\phi_2 - \phi_1)(G)$ has finite index in G; the other two cases can be handled similarly. For any $A_1, A_2 \in \mathcal{A}$ and any $\epsilon > 0$, we have

$$\begin{aligned}&R_\epsilon^{\phi_1,\ldots,\phi_\ell}(X, A_1, A_2, \underbrace{X, \ldots, X}_{\ell-2 \text{ times}}) \\ &= \{g \in G \mid |\mu(X \cap T_{\phi_1(g)}A_1 \cap T_{\phi_2(g)}A_2 \cap T_{\phi_3(g)}X \cap \cdots \cap T_{\phi_\ell(g)}X) \\ &\quad - \mu(A_1)\mu(A_2)| < \epsilon\} \\ &= \{g \in G \mid |\mu(T_{\phi_1(g)}A_1 \cap T_{\phi_2(g)}A_2) - \mu(A_1)\mu(A_2)| < \epsilon\} = R_\epsilon^{\phi_2-\phi_1}(A_1, A_2).\end{aligned}$$

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By our assumption, for any $\epsilon > 0$ and any $A_1, A_2 \in \mathcal{A}$, the set $R_\epsilon^{\phi_2-\phi_1}(A_1, A_2)$ is a Σ_ℓ^* set and hence, by Theorem 3.1(iv), $(T_{(\phi_2-\phi_1)(g)})_{g\in G}$ is strongly mixing.

We will now prove that $(T_g)_{g\in G}$ is strongly mixing by showing that for any sequence $(g_k)_{k\in\mathbb{N}}$ in G with $\lim_{k\to\infty} g_k = \infty$, there exists an increasing sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ with the property that for any $A_0, A_1 \in \mathcal{A}$,

$$\lim_{j\to\infty} \mu(A_0 \cap T_{g_{k_j}} A_1) = \mu(A_0)\mu(A_1).$$

Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in G with $\lim_{k\to\infty} g_k = \infty$. By assumption, $(\phi_2 - \phi_1)(G)$ has finite index in G, so there exist an increasing sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ and an element $\tau \in G$ for which $\{g_{k_j} + \tau \mid j \in \mathbb{N}\} \subseteq (\phi_2 - \phi_1)(G)$. Since $(T_{(\phi_2-\phi_1)(g)})_{g\in G}$ is strongly mixing, for any $A_0, A_1 \in \mathcal{A}$,

$$\lim_{j\to\infty} \mu(A_0 \cap T_{g_{k_j}} A_1) = \lim_{j\to\infty} \mu(A_0 \cap T_{g_{k_j}+\tau}(T_{-\tau} A_1)) = \mu(A_0)\mu(A_1),$$

completing the proof. □

The following proposition shows that the assumption made in Theorem 4.2 that G is finitely generated cannot be removed.

PROPOSITION 4.7. *Let $G = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}$ and let $\ell \in \mathbb{N}$. There exist a measure-preserving system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ and homomorphisms $\phi_1, \ldots, \phi_\ell : G \to G$ satisfying (a) for any $j \in \{1, \ldots, \ell\}$, $\ker(\phi_j)$ is finite, and (b) for any $i \neq j$, $\ker(\phi_j - \phi_i)$ is also finite, and such that every set of the form $R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A_0, \ldots, A_\ell)$ is Σ_ℓ^* but $(T_g)_{g\in G}$ is not strongly mixing.*

Proof. We will only carry out the proof for $\ell = 2$; the general case can be handled similarly. Let $\phi_1 : G \to G$ be the homomorphism given by

$$\phi_1((a_1, a_2, \ldots, a_n, \ldots)) = (0, a_1, 0, a_2, \ldots, 0, a_n, \ldots).$$

Note that ϕ_1 is injective (and so $\ker(\phi_1)$ is trivial).

Let $X = \{0, 1\}^G$ be endowed with the product topology, let μ be the $(\frac{1}{2}, \frac{1}{2})$ product measure on $\mathcal{A} = \text{Borel}(X)$ and, for each $g \in G$, let $S_g : X \to X$ be the map defined by $(S_g(x))(h) = x(h + g)$. The system $(X, \mathcal{A}, \mu, (S_g)_{g\in G})$ is strongly mixing. Define a measure-preserving G-action $(T_g)_{g\in G}$ on $(X, \mathcal{A}, \mu)$ by

$$T_{(a_1,a_2,\ldots)} = S_{(a_2,a_4,\ldots)}$$

and let $\phi_2 : G \to G$ be defined by $\phi_2(g) = 2\phi_1(g)$. Note that, for any $g = (a_1, a_2, \ldots) \in G$,

$$T_{\phi_1(g)} = T_{\phi_1((a_1,a_2,\ldots))} = T_{(0,a_1,0,a_2,\ldots)} = S_{(a_1,a_2,\ldots)} = S_g.$$

So, for any $\epsilon > 0$ and any $A_0, A_1, A_2 \in \mathcal{A}$,

$$\begin{aligned} &R_\epsilon^{\phi_1,\phi_2}(A_0, A_1, A_2) \\ &\quad = \{g \in G \mid |\mu(A_0 \cap T_{\phi_1(g)} A_1 \cap T_{\phi_2(g)} A_2) - \mu(A_0)\mu(A_1)\mu(A_2)| < \epsilon\} \\ &\quad = \{g \in G \mid |\mu(A_0 \cap S_g A_1 \cap S_{2g} A_2) - \mu(A_0)\mu(A_1)\mu(A_2)| < \epsilon\}. \end{aligned} \tag{4.1}$$

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It follows from Theorem 4.4 that every set of the form

$$\{g \in G \mid |\mu(A_0 \cap S_g A_1 \cap S_{2g} A_2) - \mu(A_0)\mu(A_1)\mu(A_2)| < \epsilon\}$$

is Σ_2^* and hence, by (4.1), for any A_0, A_1, A_2 and any $\epsilon > 0$, $R_\epsilon^{\phi_1,\phi_2}(A_0, A_1, A_2)$ is Σ_2^*.

Noting that for each $k \in \mathbb{N}$, $T_{(k,0,0,\ldots)} = S_{(0,0,\ldots)}$ is the identity map on X, we see that $(T_g)_{g\in G}$ is not strongly mixing. We are done. □

The next result shows that Theorem 4.3 cannot be extended to arbitrary countable abelian groups.

PROPOSITION 4.8. *Let $G = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}$ and let $\ell \in \mathbb{N}$. There exist a strongly mixing system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ and homomorphisms $\phi_1, \ldots, \phi_\ell : G \to G$ satisfying (a) for any $j \in \{1, \ldots, \ell\}$, $\phi_j(G) = G$, and (b) for any $i \neq j$, $(\phi_i - \phi_j)(G) = G$, and such that, for some $A \in \mathcal{A}$ and some $\epsilon > 0$, the set $R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A, \ldots, A)$ is not Σ_ℓ^*.*

Proof. Let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a strongly mixing system and let $p_1, \ldots, p_\ell \in \mathbb{N}$ be ℓ different prime numbers. For each $j \in \{1, \ldots, \ell\}$, let $\phi_j : G \to G$ be defined by

$$\phi_j(a_1, a_2, a_3, \ldots) = (a_{p_j^1}, a_{p_j^2}, a_{p_j^3} \ldots).$$

It follows that, for any $j \in \{1, \ldots, \ell\}$, $\phi_j(G) = G$ and since, for any distinct $i, j \in \{1, \ldots, \ell\}$, the sets $\{p_i^k \mid k \in \mathbb{N}\}$ and $\{p_j^k \mid k \in \mathbb{N}\}$ are disjoint, we have that $(\phi_j - \phi_i)(G) = G$ as well.

Observe that the subgroup $G' = \{(a_1, 0, 0, \ldots) \in G \mid a_1 \in \mathbb{Z}\}$ is isomorphic to $\mathbb{Z}$ and that, for any $j \in \{1, \ldots, \ell\}$, $G' \subseteq \ker(\phi_j)$. Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in G' with $\lim_{k\to\infty} g_k = \infty$. Since, for each $k \in \mathbb{N}$, $T_{\phi_j(g_k)} = T_{(0,0,\ldots)} = \mathrm{Id}$, where Id is the identity map on X, we have that, for any $A \in \mathcal{A}$ with $\mu(A) \in (0, 1)$, and any $k_1 < \cdots < k_\ell$,

$$\mu(A \cap T_{\phi_1(g_{k_1}+\cdots+g_{k_\ell})} A \cap \cdots \cap T_{\phi_\ell(g_{k_1}+\cdots+g_{k_\ell})} A) = \mu(A) \neq \mu^{\ell+1}(A).$$

It follows that if ϵ is small enough, the set $R_\epsilon^{\phi_1,\ldots,\phi_\ell}(A, \ldots, A)$ does not intersect the Σ_ℓ set

$$\{g_{k_1} + \cdots + g_{k_\ell} \mid k_1 < \cdots < k_\ell\}$$

and hence it is not Σ_ℓ^*. This completes the proof. □

5. *Largeness properties of $\tilde{\Sigma}_m^*$ sets*

As we have seen above, any strongly mixing system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ has the property that the sets $R_\epsilon(A_0, \ldots, A_m)$ are $\tilde{\Sigma}_m^*$ (moreover, the strong mixing of $(T_g)_{g\in G}$ is characterized by this property). This section is devoted to the discussion of massivity and ubiquity of $\tilde{\Sigma}_m^*$ sets. Since strong mixing is a stronger property than those of mild and weak mixing, one should expect that the notions of largeness associated with (multiple) mild and weak mixing are 'majorized' by the notion of largeness associated with $\tilde{\Sigma}_m^*$ sets. This will be established in §5.1 and §5.2. Finally, in §5.3 we will show that $\tilde{\Sigma}_m^*$ sets are ubiquitous in the sense that they are well spread among the cosets of *admissible* subgroups of G^m (the class of admissible subgroups will be introduced in §5.3).

5.1. *Any $\tilde{\Sigma}_m^*$ set in G^d is an $\tilde{IP}^*$ set.* In this section we will introduce $\tilde{\mathrm{IP}}^*$ sets and juxtapose them with $\tilde{\Sigma}_m^*$ sets. ($\tilde{\mathrm{IP}}^*$ sets are intrinsically linked to the multiple mixing properties of mildly mixing systems. The connection between $\tilde{\mathrm{IP}}^*$ sets and mildly mixing systems will be addressed in §6.)

Let $(G, +)$ be a countable abelian group and let $\mathcal{F}$ denote the set of all non-empty finite subsets of $\mathbb{N}$. Given a sequence $(g_k)_{k\in\mathbb{N}}$ in G, define an $\mathcal{F}$-sequence $(g_\alpha)_{\alpha\in\mathcal{F}}$ by

$$g_\alpha = \sum_{j\in\alpha} g_j = g_{k_1} + \cdots + g_{k_t},\ \alpha = \{k_1, \ldots, k_t\}. \tag{5.1}$$

We will write

$$\lim_{\alpha\to\infty} g_\alpha = \infty$$

if, for every finite $K \subseteq G$, there exists an $\alpha_0 \in \mathcal{F}$ such that, for any $\alpha \in \mathcal{F}$ with $\alpha > \alpha_0$ (that is, $\min \alpha > \max \alpha_0$), $g_\alpha \notin K$.

A set $E \subseteq G$ is called an IP set if $E = \{g_\alpha \mid \alpha \in \mathcal{F}\}$ for some sequence $(g_k)_{k\in\mathbb{N}}$ in G such that $\lim_{\alpha\to\infty} g_\alpha = \infty$. A set $E \subseteq G$ is called IP* if it has a non-trivial intersection with every IP set. IP sets are often defined just as sets of the form

$$\mathrm{FS}((g_k)_{k\in\mathbb{N}}) = \{g_{k_1} + \cdots + g_{k_t} \mid k_1 < \cdots < k_t,\ t \in \mathbb{N}\} = \{g_\alpha \mid \alpha \in \mathcal{F}\}$$

(without the requirement that $\lim_{\alpha\to\infty} g_\alpha = \infty$). Our choice of definition for IP sets is dictated by our interest in the study of asymptotic properties of measure-preserving actions. The distinction between our definition and the more traditional one is rather mild: for any infinite set of the form $E = \{g_\alpha \mid \alpha \in \mathcal{F}\}$, there exists a sequence $(h_k)_{k\in\mathbb{N}}$ such that $\{h_\alpha \mid \alpha \in \mathcal{F}\} \subseteq E$ and $\lim_{\alpha\to\infty} h_\alpha = \infty$.

