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THE SECOND DUAL OF $C_0(S, A)$

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Abstract

The second dual of the vector-valued function space $C_0(S, A)$ is characterized in terms of generalized functions in the case where A^* and A^{**} have the Radon-Nikodým property. As an application we present a simple proof that $C_0(S, A)$ is Arens regular if and only if A is Arens regular in this case. A representation theorem of the measure μh is given, where $\mu \in C_0^*(S, A)$, $h \in L_\infty(|\mu|, A^{**})$ and μh is defined by the Arens product.

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1. Introduction

Let A be a Banach space and let S be a locally compact Hausdorff topological space, $\mathscr{B}(S)$ be the σ-algebra of all the Borel sets of S. The space of continuous functions from S to A vanishing at infinity, endowed with the uniform norm, is denoted by $C_0(S, A)$. The second dual of $C_0(S, A)$ is considered in the case where S is compact and the dual A^* has the Radon-Nikodým property in [3], and for the case where S is locally compact and A is a Banach algebra with a positive cone satisfying certain conditions in [5]. When $A = \mathbb{C}$, the complex numbers, the second dual of $C_0(S, A)$ is characterized by means of generalized functions in [12].

In Section 2, a characterization of $C_0^{**}(S, A)$ by means of generalized functions is given, in the case where A^* and A^{**} have the Radon-Nikodým property.

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Recall that both A^* and A^{**} have the Radon-Nikodým property if any one of the following conditions is satisfied:

(i) A is reflexive,

(ii) A^{**} is separable,

(iii) A^{**}/A is separable (see [6, p. 219], for example).

Let A be a Banach algebra and F, G be in $C_0^{**}(S, A)$. Denote by FG and $F.G$ the left and right Arens products of F and G in $C_0^{**}(S, A)$ respectively. A Banach algebra is called Arens regular if the two Arens products coincide. It has been shown recently in [11] that if S is compact and A is Arens regular, then $C(S, A)$ is Arens regular. Since the technique used in [11] is quite complicated and $C_0(S, A)$ is a very important Banach algebra, as an application of the results in Section 2, we present a simpler proof of the Arens regularity of $C_0(S, A)$ in Section 3, assuming of course A^* and A^{**} both have the Radon-Nikodým property.

Let $T : C_0(S, A) \to \mathbb{C}$ be a bounded linear operator. The representing measure $m : \mathscr{B}(S) \to A^*$ is a weakly compact measure and it is shown in [8, p. 54] that the total variation $|m|$ and the semivariation $\tilde{m}$ of m are the same in this case. Let $MW(S, A^*)$ and $D(S, A^*)$ be the collection of all weakly compact measures and dominated measures respectively. Then

$$C_0^*(S, A) = MW(S, A^*) = D(S, A^*)$$

(see [2] and [5], for example). Since $|m|(S) = \tilde{m}(S) = \|T\|$ is finite, we see that $|m| \in M(S)$, the space of all bounded regular Borel measures on S [2, Theorem 2.8].

If $\mu \in MW(S, A^*)$ and $h \in L_\infty(|\mu|, A^{**})$, then h can be viewed as an element in $C_0^{**}(S, A)$ (see Lemma 3) and μh defined by the right Arens product is an element in $C_0^*(S, A)$. A representation theorem for μh, when considered as a measure, is given in Section 4.

Throughout this paper we assume that A^* and A^{**} have the Radon-Nikodým property and we follow the standard notation of Arens product used in Duncan and Hosseinium [9]. The bilinear integration theory used in this paper is developed in Dinculeanu [8].

2. The dual of $MW(S, A^*)$

Let $L_1(|m|, A^*)$ be the space of all the equivalence classes of A^*-valued Bochner integrable functions defined on S. Then the dual

$$L_1(|m|, A^*)^* = L_\infty(|m|, A^{**})$$

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if and only if A^{**} has the Radon-Nikodým property (see, for example, Diestel and Uhl [6], where $L_\infty(|m|, A^{**})$ stands for the space of equivalence classes of A^{**}-valued Bochner integrable functions defined on S that are $|m|$-essentially bounded, that is, such that

$$\begin{aligned}\|f\|_{m,\infty} &= \inf\{\sup\{\|f(x)\| : x \notin N\} : |m|(N) = 0\} \\ &= \inf\{c > 0 : |m|(\{x \in S : \|f(x)\| > c\}) = 0\} < \infty.\end{aligned}$$

Consider the product linear space $\prod\{L_\infty(|m|, A^{**}) : m \in MW(S, A^*)\}$. An element $f = (f_m)_{m \in MW(S,A^*)}$ in this product is called a *generalized function on* S provided

(i) $\|f\| = \sup \|f_m\|_{m,\infty} : m \in MW(S, A^*) < \infty$,

(ii) if $\mu, \nu \in MW(S, A^*)$ are such that $|\mu| \ll |\nu|$, then $f_\mu = f_\nu$ $|\mu|$-a.e.