We now introduce modifications of IP and IP* sets, namely $\tilde{\mathrm{IP}}$ sets and $\tilde{\mathrm{IP}}^*$ sets, which, as will be seen in §6, are naturally linked with the properties of the sets $R_\epsilon(A_0, \ldots, A_\ell)$ in the context of mildly mixing systems.

Definition 5.1. Let $(G, +)$ be a countable abelian group and let $d \in \mathbb{N}$. We say that a set $E \subseteq G^d$ is an $\tilde{\mathrm{IP}}$ set if it is of the form

$$E = \{(g_\alpha^{(1)}, \ldots, g_\alpha^{(d)}) \mid \alpha \in \mathcal{F}\},$$

where, for each $j \in \{1, \ldots, d\}$, $\{g_\alpha^{(j)} \mid \alpha \in \mathcal{F}\}$ is generated by $(g_k^{(j)})_{k\in\mathbb{N}}$ as in (5.1) and, in addition, for any $j \in \{1, \ldots, d\}$,

$$\lim_{\alpha\to\infty} g_\alpha^{(j)} = \infty \tag{5.2}$$

and, for any $i \neq j$,

$$\lim_{\alpha\to\infty} (g_\alpha^{(j)} - g_\alpha^{(i)}) = \infty. \tag{5.3}$$

(Note that if $d = 1$, then $E \subseteq G$ is an IP set if and only if it is an $\tilde{\mathrm{IP}}$ set.)

A set $E \subseteq G^d$ is called an $\tilde{\mathrm{IP}}^*$ set if it has a non-trivial intersection with every $\tilde{\mathrm{IP}}$ set in G^d.

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Remark 5.2. Let $(G, +)$ be a countable abelian group, let $d \in \mathbb{N}$ and let $E \subseteq G^d$ be an $\tilde{\mathrm{IP}}$ set. From now on, whenever we pick a sequence $(\mathbf{g}_k)_{k\in\mathbb{N}} = (g_k^{(1)}, \ldots, g_k^{(d)})_{k\in\mathbb{N}}$ in G^d with the property that $E = \{(g_\alpha^{(1)}, \ldots, g_\alpha^{(d)}) \mid \alpha \in \mathcal{F}\}$, we will tacitly assume that $(g_k^{(1)})_{k\in\mathbb{N}}, \ldots, (g_k^{(d)})_{k\in\mathbb{N}}$ satisfy (5.2) and (5.3).

The following lemma unveils an important connection between $\tilde{\mathrm{IP}}$ and $\tilde{\Sigma}_m$ sets.

LEMMA 5.3. *Let $(G, +)$ be a countable abelian group and let $d, m \in \mathbb{N}$. Any $\tilde{IP}$ set $E \subseteq G^d$ contains a $\tilde{\Sigma}_m$ set. Namely, there exist non-degenerated and essentially distinct sequences*

$$(\mathbf{g}_k^{(j)}) = (g_{k,1}^{(j)}, \ldots, g_{k,m}^{(j)})_{k\in\mathbb{N}}, \quad j \in \{1, \ldots, d\},$$

in G^m with the property that $\{(g_\alpha^{(1)}, \ldots, g_\alpha^{(d)}) \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq E$, where, for each $j \in \{1, \ldots, d\}$ and each $\alpha = \{k_1, \ldots, k_m\} \in \mathbb{N}^{(m)}$, $g_\alpha^{(j)} = g_{k_1,1}^{(j)} + \cdots + g_{k_m,m}^{(j)}$.

Proof. Let E be an $\tilde{\mathrm{IP}}$ set and let $(\mathbf{h}_k)_{k\in\mathbb{N}} = (h_k^{(1)}, \ldots, h_k^{(d)})_{k\in\mathbb{N}}$ be such that

$$E = \{\mathbf{h}_\alpha \mid \alpha \in \mathcal{F}\} = \{(h_\alpha^{(1)}, \ldots, h_\alpha^{(d)}) \mid \alpha \in \mathcal{F}\}.$$

Following the stipulation made in Remark 5.2, for any finite set $F \subseteq G$, we can find an $\alpha_F \in \mathcal{F}$ such that, for any $\alpha \in \mathcal{F}$ with $\alpha > \alpha_F$ and any distinct $i, j \in \{1, \ldots d\}$, $h_\alpha^{(j)} \notin F$ and $(h_\alpha^{(j)} - h_\alpha^{(i)}) \notin F$. In particular, for any distinct $i, j \in \{1, \ldots, d\}$,

$$\lim_{k\to\infty} h_k^{(j)} = \infty \quad \text{and} \quad \lim_{k\to\infty} (h_k^{(j)} - h_k^{(i)}) = \infty. \tag{5.4}$$

For each $j \in \{1, \ldots, d\}$ and each $k \in \mathbb{N}$, we let

$$\mathbf{g}_k^{(j)} = \underbrace{(h_k^{(j)}, \ldots, h_k^{(j)})}_{m \text{ times}}. \tag{5.5}$$

Note that, by (5.4), the sequences $(\mathbf{g}_k^{(1)})_{k\in\mathbb{N}}, \ldots, (\mathbf{g}_k^{(d)})_{k\in\mathbb{N}}$ are non-degenerated and essentially distinct. It follows now from (5.5) that, for any $\alpha = \{k_1, \ldots, k_m\} \in \mathbb{N}^{(m)}$,

$$(g_\alpha^{(1)}, \ldots, g_\alpha^{(d)}) = \left(\sum_{j=1}^m h_{k_j}^{(1)}, \ldots, \sum_{j=1}^m h_{k_j}^{(d)}\right) = \left(h_{\{k_1,\ldots,k_m\}}^{(1)}, \ldots, h_{\{k_1,\ldots,k_m\}}^{(d)}\right) \in E,$$

which completes the proof. □

Remark 5.4. The proof of Lemma 5.3 actually shows that any $\tilde{\mathrm{IP}}$ set is a union of $\tilde{\Sigma}_t$ sets. Let $E \subseteq G^d$ be an $\tilde{\mathrm{IP}}$ set and let $(\mathbf{g}_k)_{k\in\mathbb{N}}$ be a sequence such that $E = \{\mathbf{g}_\alpha \mid \alpha \in \mathcal{F}\}$. The proof of Lemma 5.3 shows that, for each $t \in \mathbb{N}$, $\{\mathbf{g}_{k_1} + \cdots + \mathbf{g}_{k_t} \mid k_1 < \cdots < k_t\}$ is a $\tilde{\Sigma}_t$ set. Hence,

$$E = \bigcup_{t\in\mathbb{N}} \{\mathbf{g}_{k_1} + \cdots + \mathbf{g}_{k_t} \mid k_1 < \cdots < k_t\}.$$

As an immediate consequence of Lemma 5.3 we have the following result.

COROLLARY 5.5. *Let $(G, +)$ be a countable abelian group and let $d, m \in \mathbb{N}$. Every $\tilde{\Sigma}_m^*$ set in G^d is an $\tilde{IP}^*$ set.*

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Proof. Let $E \subseteq G^d$ be a $\tilde{\Sigma}_m^*$ set and let $D \subseteq G^d$ be an $\tilde{\mathrm{IP}}$ set. By Lemma 5.3, we have that D contains a $\tilde{\Sigma}_m$ set and hence $E \cap D \neq \emptyset$. Since D was arbitrary, this shows that E is an $\tilde{\mathrm{IP}}^*$ set. □

5.2. *Any $\tilde{\Sigma}_m^*$ set in G^d has uniform density one.* We start with defining the notions of *upper density* and *uniform density one* in countable abelian groups.

Definition 5.6. Let $(G, +)$ be a countable abelian group, let $E \subseteq G$ and let $(F_k)_{k\in\mathbb{N}}$ be a Følner sequence in G. (A sequence $(F_k)_{k\in\mathbb{N}}$ of non-empty finite subsets of G is a Følner sequence if, for any $g \in G$,

$$\lim_{k\to\infty} \frac{|(g + F_k) \cap F_k|}{|F_k|} = 1,$$

where, for a finite set A, $|A|$ denotes its cardinality. It is well known that every countable abelian group contains a Følner sequence.) The *upper density* of E with respect to $(F_k)_{k\in\mathbb{N}}$ is defined by

$$\overline{d}_{(F_k)}(E) = \limsup_{k\to\infty} \frac{|E \cap F_k|}{|F_k|}.$$

A set $E \subseteq G$ has *uniform density one* if, for every Følner sequence $(F_k)_{k\in\mathbb{N}}$, $\overline{d}_{(F_k)}(E) = 1$.

Sets of uniform density one are intrinsically connected with weakly mixing measure-preserving systems. Recall that a measure-preserving action $(T_g)_{g\in G}$ on a probability space $(X, \mathcal{A}, \mu)$ is called weakly mixing if the diagonal action $(T_g \times T_g)_{g\in G}$ on $X \times X$ is ergodic. When G is an amenable group, the notion of weak mixing can be equivalently defined with the help of strong Césaro limits along Følner sequences. Namely, $(T_g)_{g\in G}$ is weakly mixing if and only if, for any Følner sequence $(F_k)_{k\in\mathbb{N}}$ and any $A_0, A_1 \in \mathcal{A}$,

$$\lim_{k\to\infty} \frac{1}{|F_k|} \sum_{g\in F_k} |\mu(A_0 \cap T_g A_1) - \mu(A_0)\mu(A_1)| = 0.$$

It follows that $(T_g)_{g\in G}$ is weakly mixing if and only if the sets

$$R_\epsilon(A_0, A_1) = \{g \in G \mid |\mu(A_0 \cap T_g A_1) - \mu(A_0)\mu(A_1)| < \epsilon\}$$

have uniform density one. The reader will find a few more equivalent forms of weak mixing in Proposition 6.8 below.

In order to derive the main result of this subsection, namely the fact that every $\tilde{\Sigma}_m^*$ set has uniform density one, we need first to prove two auxiliary propositions.

PROPOSITION 5.7. *Let $(G, +)$ be a countable abelian group, let $d \in \mathbb{N}$ and let $(F_k)_{k\in\mathbb{N}}$ be a Følner sequence in G^d. For any $E \subseteq G^d$ with $\overline{d}_{(F_k)}(E) > 0$ and any $\tilde{\mathrm{IP}}$ set $D \subseteq G^d$, there exists a sequence $(\mathbf{g}_k)_{k\in\mathbb{N}} = (g_k^{(1)}, \ldots, g_k^{(d)})$ in G^d such that (a) $\{\mathbf{g}_\alpha \mid \alpha \in \mathcal{F}\} \subseteq D$, (b) for any distinct $i, j \in \{1, \ldots, d\}$, (5.2) and (5.3) hold, and (c) for any $\alpha \in \mathcal{F}$,*

$$\overline{d}_{(F_k)}\bigg(\bigcap_{\beta\subseteq\alpha,\, \beta\neq\emptyset} (E - \mathbf{g}_\beta)\bigg) > 0. \tag{5.6}$$

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In other words, for each $\alpha \in \mathcal{F}$, *the set* $E_\alpha = \{\boldsymbol{h} \in G^d \mid$ *for all* $\beta \subseteq \alpha,\ \beta \neq \emptyset,\ \boldsymbol{h} + \boldsymbol{g}_\beta \in E\}$ *satisfies* $\overline{d}_{(F_k)}(E_\alpha) > 0$.

Proof. Let $D = \{\mathbf{h}_\alpha \mid \alpha \in \mathcal{F}\}$ be an $\widetilde{\mathrm{IP}}$ set in G^d generated by the sequence $(\mathbf{h}_k)_{k\in\mathbb{N}} = (h_{k,1}, \ldots, h_{k,d})_{k\in\mathbb{N}}$. We claim that, for any $M \in \mathbb{N}$ with $M > 1/\overline{d}_{(F_k)}(E)$, there exist $L, R \in \mathbb{N}$, $L < R \leq M$, for which $\overline{d}_{(F_k)}(E \cap (E - \mathbf{h}_{\{L+1,L+2,\ldots,R\}})) > 0$. To see this, suppose for the sake of contradiction that, for any distinct $R, L \in \{1, \ldots, M\}$, $R > L$, $\overline{d}_{(F_k)}(E \cap (E - \mathbf{h}_{\{L+1,\ldots,R\}})) = 0$. Since $\overline{d}_{(F_k)}$ is translation invariant and for any $L, R \in \{1, \ldots, M\}$, $L < R$, $\mathbf{h}_{\{L+1,\ldots,R\}} = \mathbf{h}_{\{1,\ldots,R\}} - \mathbf{h}_{\{1,\ldots,L\}}$, we have that

$$\overline{d}_{(F_k)}(E \cap (E - \mathbf{h}_{\{L+1,\ldots,R\}})) = \overline{d}_{(F_k)}((E - \mathbf{h}_{\{1,\ldots,L\}}) \cap (E - \mathbf{h}_{\{1,\ldots,R\}})) = 0.$$

It follows that

$$\overline{d}_{(F_k)}\bigg(\bigcup_{R=1}^{M}(E - \mathbf{h}_{\{1,\ldots,R\}})\bigg) = \sum_{R=1}^{M} \overline{d}_{(F_k)}(E - \mathbf{h}_{\{1,\ldots,R\}}) = M\overline{d}_{(F_k)}(E) > 1,$$

a contradiction. Thus, there exist $L, R \in \mathbb{N}$ with $L < R \leq M$ such that $\overline{d}_{(F_k)}(E \cap (E - \mathbf{h}_{\{L+1,\ldots,R\}})) > 0$. We will let $\gamma_1 = \{L+1, \ldots, , R\}$.

Now let $E_1 = E \cap (E - \mathbf{h}_{\gamma_1})$. Repeating the above argument, we find $L', R' \in \mathbb{N}$, $R < L' < R'$, such that $\gamma_2 = \{L'+1, \ldots, R'\}$ satisfies $\overline{d}_{(F_k)}(E_1 \cap (E_1 - \mathbf{h}_{\gamma_2})) > 0$. It follows that $\gamma_1 < \gamma_2$ and that $\mathbf{h}_{\gamma_1 \cup \gamma_2} = \mathbf{h}_{\gamma_1} + \mathbf{h}_{\gamma_2}$. Hence,

$$\overline{d}_{(F_k)}(E \cap (E - \mathbf{h}_{\gamma_1}) \cap (E - \mathbf{h}_{\gamma_2}) \cap (E - \mathbf{h}_{\gamma_1\cup\gamma_2})) > 0.$$

Continuing in this way, we can find a sequence $(\gamma_k)_{k\in\mathbb{N}}$ with $\gamma_k < \gamma_{k+1}$ for each $k \in \mathbb{N}$ and the property that, for any $\alpha \in \mathcal{F}$,

$$\overline{d}_{(F_k)}\bigg(\bigcap_{\beta\subseteq\alpha,\,\beta\neq\emptyset}(E - \mathbf{h}_{\bigcup_{k\in\beta}\gamma_k})\bigg) > 0.$$

For each $k \in \mathbb{N}$, let $\mathbf{g}_k = \mathbf{h}_{\gamma_k}$ and, for each $\alpha \in \mathcal{F}$, let $\mathbf{g}_\alpha = \sum_{j\in\alpha} \mathbf{g}_j = \mathbf{h}_{\bigcup_{j\in\alpha}\gamma_j}$. Observe that the sequence $(\mathbf{g}_\alpha)_{\alpha\in\mathcal{F}}$ satisfies (5.6). Let $D' = \{\mathbf{g}_\alpha \mid \alpha \in \mathcal{F}\}$. Clearly $D' \subseteq D$. To finish the proof observe that

$$(\mathbf{g}_\alpha)_{\alpha\in\mathcal{F}} = (g_{\alpha,1}, \ldots, g_{\alpha,d})_{\alpha\in\mathcal{F}} = (h_{(\bigcup_{k\in\alpha}\gamma_k),1}, \ldots, h_{(\bigcup_{k\in\alpha}\gamma_k),d})_{\alpha\in\mathcal{F}}$$

satisfies (5.2) and (5.3). Indeed, in view of Remark 5.2, for any $j \in \{1, \ldots, d\}$,

$$\lim_{\alpha\to\infty} g_{\alpha,j} = \lim_{\alpha\to\infty} h_{(\bigcup_{k\in\alpha}\gamma_k),j} = \infty$$

and, for $i \neq j$,

$$\lim_{\alpha\to\infty}(g_{\alpha,j} - g_{\alpha,i}) = \lim_{\alpha\to\infty}(h_{(\bigcup_{k\in\alpha}\gamma_k),j} - h_{(\bigcup_{k\in\alpha}\gamma_k),i}) = \infty.$$ □

PROPOSITION 5.8. *Let* $(G, +)$ *be a countable abelian group, let* $d, m \in \mathbb{N}$ *and let* $(F_k)_{k\in\mathbb{N}}$ *be a Følner sequence in* G^d. *Any* $E \subseteq G^d$ *with* $\overline{d}_{(F_k)}(E) > 0$ *contains a* $\tilde{\Sigma}_m$ *set. Namely, there exist non-degenerated and essentially distinct sequences*

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$$(\mathbf{g}_k^{(j)}) = (g_{k,1}^{(j)}, \ldots, g_{k,m}^{(j)})_{k\in\mathbb{N}}, \quad j \in \{1, \ldots, d\},$$

in G^m *with the property that* $\{(g_\alpha^{(1)}, \ldots, g_\alpha^{(d)}) \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq E$.

Proof. Fix $d \in \mathbb{N}$ and let D be an $\widetilde{\mathrm{IP}}$ set in G^d. Let $(\mathbf{h}_k)_{k\in\mathbb{N}} = (h_k^{(1)}, \ldots, h_k^{(d)})_{k\in\mathbb{N}}$ be a sequence in G^d with $D = \{\mathbf{h}_\alpha \mid \alpha \in \mathcal{F}\}$. Invoking Proposition 5.7 and passing, if needed, to a sub-$\widetilde{\mathrm{IP}}$ set in D, we can assume that, for any $\alpha \in \mathcal{F}$,

$$\overline{d}_{(F_k)}\bigg(\bigcap_{\beta\subseteq\alpha,\, \beta\neq\emptyset} (E - \mathbf{h}_\beta)\bigg) > 0 \tag{5.7}$$

and that $(\mathbf{h}_k)_{k\in\mathbb{N}}$ satisfies (5.2) and (5.3).

Let $m = 1$. There exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ in $\mathcal{F}$ such that, for each $k \in \mathbb{N}$, $\alpha_k < \alpha_{k+1}$ and such that, for any distinct $k, k' \in \mathbb{N}$ and any distinct $i, j \in \{1, \ldots, d\}$,

$$h_{\alpha_k}^{(j)} \neq h_{\alpha_{k'}}^{(j)} \quad \text{and} \quad h_{\alpha_k}^{(j)} - h_{\alpha_k}^{(i)} \neq h_{\alpha_{k'}}^{(j)} - h_{\alpha_{k'}}^{(i)}. \tag{5.8}$$

Pick a sequence $(A_k)_{k\in\mathbb{N}}$ of finite subsets of G with the properties that, for each $k \in \mathbb{N}$, (a) $|A_k| = k$, (b) $A_k \subseteq A_{k+1}$, and (c) $\bigcup_{k\in\mathbb{N}} A_k = G$. By (5.7), for each $k \in \mathbb{N}$ we can find $\mathbf{b}_k = (b_{k,1}, \ldots, b_{k,d})$ in G^d such that, for any $t \in \{1, \ldots, kd^2 + 1\}$, $\mathbf{b}_k + \mathbf{h}_{\alpha_t} \in E$. By (5.8), for any $k \in \mathbb{N}$ and any $j \in \{1, \ldots, d\}$, there exist at most k natural numbers t for which $b_{k,j} + h_{\alpha_t}^{(j)} \in A_k$. Similarly, for any distinct $i, j \in \{1, \ldots, d\}$, one has $(b_{k,j} - b_{k,i}) + (h_{\alpha_t}^{(j)} - h_{\alpha_t}^{(i)}) \in A_k$ for at most k natural numbers t.

We claim that there exists $t \in \{1, \ldots, kd^2 + 1\}$ such that, for any $j \in \{1, \ldots, d\}$, $b_{k,j} + h_{\alpha_t}^{(j)} \notin A_k$ and, for any $i \neq j$, $(b_{k,j} - b_{k,i}) + (h_{\alpha_t}^{(j)} - h_{\alpha_t}^{(i)}) \notin A_k$. Suppose for contradiction that this is not the case. Since there are $d^2 - d$ pairs (i, j) with distinct $i, j \in \{1, \ldots, d\}$, there exist at least $k + 1$ natural numbers t for which, say, $b_{k,1} + h_{\alpha_t}^{(1)} \in A_k$, a contradiction.

Thus, there exists a sequence $(k_t)_{t\in\mathbb{N}}$ in $\mathbb{N}$ for which the sequences

$$(b_{t,j} + h_{\alpha_{k_t}}^{(j)})_{t\in\mathbb{N}}, \quad j \in \{1, \ldots, d\}$$

are non-degenerated and essentially distinct, and

$$\{(b_{t,1} + h_{\alpha_{k_t}}^{(1)}, \ldots, b_{t,d} + h_{\alpha_{k_t}}^{(d)}) \mid t \in \mathbb{N}\} \subseteq E.$$

Now let $m > 1$. By Lemma 5.3 there exist non-degenerated and essentially distinct sequences $(\mathbf{f}_k^{(j)})_{k\in\mathbb{N}} = (f_{k,1}^{(j)}, \ldots, f_{k,m-1}^{(j)})_{k\in\mathbb{N}}$, $j \in \{1, \ldots, d\}$, with the property that $\{(f_\alpha^{(1)}, \ldots, f_\alpha^{(d)}) \mid \alpha \in \mathbb{N}^{(m-1)}\} \subseteq D$. For each $k \in \mathbb{N}$, let

$$E_k = \bigcap_{\alpha\subseteq\{1,\ldots,k+m-1\},\, |\alpha|=m-1} (E - (f_\alpha^{(1)}, \ldots, f_\alpha^{(d)})). \tag{5.9}$$

By (5.7), for each $k \in \mathbb{N}$, $\overline{d}_{(F_k)}(E_k) > 0$. It follows from the case $m = 1$ that there exist sequences

$$(g_{k,j})_{k\in\mathbb{N}}, \quad j \in \{1, \ldots, d\},$$

with the properties that (a) for any $k \in \mathbb{N}$, $(g_{k,1}, \ldots, g_{k,d}) \in E_k$, (b) for any $j \in \{1, \ldots, d\}$, $\lim_{k\to\infty} g_{k,j} = \infty$, and (c) for any distinct $i, j \in \{1, \ldots, d\}$, $\lim_{k\to\infty} g_{k,i} - g_{k,j} = \infty$. For each $j \in \{1, \ldots, d\}$, form the sequence

$$(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}} = (f_{k,1}^{(j)}, \ldots, f_{k,m-1}^{(j)}, g_{k,j}) = (g_{k,1}^{(j)}, \ldots, g_{k,m}^{(j)}).$$

By (5.9) and (a), we have that, for any $k \in \mathbb{N}$ and any $\alpha \subseteq \{1, \ldots, k-1\}$ with $|\alpha| = m-1$, $(g_{k,1}, \ldots, g_{k,d}) + (f_\alpha^{(1)}, \ldots, f_\alpha^{(d)}) \in E$ and hence

$$\{(f^{(1)}_{\{k_1,\ldots,k_{m-1}\}} + g_{k_m,1}, \ldots, f^{(d)}_{\{k_1,\ldots,k_{m-1}\}} + g_{k_m,d}) \mid k_1 < \cdots < k_{m-1} < k_m\} \subseteq E.$$

By (b) and (c), the sequences $(\mathbf{g}_k^{(1)})_{k\in\mathbb{N}}, \ldots, (\mathbf{g}_k^{(d)})_{k\in\mathbb{N}}$ are non-degenerated and essentially distinct. We are done. □

COROLLARY 5.9. *Let $(G, +)$ be a countable abelian group and let $d, m \in \mathbb{N}$. Every $\tilde{\Sigma}_m^*$ set in G^d has uniform density one.*

Proof. We will assume that $D \subseteq G^d$ does not have uniform density one and show that D is not a $\tilde{\Sigma}_m^*$ set. Indeed, if D does not have uniform density one, then there exists a Følner sequence $(F_k)_{k\in\mathbb{N}}$ in G^d for which $\overline{d}_{(F_k)}(D) < 1$. Let $E = G^d \setminus D$ and note that $\overline{d}_{(F_k)}(E) > 0$. By Proposition 5.8, E contains a $\tilde{\Sigma}_m$ set. This implies that D is not a $\tilde{\Sigma}_m^*$. □

5.3. *The ubiquity of $\tilde{\Sigma}_m^*$ sets.* In this subsection we will show that there exists a broad class of subgroups of G^d with the property that, for each group H from this class, any $\tilde{\Sigma}_m^*$ set in G^d has a large intersection with H. In fact, we will show that either a subgroup H belongs to this class or $G^d \setminus H$ is a $\tilde{\Sigma}_m^*$ set for any $m \in \mathbb{N}$.