Here $|\mu| \ll |\nu|$ means $|\mu|$ is absolutely continuous with respect to $|\nu|$. It is easy to see that condition (ii) above is meaningful for the equivalence classes of functions.

Let $GL(S, A^{**})$ denote the linear subspace of A^{**}-valued generalized functions on S. It is easy to verify that $GL(S, A^{**})$ is a Banach space with norm $\|f\| = \sup\{\|f_\mu\|_{\mu,\infty} : \mu \in MW(S, A^*)\}$.

DEFINITION. Let $\mu, \nu : \mathscr{B}(S) \to A^*$ be in $MW(S, A^*)$. If $\lim_{|\mu|(E)\to 0} \nu(E) = 0$, then ν is called $|\mu|$-*continuous* and is denoted by $\nu \ll |\mu|$.

THEOREM 1. *for each bounded linear functional* $F \in MW(S, A^*)^*$, *there is a unique generalized function* $f \in GL(S, A^*)$ *such that*

$$F(\mu) = \int f_\mu \, d\mu \qquad (\mu \in MW(S, A^*))$$

and $\|F\| = \|f\|$.

PROOF. For $\mu \in MW(S, A^*)$, F induces a bounded linear functional F_μ on

$$L_1(|\mu|, A^*) = \{\nu \in MW(S, A^*) : \nu \ll |\mu|\},$$

since A^* has the Radon-Nikodým property. Let $\nu \leftrightarrow f^\nu$ be the corresponding mapping. Now, since A^{**} has the Radon-Nikodým property, we see that $L_1(|\mu|, A^*)^* = L_\infty(|\mu|, A^{**})$ and that there is $f_\mu \in L_\infty(|\mu|, A^{**})$ such that

$$F_\mu(\nu) = F(\nu) = \int \langle f^\nu, f_\mu \rangle \, d|\mu|,$$

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for any $\nu \in L_1(|\mu|, A^*)$ [6, pages 98–99]. Since

$$\nu(E) = \int_E f^\nu \, d|\mu|,$$

and the countably valued functions in $L_\infty(|\mu|, A^{**})$ are dense in $L_\infty(|\mu|, A^{**})$ (see [6, p. 97], for example), we see that

$$\int \langle f^\nu, f_\mu \rangle \, d|\mu| = \int f_\mu \, d\nu.$$

In particular $F(\mu) = \int f_\mu \, d\mu$.

We shall show that $f = (f_\mu)_{\mu \in MW(S,A^*)}$ is a generalized function. Let $\mu, \nu \in MW(S, A^*)$ such that $|\mu| \ll |\nu|$. For $\gamma \in L_1(|\mu|, A^*)$, we have $\gamma \ll |\mu|$, $\gamma \ll |\nu|$. Hence

$$\int f_\mu \, d\gamma = F_\mu(\gamma) = F(\gamma) = F_\nu(\gamma) = \int f_\nu \, d\gamma.$$

Thus $f_\mu = f_\nu$ $|\mu|$-a.e. Also, for $\mu \in MW(S, A^*)$,

$$\|f_\mu\|_{\mu,\infty} = \|F_\mu\| = \sup\{|F_\mu(\nu)| : \nu \in L_1(|\mu|, A^*), \|\nu\| \le 1\} \le \|F\|.$$

On the other hand,

$$\begin{aligned} \|F\| &= \sup\{|F(\nu)| : \nu \in L_1(|\mu|, A^*), \ \|\nu\| \le 1\} \\ &= \sup\{|\int f_\nu \, d\nu| : \nu \in L_1(|\mu|, A^*), \ \|\nu\| \le 1\} \\ &\le \sup\{\|f_\nu\|_{\nu,\infty} \cdot \|\nu\|\} \le \|f\|. \end{aligned}$$

Hence $f \in GL(S, A^{**})$ and $\|F\| = \|f\|$.