Definition 5.10. Let $(G, +)$ be a countable abelian group, let $d \in \mathbb{N}$ and let H be a subgroup of G^d. We say that H is an *admissible subgroup of* G^d if there exist non-degenerated and essentially distinct sequences $(g_k^{(1)})_{k\in\mathbb{N}}, \ldots, (g_k^{(d)})_{k\in\mathbb{N}}$ in G such that

$$\{(g_k^{(1)}, \ldots, g_k^{(d)}) \mid k \in \mathbb{N}\} \subseteq H.$$

Example 5.11. Let $(G, +)$ be a countable abelian group and let $H = \{(g, h, 0) \mid g, h \in G\} \subseteq G^3$. Clearly, H is not an admissible subgroup of G^3.

Example 5.12. Let $(G, +)$ be a countable abelian group with an element g of infinite order. For any $d \in \mathbb{N}$ and any distinct $a_1, \ldots, a_d \in \mathbb{Z} \setminus \{0\}$, the set $\{(ka_1 g, ka_2 g, \ldots, ka_d g) \mid k \in \mathbb{Z}\}$ is an admissible subgroup of G^d.

Example 5.13. Let $(G, +)$ be a countable abelian torsion group (that is, each of its elements has finite order). There exist a sequence $(g_k)_{k\in\mathbb{N}}$ in G and a nested sequence of finite subgroups $(G_N)_{N\in\mathbb{N}}$ with the following properties: (i) G_N is generated by $\{g_1, \ldots, g_N\}$; (ii) for each $k \in \mathbb{N}$, $g_{k+1} \notin G_k$. Then, for any $d \in \mathbb{N}$ and any distinct $a_1, \ldots, a_d \in \mathbb{N}$, the group generated by the set $\{(g_{a_1 k}, g_{a_2 k}, \ldots, g_{a_d k}) \mid k \in \mathbb{N}\}$ is an admissible subgroup of G^d. Indeed, note that, for any $k \in \mathbb{N}$ and any

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$a, b \in \mathbb{N}$ with $a < b$, $g_{ak} \notin G_{ak-1}$ and $(g_{bk} - g_{ak}) \notin G_{ak}$. So $\lim_{k\to\infty} g_{ak} = \infty$ and $\lim_{k\to\infty}(g_{bk} - g_{ak}) = \infty$.

The following proposition provides a useful characterization of admissible subgroups.

PROPOSITION 5.14. *Let $(G, +)$ be a countable abelian group, let $d \in \mathbb{N}$ and let H be a subgroup of G^d. The following statements are equivalent.*

(i) *H is an admissible subgroup of G^d.*
(ii) *There exist an $m \in \mathbb{N}$ and a $\tilde{\Sigma}_m$ set $E \subseteq G^d$ such that $E \subseteq H$.*
(iii) *For any $m \in \mathbb{N}$, there exists a $\tilde{\Sigma}_m$ set $E \subseteq G^d$ such that $E \subseteq H$.*
(iv) *There exists an $\tilde{IP}$ set $E \subseteq G^d$ such that $E \subseteq H$.*
(v) *For any $j \in \{1, \ldots, d\}$, $\pi_j(H)$ is infinite and, for any $i \neq j$, $(\pi_j - \pi_i)(H)$ is also infinite, where for each $j \in \{1, \ldots, d\}$, $\pi_j : H \to G$ is defined by $\pi_j(g_1, \ldots, g_d) = g_j$.*

Proof. It is not hard to see that (i) and (ii) are equivalent. The implications (i) $\Longrightarrow$ (iii), (iii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (v) are trivial. We will now prove (v) $\Longrightarrow$ (i).

Let $P = \{\pi_j \mid j \in \{1, \ldots, d\}\} \cup \{\pi_j - \pi_i \mid i, j \in \{1, \ldots, d\},\ i \neq j\}$ and let M be the largest non-negative integer for which there exist an $F \subseteq P$ with $|F| = M$ and a sequence $(\mathbf{g}_k)_{k\in\mathbb{N}}$ in H such that, for any $\pi \in F$, $\lim_{k\to\infty} \pi(\mathbf{g}_k) = \infty$. Since $|P| = d^2$, we have $M \leq d^2$. Also, since, for each $\pi \in P$, $\pi(H)$ is infinite, $M \geq 1$. If $M = d^2$, then (i) holds. So, assume for contradiction that $M < d^2$.

By the definition of M, there exist a set $F_0 \subseteq P$ with $|F_0| = M$ and a sequence $(\mathbf{g}_k)_{k\in\mathbb{N}}$ in H such that if $\pi \in F_0$, $\lim_{k\to\infty} \pi(\mathbf{g}_k) = \infty$ and if $\pi \in (P \setminus F_0)$, then there exists a finite set $A_\pi \subseteq G$ such that $\{\pi(\mathbf{g}_k) \mid k \in \mathbb{N}\} \subseteq A_\pi$. By passing, if needed, to a subsequence, we can assume that, for each $\pi \in (P \setminus F_0)$, there exists a $g_\pi \in G$ such that $\lim_{k\to\infty} \pi(\mathbf{g}_k) = g_\pi$. Let $\pi_0 \in (P \setminus F_0)$. By (v), there exists a sequence $(\mathbf{g}'_k)_{k\in\mathbb{N}}$ in H such that $\lim_{k\to\infty} \pi_0(\mathbf{g}'_k) = \infty$. Note that, for any finite set $A \subseteq H$, any $\pi \in F_0$ and any $t \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that, for any $k' > k$,

$$\pi(\mathbf{g}_{k'} + \mathbf{g}'_t) = \pi(\mathbf{g}_{k'}) + \pi(\mathbf{g}'_t) \notin A.$$

Also, note that there exists a $k_0 \in \mathbb{N}$ such that, for any $k > k_0$, $\pi_0(\mathbf{g}_k) = g_{\pi_0}$. It follows that we can find an increasing sequence $(k_t)_{t\in\mathbb{N}}$ in $\mathbb{N}$ for which $\lim_{t\to\infty} \pi(\mathbf{g}_{k_t} + \mathbf{g}'_t) = \infty$ for each $\pi \in F_0 \cup \{\pi_0\}$. This contradicts the definition of M, completing the proof. □

COROLLARY 5.15. *Let $(G, +)$ be a countable abelian group and let $d \in \mathbb{N}$. A subgroup H of G^d is either admissible or, for any $m \in \mathbb{N}$, $G^d \setminus H$ is a $\tilde{\Sigma}^*_m$ set.*

Proof. If H is not an admissible subgroup, Proposition 5.14(ii) implies that, for each $m \in \mathbb{N}$, H does not contain any $\tilde{\Sigma}_m$ set in G^d. Thus, $G^d \setminus H$ is a $\tilde{\Sigma}^*_m$ set for each $m \in \mathbb{N}$. □

Before stating and proving one of the main results of this subsection which deals with the ubiquity of $\tilde{\Sigma}^*_m$ sets in admissible subgroups (Theorem 5.20 below), we need one more definition and a technical lemma.

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Definition 5.16. Let $(G, +)$ be a countable abelian group, let $d, m \in \mathbb{N}$ and let $H \subseteq G^d$ be an admissible subgroup. A set $E \subseteq H$ is called an H-$\tilde{\Sigma}_m^*$ set if it has a non-trivial intersection with every $\tilde{\Sigma}_m$ set contained in H. Similarly, a set $E \subseteq H$ is called an H-$\tilde{\mathrm{IP}}^*$ set if it has a non-trivial intersection with every $\tilde{\mathrm{IP}}$ set contained in H.

Remark 5.17. Let $(G, +)$ be a countable abelian group, let $d \in \mathbb{N}$ and let $H \subseteq G^d$ be an admissible subgroup of G^d. It is useful to perceive H-$\tilde{\Sigma}_m^*$ sets as relative versions of $\tilde{\Sigma}_m^*$ sets in G^d. Note that if H is a proper subgroup of G^d, H-$\tilde{\Sigma}_m^*$ sets are not $\tilde{\Sigma}_m^*$. Indeed, since, for each $m \in \mathbb{N}$, any translation of a $\tilde{\Sigma}_m$ set in G^d is again a $\tilde{\Sigma}_m$ set, every coset of H contains a $\tilde{\Sigma}_m$ set in G^d. It follows that $G^d \setminus H$ contains a $\tilde{\Sigma}_m$ set for each $m \in \mathbb{N}$. Hence, no H-$\tilde{\Sigma}_m^*$ set is a $\tilde{\Sigma}_m^*$ set.

Remark 5.18. Let $(G, +)$ be a countable abelian group, let $d, m \in \mathbb{N}$, let $H \subseteq G^d$ be an admissible subgroup and let E be a $\tilde{\Sigma}_m^*$ set in G^d. It follows from the definition that $E \cap H$ is a H-$\tilde{\Sigma}_m^*$ set. Indeed, let $D \subseteq H$ be a $\tilde{\Sigma}_m$ set. We have $(E \cap H) \cap D = E \cap D \neq \emptyset$. Note also that, for any $\mathbf{g} \in G^d$, $E \cap (\mathbf{g} + H)$ is the translation of the H-$\tilde{\Sigma}_m^*$ set $(-\mathbf{g} + E) \cap H$. Thus, the cosets of H have a large intersection with E as well.

LEMMA 5.19. *Let $(G, +)$ be a countable abelian group, let $d, m \in \mathbb{N}$, let H be an admissible subgroup of G^d and let $(F_k)_{k \in \mathbb{N}}$ be a Følner sequence in H. Any $E \subseteq H$ with $\overline{d}_{(F_k)}(E) > 0$ contains a $\tilde{\Sigma}_m$ set.*

Proof. Since H is admissible, there exists an $\tilde{\mathrm{IP}}$ set $D' \subseteq H$. The result in question follows by replacing D by D' in the proof of Proposition 5.8 and applying an adequate modification of Proposition 5.7. □

THEOREM 5.20. *Let $(G, +)$ be a countable abelian group, let $d, m \in \mathbb{N}$ and let $H \subseteq G^d$ be an admissible subgroup. Any H-$\tilde{\Sigma}_m^*$ set is an H-$\tilde{\mathrm{IP}}^*$ set and has uniform density one in H.*

Proof. Let $E' \subseteq H$ be an H-$\tilde{\Sigma}_m^*$ set. By Lemma 5.3, every $\tilde{\mathrm{IP}}$ set contains a $\tilde{\Sigma}_m$ set. It follows that E' is an H-$\tilde{\mathrm{IP}}^*$ set. By Lemma 5.19, we can argue as in the proof of Corollary 5.9 to show that E' has uniform density one in H. □

COROLLARY 5.21. *Let $(G, +)$ be a countable abelian group, let $d \in \mathbb{N}$, let H be an admissible subgroup of G^d and let $(X, \mathcal{A}, \mu, (T_g)_{g \in G})$ be a strongly mixing system. For any $\mathbf{g} \in G^d$, each set of the form $R_\epsilon(A_0, \ldots, A_\ell) \cap (\mathbf{g} + H)$ is the translation of a set with uniform density one in H.*

Proof. This result follows from Theorem 3.1, Remark 5.18 and Theorem 5.20. □

A natural class of admissible subgroups in G^d is provided by the one-parameter subgroups of the form

$$H_{\phi_1,\ldots,\phi_d} = \{(\phi_1(g), \ldots, \phi_d(g)) \mid g \in G\},$$

where $\phi_1, \ldots, \phi_d : G \to G$ are homomorphisms such that, for any $j \in \{1, \ldots, d\}$, $|\ker(\phi_j)| < \infty$ and, for any $i \neq j$, $|\ker(\phi_j - \phi_i)| < \infty$. The following proposition,

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alluded to in Remark 4.5, involves preimages of sets in G^d via the elements of $H_{\phi_1,\ldots,\phi_d}$ and provides an alternative proof of Theorem 4.4.

PROPOSITION 5.22. *Let $(G,+)$ be a countable abelian group, let $d, m \in \mathbb{N}$ and let $\phi_1, \ldots, \phi_d : G \to G$ be homomorphisms such that, for any $j \in \{1, \ldots, d\}$,* $\ker(\phi_j)$ *is finite and, for any $i \neq j$,* $\ker(\phi_j - \phi_i)$ *is also finite. If $E \subseteq G^d$ is a $\tilde{\Sigma}_m^*$ set, then $E' = \{g \in G \mid (\phi_1(g), \ldots, \phi_d(g)) \in E\}$ is a Σ_m^* set in G.*

Proof. Let $D \subseteq G$ be the Σ_m set in G generated by the non-degenerated sequence $(\mathbf{g}_k)_{k\in\mathbb{N}} = (g_{k,1}, \ldots, g_{k,m})_{k\in\mathbb{N}}$ in G^m (that is, $D = \{g_\alpha \mid \alpha \in \mathbb{N}^{(m)}\}$). We will show that $D \cap E' \neq \emptyset$.

By our assumption on $\phi_1, \ldots, \phi_d$, for each $j \in \{1, \ldots, m\}$, the sequences $(\phi_1(g_{k,j}))_{k\in\mathbb{N}}, \ldots, (\phi_d(g_{k,j}))_{k\in\mathbb{N}}$ are non-degenerated and essentially distinct. Thus, the set $D' = \{(\phi_1(g_\alpha), \ldots, \phi_d(g_\alpha)) \mid \alpha \in \mathbb{N}^{(m)}\}$ is a $\tilde{\Sigma}_m$ set in G^d. Noting that $D' \cap E \neq \emptyset$, we obtain $D \cap E' \neq \emptyset$. □

So far we have been focusing on the massivity and ubiquity of *general* $\tilde{\Sigma}_\ell^*$ sets. However the ‘dynamical’ $\tilde{\Sigma}_\ell^*$ sets $R_\epsilon(A_0, \ldots, A_\ell)$, are even more prevalent in G^ℓ. For example, assuming for convenience that $G = \mathbb{Z}$, one can show that the sets of the form $R_\epsilon(A_0, \ldots, A_\ell)$ have an ample presence in ‘polynomial’ subsets of $\mathbb{Z}^\ell$. This is illustrated by the following polynomial extension of Proposition 4.1 (which is proved in a companion paper [9]).

THEOREM 5.23. *Let $\ell \in \mathbb{N}$ and let $p_1, \ldots, p_\ell \in \mathbb{Z}[x]$ be non-constant polynomials such that, for any distinct $i, j \in \{1, \ldots, \ell\}$,* $\deg(p_j - p_i) > 0$. *There exists an $m \in \mathbb{N}$ such that, for any strongly mixing system $(X, \mathcal{A}, \mu, T)$, any $\epsilon > 0$ and any $A_0, \ldots, A_\ell \in \mathcal{A}$, the set*

$$R_\epsilon^{p_1,\ldots,p_\ell}(A_0, \ldots, A_\ell) = \left\{ n \in \mathbb{Z} \;\middle|\; \left| \mu(A_0 \cap T^{p_1(n)} A_1 \cap \cdots \cap T^{p_\ell(n)} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j) \right| < \epsilon \right\} \tag{5.10}$$

is Σ_m^*.

The following proposition shows that, in general, $\tilde{\Sigma}_\ell^*$ sets, unlike the sets of the form $R_\epsilon(A_0, \ldots, A_\ell)$, can be disjoint from the polynomial sets $H_{p_1,\ldots,p_\ell} = \{(p_1(n), \ldots, p_\ell(n)) \mid n \in \mathbb{Z}\}$, where $p_1, \ldots, p_\ell \in \mathbb{Z}[x]$.

PROPOSITION 5.24. *Let $\ell \in \mathbb{N}$ and let $p_1, \ldots, p_\ell \in \mathbb{Z}[x]$ be non-constant polynomials such that, for any distinct $i, j \in \{1, \ldots, \ell\}$,* $\deg(p_j - p_i) > 0$. *Suppose that* $\deg(p_1) > 1$. *Then, for any $m \geq 2$, $H_{p_1,\ldots,p_\ell}$ contains no $\tilde{\Sigma}_m$ sets. Equivalently, $\mathbb{Z}^\ell \setminus H_{p_1,\ldots,p_\ell}$ is a $\tilde{\Sigma}_m^*$ set for each $m \geq 2$.*

Proof. Since the projection onto the first coordinate of any $\tilde{\Sigma}_m$ set $E \subseteq \mathbb{Z}^\ell$ is a Σ_m set in $\mathbb{Z}$, it suffices to show that the set $\{p_1(n) \mid n \in \mathbb{Z}\}$ contains no Σ_m sets. Suppose for contradiction that $\{p_1(n) \mid n \in \mathbb{Z}\}$ contains a Σ_m set

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$$D = \{n_{k_1}^{(1)} + \cdots + n_{k_m}^{(m)} \mid k_1 < \cdots < k_m\},$$

where $(n_k^{(1)})_{k\in\mathbb{N}}, \ldots, (n_k^{(m)})_{k\in\mathbb{N}}$ are non-degenerated sequences in $\mathbb{Z}$.

Choose $t_1, t_2, t_3 \in \mathbb{N}$ to be such that $n_{t_1}^{(1)} < n_{t_2}^{(1)} < n_{t_3}^{(1)}$ and let

$$I = \{n_{t_1}^{(1)} + n_{k_2}^{(2)} + \cdots + n_{k_m}^{(m)} \mid \max\{t_1, t_2, t_3\} < k_2 < \cdots < k_m\}.$$

Clearly, I is an infinite subset of D. So, letting $a = n_{t_2}^{(1)} - n_{t_1}^{(1)}$ and $b = n_{t_3}^{(1)} - n_{t_1}^{(1)}$, we have $a + I \subseteq D$ and $b + I \subseteq D$.

Let $(n_k)_{k\in\mathbb{N}}$ be an enumeration of the elements of I. One can find an increasing sequence $(k_j)_{j\in\mathbb{N}}$ for which at least two of the sets $\{n_{k_j} \mid j \in \mathbb{N}\}$, $\{a + n_{k_j} \mid j \in \mathbb{N}\}$ and $\{b + n_{k_j} \mid j \in \mathbb{N}\}$ are contained in at least one of the sets $\{p_1(n) \mid n \in \mathbb{N}\}$ and $\{p_1(-n) \mid n \in \mathbb{N}\}$. We will assume that $\{a + n_{k_j} \mid j \in \mathbb{N}\}$ and $\{b + n_{k_j} \mid j \in \mathbb{N}\}$ are contained in $\{p_1(n) \mid n \in \mathbb{N}\}$ (the other cases can be handled similarly). It follows that there exist infinitely many pairs $(n, m) \in \mathbb{N} \times \mathbb{N}$ such that $p_1(n) - p_1(m) = b - a$. Since $b > a$, this contradicts the fact that $\deg(p_1) > 1$. □

6. *Multiple recurrence for mildly and weakly mixing systems via $\mathcal{R}$-limits*

As we saw above, $\mathcal{R}$-limits can be successfully used to characterize strong mixing and establish higher-order mixing properties. In this section, we will show that $\mathcal{R}$-limits can be also useful in dealing with mildly and weakly mixing systems. In particular, we will obtain analogues of Theorem 3.1 for mildly and weakly mixing systems.

6.1. *Mildly mixing systems.* In this subsection we will deal with mildly mixing systems (see Definition 6.4 below) from the perspective of $\mathcal{R}$-limits. The notion of mild mixing, which lies between weak and strong mixing, was introduced by Walters in 1972 [**32**] and rediscovered by Furstenberg and Weiss in 1978 [**16**]. Mild mixing has multiple equivalent forms (see [**3**, **4**, **14**, **16**]) and plays a fundamental role in IP ergodic theory and its applications, including various refinements of the classical Szemerédi theorem (see [**5**, **15**]). The multiple recurrence theorems for mildly mixing systems (see [**5**, **14**, **15**]) utilize the notion of IP-limit which we will presently define. We will then establish a connection between IP-limits and $\mathcal{R}$-limits and, finally, prove an analogue of Theorem 3.1 for mildly mixing actions.

Definition 6.1. (Cf. [**15**, Definitions 1.1 and 1.3]) Let (X, d) be a compact metric space and let $(x_\alpha)_{\alpha\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in X. A set $\mathcal{F}^{(1)} \subseteq \mathcal{F}$ is an IP-ring if there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ in $\mathcal{F}$ with $\alpha_k < \alpha_{k+1}$, for each $k \in \mathbb{N}$, for which

$$\mathcal{F}^{(1)} = \left\{ \bigcup_{j\in\alpha} \alpha_j \,\middle|\, \alpha \in \mathcal{F} \right\}.$$

For any IP-ring $\mathcal{F}^{(1)}$, we write

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\, x_\alpha = x$$

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if, for every $\epsilon > 0$, there exists an $\alpha_0 \in \mathcal{F}^{(1)}$ such that, for any $\alpha \in \mathcal{F}^{(1)}$ with $\alpha > \alpha_0$,

$$d(x_\alpha, x) < \epsilon.$$

It follows from a result of Hindman [17] that if $(x_\alpha)_{\alpha\in\mathcal{F}}$ is an $\mathcal{F}$-sequence in a compact metric space X, then, for any IP-ring $\mathcal{F}^{(1)} \subseteq \mathcal{F}$, one can always find an $x \in X$ and an IP-ring $\mathcal{F}^{(2)} \subseteq \mathcal{F}^{(1)}$ such that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} x_\alpha = x \tag{6.1}$$

(see [14, Theorem 8.14]). In particular, for any countable abelian group $(G, +)$, any sequence $(g_k)_{k\in\mathbb{N}}$ in G and any probability measure-preserving system $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$, there exists an IP-ring $\mathcal{F}^{(1)}$ for which

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} T_{g_\alpha}$$

exists in the weak operator topology of $L^2(\mu)$. This implies (and is equivalent to) the fact that for any $A_0, A_1 \in \mathcal{A}$,

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} \mu(A_0 \cap T_{g_\alpha} A_1)$$

exists.