Let i be the identity of A. It is shown in [4, Lemma 2.4] that $(1_S \otimes i)$ is the identity of $C_0^{**}(S, A)$. Let $I = (I_\mu)_{\mu \in MW(S,A^*)}$ be defined by $I_\mu = i$ for all $\mu \in MW(S, A^*)$. Then I is the identity in $GL(S, A^{**})$.

THEOREM 2. *Let $T : GL(S, A^{**}) \to MW(S, A^*)^*$ be defined by*

$$Tf(\mu) = \int f_\mu \, d\mu \qquad (\mu \in MW(S, A^*), \ f \in GL(S, A^{**})).$$

Then T is an isometric isomorphism of $GL(S, A^{**})$ onto $MW(S, A^*)^*$ and $TI = (1_S \otimes i)$.

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PROOF. Let $f \in GL(S, A^{**})$. For $\mu, \nu \in MW(S, A^*)$, we see $|\mu|, |\nu| \in M(S)$. Let x^* be any element in A^* with $\|x^*\| = 1$. Set $\tau = (|\mu| + |\nu|)x^*$. Then $\tau \in MW(S, A^*)$. Clearly $|\mu + \nu| \ll |\tau|$, $|\mu| \ll |\tau|$, $|\nu| \ll |\tau|$. Hence

$$\begin{aligned} Tf(\mu+\nu) &= \int f_{\mu+\nu}\, d(\mu+\nu) = \int f_\tau\, d(\mu+\nu) \\ &= \int f_\tau\, d\mu + \int f_\tau\, d\nu \\ &= \int f_\mu\, d\mu + \int f_\nu\, d\nu \\ &= Tf(\mu) + Tf(\nu) \end{aligned}$$

and, for $\alpha \in \mathbb{C}$,

$$\begin{aligned} Tf(\alpha\mu) &= \int f_{\alpha\mu}\, d(\alpha\mu) = \alpha \int f_{\alpha\mu}\, d\mu \\ &= \alpha \int f_\mu\, d\mu = \alpha Tf(\mu), \end{aligned}$$

since $|\mu| \ll |\alpha\mu|$. Thus Tf is linear. Now

$$\begin{aligned} \|Tf\| &= \sup\{|Tf(\mu)| : \mu \in MW(S, A^*),\ \|\mu\| \leq 1\} \\ &\leq \sup\{\|f_\mu\|_{\mu,\infty} \cdot \|\mu\| : \|\mu\| \leq 1\} \leq \|f\|. \end{aligned}$$

Thus Tf is a bounded linear functional on $MW(S, A^*)$. Since T is onto, Theorem 1 shows that it is an isometric isomorphism.

Furthermore, for $\mu \in MW(S, A^*)$,

$$\begin{aligned} (TI)(\mu) &= \int I_\mu\, d\mu = i(\mu(S)) = \mu(S)i \\ &= \mu^{**}(1_S \otimes i) = (1_S \otimes i)(\mu) \end{aligned}$$

(the above first equality is implicitly implied in the proof of [2, Theorem 2.2]). Thus $TI = 1_S \otimes i$.

3. Arens regularity of $C_0(S, A)$

Although $L_\infty(|\mu|, A^{**})$ consists of equivalence classes of functions, the next lemma shows that we can view each element in $L_\infty(|\mu|, A^{**})$ as an element in $C_0^{**}(S, A)$.

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LEMMA 3. *Every element f in $L_\infty(|\mu|, A^{**})$ can be considered as an element f in $C_0^{**}(S, A)$. If $F, G \in C_0^{**}(S, A)$, then $\mu^{**}(F) = \mu^{**}(G)$ if and only if F and G agree on $L_1(|\mu|, A^*)$, where μ^{**} is the second adjoint of μ, as a linear functional on $C_0^{**}(S, A)$.*

PROOF. Since A^* and A^{**} have the Radon-Nikodým property,

$$\begin{aligned} L_\infty(|\mu|, A^{**}) &= L_1(|\mu|, A^*)^*, \\ L_1(|\mu|, A^*) &= \{\nu \in MW(S, A^*) : \nu \ll |\mu|\} \\ &\subseteq MW(S, A) = C_0^*(S, A). \end{aligned}$$

Thus by the Hahn-Banach theorem, each $f \in L_\infty(|\mu|, A^{**})$ can be extended to an element in $C_0^{**}(S, A)$. For simplicity we again use f to denote the extension. Note that the extension is not unique and that all these extensions agree on $L_1(|\mu|, A^*)$.