THEOREM 6.2. *Let (X, d) be a compact metric space, let $(G, +)$ be a countable abelian group, let $(x_g)_{g\in G}$ be a sequence in X, let $x_0 \in X$ and let $(g_k)_{k\in\mathbb{N}}$ be a sequence in G. The following statements are equivalent.*

(i) *For any IP-ring $\mathcal{F}^{(1)} \subseteq \mathcal{F}$ for which* $\text{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} x_{g_\alpha}$ *exists, one has*

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} x_{g_\alpha} = x_0. \tag{6.2}$$

(ii) *For any IP-ring $\mathcal{F}^{(1)} \subseteq \mathcal{F}$, there exist an $m \in \mathbb{N}$ and a sequence $(h_{k,1}, \ldots, h_{k,m})_{k\in\mathbb{N}}$ in G^m such that $\{h_\alpha \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq \{g_\alpha \mid \alpha \in \mathcal{F}^{(1)}\}$ and*

$$\operatorname*{\mathcal{R}-lim}_{\alpha\in\mathbb{N}^{(m)}} x_{h_\alpha} = x_0. \tag{6.3}$$

Proof. (i) $\Longrightarrow$ (ii): Let $\mathcal{F}^{(1)}$ be an IP-ring. Since X is compact, we can assume (by passing, if needed, to a sub-IP-ring) that $\text{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} x_{g_\alpha}$ exists. Thus, by (i), (6.2) holds. It follows from the definition of an IP-limit that there exists a sequence $(h_k)_{k\in\mathbb{N}}$ in G such that $\{h_k \mid k \in \mathbb{N}\} \subseteq \{g_\alpha \mid \alpha \in \mathcal{F}^{(1)}\}$ and $\lim_{k\to\infty} x_{h_k} = x_0$. This completes the proof of (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i): Let $\mathcal{F}^{(1)}$ be an IP-ring for which $\text{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} x_{g_\alpha} = y$ for some $y \in X$. Suppose for contradiction that there exists an $\epsilon > 0$ for which $d(y, x_0) > \epsilon$. By the definition of an IP-limit, there exists $\alpha_0 \in \mathcal{F}$ such that, for any $\alpha \in \mathcal{F}^{(1)}$ with $\alpha > \alpha_0$, $d(x_{g_\alpha}, x_0) > \epsilon$. Since $\{\alpha \in \mathcal{F}^{(1)} \mid \alpha > \alpha_0\}$ is an IP-ring, it follows from (ii) that there exist an $m \in \mathbb{N}$ and a sequence $(h_{k,1}, \ldots, h_{k,m})_{k\in\mathbb{N}}$ in G^m such that $\{h_\alpha \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq \{g_\alpha \mid \alpha \in \mathcal{F}^{(1)} \text{ and } \alpha > \alpha_0\}$ and $\mathcal{R}\text{-lim}_{\alpha\in\mathbb{N}^{(m)}} x_{h_\alpha} = x_0$. In particular, there exists an $h \in \{g_\alpha \mid \alpha \in \mathcal{F}^{(1)} \text{ and } \alpha > \alpha_0\}$ for which $d(x_h, x_0) < \epsilon$, a contradiction. □

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Remark 6.3. Theorem 6.2 shows that IP-limits can be attained via $\mathcal{R}$-limits. The following example demonstrates that this is not the case the other way around. Let $G=\mathbb{Z}$, let $X=\{0,1\}$, let $m\in\mathbb{N}$ and consider the Σ_m set $E=\{3^{k_1}+\cdots+3^{k_m}\mid k_1<\cdots<k_m\}$. The set E is comprised of all the elements of $3\mathbb{N}$ whose base 3 expansion has exactly m non-zero entries, all of which are 1. It follows that there are no $a,b,c\in E$ for which $a+b=c$. This, in turn, implies that E contains no IP sets and hence $\mathbb{Z}\setminus E$ is an IP* set. Let $(n_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{Z}$ and let $\mathcal{F}^{(1)}\subseteq\mathcal{F}$ be an IP-ring for which $\text{IP-lim}_{\alpha\in\mathcal{F}^{(1)}}\mathbb{1}_E(n_\alpha)$ exists. Since $0\notin E$ and $\mathbb{Z}\setminus E$ is IP*, one has $\text{IP-lim}_{\alpha\in\mathcal{F}^{(1)}}\mathbb{1}_E(n_\alpha)=0$. On the other hand, since for any $k_1<\cdots<k_m$, $\mathbb{1}_E(3^{k_1}+\cdots+3^{k_m})=1$, one has that, for any infinite set $S\subseteq\mathbb{N}$,

$$\underset{\{k_1,\ldots,k_m\}\in S^{(m)}}{\mathcal{R}\text{-lim}}\mathbb{1}_E(3^{k_1}+\cdots+3^{k_m})=1.$$

Definition 6.4. Let $(G,+)$ be a countable abelian group and let $(X,\mathcal{A},\mu,(T_g)_{g\in G})$ be a measure-preserving system. $(T_g)_{g\in G}$ is mildly mixing if, for any sequence $(g_k)_{k\in\mathbb{N}}$ in G for which $\lim_{\alpha\to\infty}g_\alpha=\infty$, there exists an IP-ring $\mathcal{F}^{(1)}$ such that, for any $f\in L^2(\mu)$,

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\,T_{g_\alpha}f=\int_X f\,\mathrm{d}\mu \tag{6.4}$$

weakly.

We are now ready to state and prove the main theorem of this subsection. It can be viewed as an analogue of Theorem 3.1 for mildly mixing actions. We remind the reader that a sequence of measure-preserving transformations $(T_k)_{k\in\mathbb{N}}$ of a probability space $(X,\mathcal{A},\mu)$ has the mixing property if, for every $A_0,A_1\in\mathcal{A}$, $\lim_{k\to\infty}\mu(A_0\cap T_k^{-1}A_1)=\mu(A_0)\mu(A_1)$.

THEOREM 6.5. *Let $\ell\in\mathbb{N}$, let $(G,+)$ be a countable abelian group and let $(X,\mathcal{A},\mu,(T_g)_{g\in G})$ be a measure-preserving system. The following statements are equivalent.*

(i) *$(T_g)_{g\in G}$ is mildly mixing.*

(ii) *For any $\widetilde{IP}$ set $E\subseteq G^\ell$ and any $m\in\mathbb{N}$, there exist non-degenerated and essentially distinct sequences $(\boldsymbol{g}_k^{(j)})_{k\in\mathbb{N}}=(g_{k,1}^{(j)},\ldots,g_{k,m}^{(j)})_{k\in\mathbb{N}}$, $j\in\{1,\ldots,\ell\}$, in G^m with the following properties.*

  (a) $\{(g_\alpha^{(1)},\ldots,g_\alpha^{(\ell)})\mid\alpha\in\mathbb{N}^{(m)}\}\subseteq E$.

  (b) *For any $t\in\{1,\ldots,m\}$ and any $j\in\{1,\ldots,\ell\}$, $(T_{g_{k,t}^{(j)}})_{k\in\mathbb{N}}$ has the mixing property.*

  (c) *For any t and any $i\neq j$, $(T_{g_{k,t}^{(j)}-g_{k,t}^{(i)}})_{k\in\mathbb{N}}$ has the mixing property.*

(iii) *For any $\widetilde{IP}$ set $E\subseteq G^\ell$, there exist an $m\in\mathbb{N}$ and non-degenerated and essentially distinct sequences $(\boldsymbol{g}_k^{(1)})_{k\in\mathbb{N}},\ldots,(\boldsymbol{g}_k^{(\ell)})_{k\in\mathbb{N}}$ in G^m with $\{(g_\alpha^{(1)},\ldots,g_\alpha^{(\ell)})\mid\alpha\in\mathbb{N}^{(m)}\}\subseteq E$ and such that, for any $A_0,\ldots,A_\ell\in\mathcal{A}$,*

$$\underset{\alpha\in\mathbb{N}^{(m)}}{\mathcal{R}\text{-lim}}\,\mu(A_0\cap T_{g_\alpha^{(1)}}A_1\cap\cdots\cap T_{g_\alpha^{(\ell)}}A_\ell)=\prod_{j=0}^{\ell}\mu(A_j). \tag{6.5}$$

(iv) *Given sequences $(g_k^{(1)})_{k\in\mathbb{N}},\ldots,(g_k^{(\ell)})_{k\in\mathbb{N}}$ in G such that, for any $j\in\{1,\ldots,\ell\}$, $\lim_{\alpha\to\infty}g_\alpha^{(j)}=\infty$ and, for any $i\neq j$, $\lim_{\alpha\to\infty}g_\alpha^{(j)}-g_\alpha^{(i)}=\infty$ (and so*

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$E = \{(g_\alpha^{(1)}, \ldots, g_\alpha^{(\ell)}) \mid \alpha \in \mathcal{F}\}$ is an $\widetilde{\mathrm{IP}}$ set), there exists an IP*-ring $\mathcal{F}^{(1)}$ such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$,*

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\, \mu(A_0 \cap T_{g_\alpha^{(1)}} A_1 \cap \cdots \cap T_{g_\alpha^{(\ell)}} A_\ell) = \prod_{j=1}^{\ell} \mu(A_j). \tag{6.6}$$

(v) *For any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set*

$$R_\epsilon(A_0, \ldots, A_\ell) = \bigg\{(g_1, \ldots, g_\ell) \in G^\ell \;\bigg|\; \bigg|\mu(A_0 \cap T_{g_1} A_1 \cap \cdots \cap T_{g_\ell} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\bigg| < \epsilon\bigg\}$$

is an $\widetilde{\mathrm{IP}}^*$ set.

Proof. (i) $\Longrightarrow$ (ii): Let $m \in \mathbb{N}$, let $E \subseteq G^\ell$ be an $\widetilde{\mathrm{IP}}$ set and let the sequences $(h_k^{(1)})_{k\in\mathbb{N}}, \ldots, (h_k^{(\ell)})_{k\in\mathbb{N}}$ in G be such that $E = \{(h_\alpha^{(1)}, \ldots, h_\alpha^{(\ell)}) \mid \alpha \in \mathcal{F}\}$. By the stipulation made in Remark 5.2, for any IP-ring $\mathcal{F}^{(1)} \subseteq \mathcal{F}$, the set $\{(h_\alpha^{(1)}, \ldots, h_\alpha^{(\ell)}) \mid \alpha \in \mathcal{F}^{(1)}\}$ is again an $\widetilde{\mathrm{IP}}$ set. Pick $\mathcal{F}^{(1)}$ to be an IP-ring such that, for any $A_0, A_1 \in \mathcal{A}$ and any $i, j \in \{1, \ldots, \ell\}$,

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\, \mu(A_0 \cap T_{h_\alpha^{(j)}} A_1) \quad \text{and, if } i \neq j, \quad \underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\, \mu(A_0 \cap T_{h_\alpha^{(j)} - h_\alpha^{(i)}} A_1) \tag{6.7}$$

exist. Let $(\alpha_k)_{k\in\mathbb{N}}$ be the sequence in $\mathcal{F}$ generating $\mathcal{F}^{(1)}$ (so, in particular, $\alpha_k < \alpha_{k+1}$ for each $k \in \mathbb{N}$). It follows from (i) that each of the limits appearing in (6.7) equals $\mu(A_0)\mu(A_1)$ (otherwise, we would have a contradiction with formula (6.4)). Thus, for any $A_0, A_1 \in \mathcal{A}$ and any $i, j \in \{1, \ldots, \ell\}$,

$$\lim_{k\to\infty} \mu(A_0 \cap T_{h_{\alpha_k}^{(j)}} A_1) = \mu(A_0)\mu(A_1) \quad \text{and, if } i \neq j, \quad \lim_{k\to\infty} \mu(A_0 \cap T_{h_{\alpha_k}^{(j)} - h_{\alpha_k}^{(i)}} A_1) = \mu(A_0)\mu(A_1).$$

For each $j \in \{1, \ldots, \ell\}$, let $(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}} = (\underbrace{h_{\alpha_k}^{(j)}, \ldots, h_{\alpha_k}^{(j)}}_{m \text{ times}})$. It is now easy to check that the sequences $(\mathbf{g}_k^{(1)})_{k\in\mathbb{N}}, \ldots, (\mathbf{g}_k^{(\ell)})_{k\in\mathbb{N}}$ are non-degenerated, essentially distinct, and satisfy (a)–(c), completing the proof of (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii): This follows from Proposition 2.9.

(iii) $\Longrightarrow$ (iv): We will prove (iv) by applying Theorem 6.2 to the G^ℓ-sequence

$$x_{(g_1,\ldots,g_\ell)} = \mu(A_0 \cap T_{g_1} A_1 \cap \cdots \cap T_{g_\ell} A_\ell),\ (g_1, \ldots, g_\ell) \in G^\ell,$$

and the sequence $(g_k^{(1)}, \ldots, g_k^{(\ell)})_{k\in\mathbb{N}}$ in G^ℓ.

Note that for any IP-ring $\mathcal{F}^{(2)}$, $\{(g_\alpha^{(1)}, \ldots, g_\alpha^{(\ell)}) \mid \alpha \in \mathcal{F}^{(2)}\}$ is an $\widetilde{\mathrm{IP}}$ set. By (iii), there exist an $m \in \mathbb{N}$ and non-degenerated and essentially distinct sequences $(\mathbf{h}_k^{(1)})_{k\in\mathbb{N}}, \ldots, (\mathbf{h}_k^{(\ell)})_{k\in\mathbb{N}}$ in G^m with

$$\{(h_\alpha^{(1)}, \ldots, h_\alpha^{(\ell)}) \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq \{(g_\alpha^{(1)}, \ldots, g_\alpha^{(\ell)}) \mid \alpha \in \mathcal{F}^{(2)}\}$$

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for which (6.5) holds. Letting $\mathcal{F}^{(1)}$ be an IP-ring for which the left-hand side of (6.6) exists for any $A_0, \ldots, A_\ell \in \mathcal{A}$, we obtain by Theorem 6.2 that (6.6) holds.

(iv) $\Longrightarrow$ (v): This implication follows from the definition of $\tilde{\mathrm{IP}}^*$.

(v) $\Longrightarrow$ (i): Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in G with the property that $\lim_{\alpha\to\infty} g_\alpha = \infty$. It suffices to show that, for some IP-ring $\mathcal{F}^{(1)}$ and any $A_0, A_1 \in \mathcal{A}$,

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\, \mu(A_0 \cap T_{g_\alpha} A_1) = \mu(A_0)\mu(A_1).$$

By (6.1), there exists an IP-ring $\mathcal{F}^{(1)} \subseteq \mathcal{F}$ such that, for any $A_0, A_1 \in \mathcal{A}$,

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\, \mu(A_0 \cap T_{g_\alpha} A_1) \tag{6.8}$$

exists. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}^{(1)}$ with $\gamma_k < \gamma_{k+1}$, for each $k \in \mathbb{N}$, and such that the sequences $(h_k^{(j)})_{k\in\mathbb{N}} = (g_{\gamma_{j+\ell k}})_{k\in\mathbb{N}}$, $j \in \{1, \ldots, \ell\}$, in G satisfy that (a) for any $j \in \{1, \ldots, \ell\}$, $\lim_{\alpha\to\infty} h_\alpha^{(j)} = \infty$, and (b) for any $i \neq j$, $\lim_{\alpha\to\infty} h_\alpha^{(j)} - h_\alpha^{(i)} = \infty$. For each $\alpha_0 \in \mathcal{F}$, let

$$E_{\alpha_0} = \{(h_\alpha^{(1)}, \ldots, h_\alpha^{(\ell)}) \mid \alpha \in \mathcal{F} \text{ and } \alpha > \alpha_0\}.$$

Since E_{α_0} is an $\tilde{\mathrm{IP}}$ set, (v) implies that, for any $\alpha_0 \in \mathcal{F}$, any $A_0, A_1 \in \mathcal{A}$ and any $\epsilon > 0$,

$$E_{\alpha_0} \cap R_\epsilon(A_0, A_1, X, \ldots, X) \neq \emptyset.$$

Thus, for any $\alpha_0 \in \mathcal{F}$, there exists an $\alpha > \alpha_0$ such that $h_\alpha^{(1)} \in R_\epsilon(A_0, A_1)$. Note that

$$\lim_{\alpha\to\infty} \min\left(\bigcup_{k\in\alpha} \gamma_{1+\ell k}\right) = \infty.$$

It follows that, for any $\beta_0 \in \mathcal{F}$, there is an $\alpha \in \mathcal{F}$ such that $h_\alpha^{(1)} \in R_\epsilon(A_0, A_1)$ and such that $\beta = \bigcup_{k\in\alpha} \gamma_{1+\ell k} \in \mathcal{F}^{(1)}$ satisfies $\beta > \beta_0$. But $g_\beta = g_{(\bigcup_{k\in\alpha} \gamma_{1+\ell k})} = h_\alpha^{(1)}$, so

$$|\mu(A_0 \cap T_{g_\beta} A_1) - \mu(A_0)\mu(A_1)| < \epsilon.$$

Since ϵ was arbitrary, for any $A_0, A_1 \in \mathcal{A}$,

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\, \mu(A_0 \cap T_{g_\alpha} A_1) = \mu(A_0)\mu(A_1),$$

which completes the proof. □

Remark 6.6. We saw in §4 that the versatility of $\mathcal{R}$-limits allows one to obtain from the *multiparameter* Theorem 3.1 some interesting results of *diagonal* nature. Similarly, one can obtain diagonal results from Theorem 6.5. For example, let $G = \mathbb{Z}$ and assume that $(X, \mathcal{A}, \mu, T)$ is a mildly mixing system. Then, by Theorem 6.5(iv), for any strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}$, any non-zero and distinct integers $a_1, \ldots, a_\ell$ and any IP-ring $\mathcal{F}^{(1)} \subseteq \mathcal{F}$, there exists an IP-ring $\mathcal{F}^{(2)} \subseteq \mathcal{F}^{(1)}$ such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$,

$$\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\, \mu(A_0 \cap T^{a_1 n_\alpha} A_1 \cap \cdots \cap T^{a_\ell n_\alpha} A_\ell) = \prod_{j=0}^{\ell} \mu(A_j) \tag{6.9}$$

(cf. [**14**, Theorem 9.27] and [**15**, Theorem 5.4].)

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6.2. *Weakly mixing systems.* This subsection is devoted to weakly mixing systems (which were introduced in §5.2) and has a similar structure to that of §6.1. We will first establish a technical lemma which connects $\mathcal{R}$-limits with Césaro convergence. We will then prove an analogue of Theorem 3.1 (see Theorem 6.10 below) for weakly mixing systems and derive a corollary which has diagonal nature.

LEMMA 6.7. *Let $(G, +)$ be a countable abelian group, let (X, d) be a compact metric space, let $(x_g)_{g\in G}$ be a sequence in X, let $x_0 \in X$, let $(F_k)_{k\in\mathbb{N}}$ be a Følner sequence in G and let $E \subseteq G$ be such that $\overline{d}_{(F_k)}(E) > 0$. The following statements are equivalent.*

(i)

$$\lim_{k\to\infty} \frac{1}{|F_k|} \sum_{g\in F_k} \mathbb{1}_E(g) d(x_g, x_0) = 0. \tag{6.10}$$

(ii) *For any $D \subseteq E$ with $\overline{d}_{(F_k)}(D) > 0$, there exist an $m \in \mathbb{N}$ and a sequence $(g_{k,1}, \ldots, g_{k,m})_{k\in\mathbb{N}}$ in G^m for which $\{g_\alpha \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq D$ and*

$$\mathcal{R}\text{-}\lim_{\alpha\in\mathbb{N}^{(m)}} x_{g_\alpha} = x_0. \tag{6.11}$$

Proof. (i) $\implies$ (ii): Let $D \subseteq E$ be such that $\overline{d}_{(F_k)}(D) > 0$. It follows from (6.10) that

$$\lim_{k\to\infty} \frac{1}{|F_k|} \sum_{g\in F_k} \mathbb{1}_D(g) d(x_g, x_0) = 0.$$

Let $\epsilon > 0$. There exist infinitely many $g \in D$ such that $d(x_g, x_0) < \epsilon$ (otherwise, we would have $\limsup_{k\to\infty}(1/|F_k|) \sum_{g\in F_k} \mathbb{1}_D(g) d(x_g, x_0) > 0$). Thus, for each $k \in \mathbb{N}$, there is a $g_k \in D$ with $d(x_{g_k}, x_0) < 1/k$. It follows now that

$$\mathcal{R}\text{-}\lim_{\{k\}\in\mathbb{N}^{(1)}} x_{g_{\{k\}}} = \lim_{k\to\infty} x_{g_k} = x_0.$$

(ii) $\implies$ (i): It suffices to show that, for any given $\epsilon > 0$, $\overline{d}_{(F_k)}(D_\epsilon) = 0$, where

$$D_\epsilon = \{g \in E \mid d(x_g, x_0) > \epsilon\}.$$

(This will imply that, for each $\epsilon > 0$,

$$\begin{aligned}\limsup_{k\to\infty} &\frac{1}{|F_k|} \sum_{g\in F_k} \mathbb{1}_E(g) d(x_g, x_0) \\ &\leq \limsup_{k\to\infty} \bigg(\frac{1}{|F_k|} \sum_{g\in F_k} \epsilon \mathbb{1}_{E\setminus D_\epsilon}(g) + \frac{1}{|F_k|} \sum_{g\in F_k} \mathbb{1}_{D_\epsilon}(g) d(x_g, x_0) \bigg) \leq \epsilon.)\end{aligned}$$

Fix $\epsilon > 0$ and suppose for contradiction that $\overline{d}_{(F_k)}(D_\epsilon) > 0$. It follows from (ii) that there exist an $m \in \mathbb{N}$ and a sequence $(g_{k,1}, \ldots, g_{k,m})_{k\in\mathbb{N}}$ in G^m with $\{g_\alpha \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq D_\epsilon$ for which (6.11) holds. In particular, for some $g \in D_\epsilon$, $d(x_g, x_0) < \epsilon$, which gives us the desired contradiction. □

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We collect in the following proposition some equivalent definitions of weak mixing which will be needed for the proof of Theorem 6.10. The proof is totally analogous to the classical case $G = \mathbb{Z}$ and is omitted.

PROPOSITION 6.8. *Let $(G, +)$ be a countable abelian group and let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system. The following statements are equivalent.*

(i) *$(T_g)_{g\in G}$ is weakly mixing.*

(ii) *For any ergodic probability measure-preserving system $(Y, \mathcal{B}, \nu, (S_g)_{g\in G})$, the system*

$$(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu, (T_g \times S_g)_{g\in G})$$

is ergodic.

(iii) *For any Følner sequence $(F_k)_{k\in\mathbb{N}}$ in G there exists a set $B \subseteq G$ with $\overline{d}_{(F_k)}(B) = 0$ such that, for any $A_0, A_1 \in \mathcal{A}$,*

$$\lim_{g\to\infty,\, g\notin B} \mu(A_0 \cap T_g A_1) = \mu(A_0)\mu(A_1).$$

(iv) *There exists a sequence $(g_k)_{k\in\mathbb{N}}$ in G with $\lim_{k\to\infty} g_k = \infty$ such that, for any $A_0, A_1 \in \mathcal{A}$,*

$$\lim_{k\to\infty} \mu(A_0 \cap T_{g_k} A_1) = \mu(A_0)\mu(A_1).$$

Remark 6.9. It follows from (ii) that, for any two weakly mixing systems $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ and $(Y, \mathcal{B}, \nu, (S_g)_{g\in G})$, $(T_g \times S_g)$ is again weakly mixing.

THEOREM 6.10. *Let $\ell \in \mathbb{N}$, let $(G, +)$ be a countable abelian group and let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system. The following statements are equivalent.*

(i) *$(T_g)_{g\in G}$ is weakly mixing.*

(ii) *For any Følner sequence $(F_k)_{k\in\mathbb{N}}$ in G^ℓ, any set $E \subseteq G^\ell$ with $\overline{d}_{(F_k)}(E) > 0$ and any $m \in \mathbb{N}$, there exist non-degenerated and essentially distinct sequences $(\mathbf{g}_k^{(j)})_{k\in\mathbb{N}} = (g_{k,1}^{(j)}, \ldots, g_{k,m}^{(j)})_{k\in\mathbb{N}}$, $j \in \{1, \ldots, \ell\}$, in G^m with the following properties.*

   (a) *$\{(g_\alpha^{(1)}, \ldots, g_\alpha^{(\ell)}) \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq E$.*

   (b) *For any $t \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, \ell\}$, $(T_{g_{k,t}^{(j)}})_{k\in\mathbb{N}}$ has the mixing property.*

   (c) *For any t and any $i \neq j$, $(T_{g_{k,t}^{(j)} - g_{k,t}^{(i)}})_{k\in\mathbb{N}}$ has the mixing property.*

(iii) *For any Følner sequence $(F_k)_{k\in\mathbb{N}}$ in G^ℓ and any set $E \subseteq G^\ell$ with $\overline{d}_{(F_k)}(E) > 0$, there exist an $m \in \mathbb{N}$ and sequences $(\mathbf{g}_k^{(1)})_{k\in\mathbb{N}}, \ldots, (\mathbf{g}_k^{(\ell)})_{k\in\mathbb{N}}$ in G^m with $\{(g_\alpha^{(1)}, \ldots, g_\alpha^{(\ell)}) \mid \alpha \in \mathbb{N}^{(m)}\} \subseteq E$ and such that, for any $A_0, \ldots, A_\ell \in \mathcal{A}$,*

$$\mathcal{R}\text{-}\lim_{\alpha\in\mathbb{N}^{(m)}} \mu(A_0 \cap T_{g_\alpha^{(1)}} A_1 \cap \cdots \cap T_{g_\alpha^{(\ell)}} A_\ell) = \prod_{j=0}^{\ell} \mu(A_j).$$

(iv) *For any $A_0, \ldots, A_\ell \in \mathcal{A}$ and any $\epsilon > 0$, the set*

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$$R_\epsilon(A_0, \ldots, A_\ell) = \bigg\{(g_1, \ldots, g_\ell) \in G^\ell \;\bigg|\; \bigg|\mu(A_0 \cap T_{g_1}A_1 \cap \cdots \cap T_{g_\ell}A_\ell) - \prod_{j=0}^{\ell}\mu(A_j)\bigg| < \epsilon\bigg\}$$

has uniform density one.

Proof. (i) $\Longrightarrow$ (ii): For each $j \in \{1, \ldots, \ell\}$, let $\pi_j : G^\ell \to G$ be defined by $\pi_j(g_1, \ldots, g_\ell) = g_j$. Note that $(T_{\pi_j(\mathbf{g})})_{\mathbf{g}\in G^\ell}$ is a weakly mixing action and, for any $i \neq j$, $(T_{(\pi_j-\pi_i)(\mathbf{g})})_{\mathbf{g}\in G^\ell}$ is also weakly mixing. Moreover (see Remark 6.9),

$$(S_{\mathbf{g}})_{\mathbf{g}\in G^\ell} = \bigg(\prod_{j=1}^{\ell} T_{\pi_j(\mathbf{g})} \times \prod_{i\neq j} T_{(\pi_j-\pi_i)(\mathbf{g})}\bigg)_{\mathbf{g}\in G^\ell}$$

is a weakly mixing G^ℓ-action on the probability space

$$\bigg(X^{\ell^2}, \bigotimes_{j=1}^{\ell^2} \mathcal{A}, \nu\bigg),$$

where $\nu = \underbrace{\mu \times \cdots \times \mu}_{\ell^2 \text{ times}}$.

By Proposition 6.8(iii), there exists a set $B \subseteq G^\ell$ with $\overline{d}_{(F_k)}(B) = 0$ such that, for any $A_0, A_1 \in \bigotimes_{j=1}^{\ell^2} \mathcal{A}$,

$$\lim_{\mathbf{g}\to\infty,\, \mathbf{g}\notin B} \nu(A_0 \cap S_{\mathbf{g}}A_1) = \nu(A_0)\nu(A_1). \tag{6.12}$$

We start with proving (ii) for $m = 1$. Let $E \subseteq G^\ell$ with $\overline{d}_{(F_k)}(E) > 0$. By Proposition 5.8 (applied to $d = \ell$, $m = 1$ and the set $(E \setminus B) \subseteq G^\ell$) there exist non-degenerated and essentially distinct sequences $(g_k^{(1)})_{k\in\mathbb{N}}, \ldots, (g_k^{(\ell)})_{k\in\mathbb{N}}$ in G with the property that, for each $k \in \mathbb{N}$, $\mathbf{g}_k = (g_k^{(1)}, \ldots, g_k^{(\ell)}) \in E \setminus B$. It follows now from (6.12) that $(S_{\mathbf{g}_k})_{k\in\mathbb{N}}$ has the mixing property and hence, for any $j \in \{1, \ldots, \ell\}$, $(T_{g_k^{(j)}})_{k\in\mathbb{N}}$ has the mixing property and, for any $i \neq j$, $(T_{g_k^{(j)}-g_k^{(i)}})_{k\in\mathbb{N}}$ has the mixing property as well.

Assume now that $m > 1$. Let $(g_k^{(1)})_{k\in\mathbb{N}}, \ldots, (g_k^{(\ell)})_{k\in\mathbb{N}}$ be non-degenerated and essentially distinct sequences in G such that, for any distinct $i, j \in \{1, \ldots, \ell\}$, $(T_{g_k^{(j)}})_{k\in\mathbb{N}}$ and $(T_{g_k^{(j)}-g_k^{(i)}})_{k\in\mathbb{N}}$ have the mixing property. Let $(\mathbf{h}_k)_{k\in\mathbb{N}} = (h_k^{(1)}, \ldots, h_k^{(\ell)})_{k\in\mathbb{N}}$ be a subsequence of $(g_k^{(1)}, \ldots, g_k^{(\ell)})_{k\in\mathbb{N}}$ such that, for any $i, j \in \{1, \ldots, \ell\}$,

$$\lim_{\alpha\to\infty} h_\alpha^{(j)} = \infty \quad \text{and, if } i \neq j, \quad \lim_{\alpha\to\infty} (h_\alpha^{(j)} - h_\alpha^{(i)}) = \infty. \tag{6.13}$$

Observe that, by (6.13), $\{(h_\alpha^{(1)}, \ldots, h_\alpha^{(\ell)}) \mid \alpha \in \mathcal{F}\}$ is an $\tilde{\mathrm{IP}}$ set. It follows from our choice of $(g_k^{(1)})_{k\in\mathbb{N}}, \ldots, (g_k^{(\ell)})_{k\in\mathbb{N}}$, that, for any $M \in \mathbb{N}$, any non-empty set $\alpha \subseteq \{1, \ldots, M\}$, any $A_0, A_1 \in \mathcal{A}$ and any $j \in \{1, \ldots, \ell\}$,

$$\lim_{k\to\infty} \mu(T_{-h_\alpha^{(j)}}A_0 \cap T_{h_k^{(j)}}A_1) = \mu(A_0)\mu(A_1), \tag{6.14}$$

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and, for any $i \neq j$,

$$\lim_{k\to\infty} \mu(T_{-(h_\alpha^{(j)}-h_\alpha^{(i)})}A_0 \cap T_{h_k^{(j)}-h_k^{(i)}}A_1) = \mu(A_0)\mu(A_1). \tag{6.15}$$

Passing, if needed, to a subsequence of $(\mathbf{h}_k)_{k\in\mathbb{N}}$, we can derive now from (6.14) and (6.15) the following equations

$$\text{IP-}\lim_{\alpha\in\mathcal{F}} \mu(A_0 \cap T_{h_\alpha^{(j)}}A_1) = \mu(A_0)\mu(A_1)$$

and, if $i \neq j$,

$$\text{IP-}\lim_{\alpha\in\mathcal{F}} \mu(A_0 \cap T_{h_\alpha^{(j)}-h_\alpha^{(i)}}A_1) = \mu(A_0)\mu(A_1).$$

We can conclude now the proof of (i) $\Longrightarrow$ (ii) by arguing as in the proof of Proposition 5.8 and imitating the constructions in the proofs of Proposition 5.7 and Lemma 5.3.

(ii) $\Longrightarrow$ (iii): This follows from Proposition 2.9.

(iii) $\Longrightarrow$ (iv): Let $E = G^\ell \setminus R_\epsilon(A_0, \dots, A_\ell)$. It suffices to show that, for any Følner sequence $(F_k)_{k\in\mathbb{N}}$ in G^ℓ, $\overline{d}_{(F_k)}(E) = 0$. To see this, note that if this was not the case, (iii) would imply that $E \cap R_\epsilon(A_0, \dots, A_\ell) \neq \emptyset$, a contradiction.

(iv) $\Longrightarrow$ (i): This implication is trivial and is omitted. □

We conclude this section with a corollary of Theorem 6.10 which has diagonal nature (this corollary can also be obtained from the main result in [6]).

COROLLARY 6.11. *Let $(G, +)$ be a countable abelian group, let $(X, \mathcal{A}, \mu, (T_g)_{g\in G})$ be a measure-preserving system and let $\phi_1, \dots, \phi_\ell : G \to G$ be homomorphisms with the property that, for any $j \in \{1, \dots, \ell\}$, $(T_{\phi_j(g)})_{g\in G}$ is weakly mixing and, for any $i \neq j$, $(T_{(\phi_j-\phi_i)(g)})_{g\in G}$ is also weakly mixing. For any Følner sequence $(F_k)_{k\in\mathbb{N}}$ in G and any $A_0, \dots, A_\ell \in \mathcal{A}$,*

$$\lim_{k\to\infty} \frac{1}{|F_k|} \sum_{g\in F_k} \left| \mu(A_0 \cap T_{\phi_1(g)}A_1 \cap \cdots \cap T_{\phi_\ell(g)}A_\ell) - \prod_{j=0}^{\ell} \mu(A_j) \right| = 0. \tag{6.16}$$

Proof. By Lemma 6.7, in order to prove (6.16), it suffices to show that for any $E \subseteq G$ with $\overline{d}_{(F_k)}(E) > 0$, there exists a non-degenerated sequence $(\mathbf{g}_k)_{k\in\mathbb{N}} = (g_{k,1}, \dots, g_{k,\ell})_{k\in\mathbb{N}}$ in G^ℓ with $\{g_\alpha \mid \alpha \in \mathbb{N}^{(\ell)}\} \subseteq E$ such that

$$\mathcal{R}\text{-}\lim_{\alpha\in\mathbb{N}^{(\ell)}} \mu(A_0 \cap T_{\phi_1(g_\alpha)}A_1 \cap \cdots \cap T_{\phi_\ell(g_\alpha)}A_\ell) = \prod_{j=0}^{\ell} \mu(A_j). \tag{6.17}$$

By Theorem 6.10(ii), applied to the weakly mixing G-action

$$(S_g)_{g\in G} = \left(\prod_{j=1}^{\ell} T_{\phi_j(g)} \times \prod_{i\neq j} T_{(\phi_j-\phi_i)(g)} \right)_{g\in G},$$

there exists a non-degenerated sequence $(g_{k,1}, \dots, g_{k,\ell})_{k\in\mathbb{N}}$ in G, with $\{g_\alpha \mid \alpha \in \mathbb{N}^{(\ell)}\} \subseteq E$, and such that, for any $t \in \{1, \dots, \ell\}$, the sequence $(S_{g_{k,t}})_{k\in\mathbb{N}}$ has the mixing property. It follows that for any $t \in \{1, \dots, \ell\}$ and any $j \in \{1, \dots, \ell\}$, $(T_{\phi_j(g_{k,t})})_{k\in\mathbb{N}}$ has the mixing

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property and, for any t and $i \neq j$, $(T_{(\phi_j-\phi_i)(g_{k,t})})_{k\in\mathbb{N}}$ has the mixing property as well. The result now follows from Proposition 2.9. □

Remark 6.12. Taking in Corollary 6.11 $G = \mathbb{Z}$, one obtains the following classical result due to Furstenberg (cf. [**14**, Theorem 4.11]).

For any weakly mixing system $(X, \mathcal{A}, \mu, T)$, *any non-zero and distinct integers* $a_1, \ldots, a_\ell$ *and any* $A_0, \ldots, A_\ell \in \mathcal{A}$,

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M+1}^{N} \left|\mu(A_0 \cap T^{a_1 n} A_1 \cap \cdots \cap T^{a_\ell n} A_\ell) - \prod_{j=0}^{\ell} \mu(A_j)\right| = 0.$$

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