Suppose now that $F, G \in C_0^{**}(S, A)$ agree on $L_1(|\mu|, A^*)$. Then, for any complex number α,

$$\mu^{**}(F)(\alpha) = F(\mu^*(\alpha)), \qquad \mu^{**}(G)(\alpha) = G(\mu^*(\alpha)).$$

Now for every $h \in C_0(S, A)$, $\mu^*(\alpha)(h) = \alpha\mu(h)$. We conclude that $\mu^*(\alpha) \ll |\mu|$, and so

$$\mu^{**}(F)(\alpha) = F(\mu^*(\alpha)) = G(\mu^*(\alpha)) = \mu^{**}(G)(\alpha).$$

Thus $\mu^{**}(F) = \mu^{**}(G)$.

On the other hand, suppose now that $\mu^{**}(F) = \mu^{**}(G)$. Let $f, g \in GL(S, A^{**})$ such that $Tf = F$, $Tg = G$. Then,

$$Tf(\mu) = \int f_\mu \, d\mu, \qquad Tg(\mu) = \int g_\mu \, d\mu.$$

Let $\nu \in L_1(|\mu|, A^*)$. Then $\nu \ll |\mu|$ and so

$$\begin{aligned} F(\nu) = Tf(\nu) &= \int f_\nu \, d\nu \\ &= \int f_\mu \, d\nu \qquad (|\nu| \ll |\mu|), \\ G(\nu) = Tg(\nu) &= \int g_\nu \, d\nu \\ &= \int g_\mu \, d\nu \qquad (|\nu| \ll |\mu|). \end{aligned}$$

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Since $\mu^{**}(F) = \mu^{**}(G)$, we see that $f_\mu = g_\mu$ $|\mu|$-a.e. Now $\nu \in L_1(|\mu|, A^*)$ implies that $f_\mu = g_\mu$ $|\nu|$-a.e. Thus

$$F(\nu) = \int f_\mu \, d\nu = \int g_\mu \, d\nu = G(\nu),$$

completing the proof.

REMARKS 4.

(1) If f and g belong to the same equivalence class in $L_\infty(|\mu|, A^{**})$, they must agree on $L_1(|\mu|, A^*)$.

(2) For $\mu \in MW(S, A^*)$, $f \in GL(S, A^{**})$, Theorems 1 and 2 imply, in view of Lemma 3, that

$$\begin{aligned} \mu^{**}(f_\mu) &= f_\mu(\mu) \\ &= \int f_\mu \, d\mu \qquad (L_\infty(|\mu|, A^{**}) = L_1(|\mu|, A^*)^*) \\ &= Tf(\mu). \end{aligned}$$

Conversely, if $h \in C_0^{**}(S, A)$ and $f \in GL(S, A^{**})$ are such that $Tf = h$, then

$$\mu^{**}(h) = h(\mu) = Tf(\mu) = \int f_\mu \, d\mu = \mu^{**}(f_\mu).$$

From Lemma 3, we see that $f_\mu = h$ on $L_1(|\mu|, A^*)$ for each $\mu \in MW(S, A^*)$.

DEFINITION. For $F, G \in C_0^{**}(S, A)$, there are $f, g \in GL(S, A^{**})$ such that $Tf = F$, $Tg = G$ by Theorem 2. For $\mu \in MW(S, A^*)$, define $F \times \mu \in C_0^{***}(S, A)$ by

$$(F \times \mu)(h) = \mu^{**}(hf_\mu), \qquad (h \in C_0^{**}(S, A)).$$

Then $F \times \mu \in C_0^*(S, A)$ in particular. Note that the above definition is independent of the extension of f_μ to $C_0^{**}(S, A)$. In fact, if $F = F'$ and $G = G'$ on $L_1(|\mu|, A^*)$, then $FG = F'G$ and $FG = Fg'$ also on $L_1(|\mu|, A^*)$ because $G\mu$ is in $L_1(|\mu|, A^*)$ and $G\mu = G'\mu$ (since $\mu f \in L_1(|\mu|, A^*)$ for $f \in C_0(S, A)$). Let $F \times G$ be the element in $MW(S, A^*)$ defined by

$$(F \times G)(\mu) = F(G \times \mu), \qquad (\mu \in MW(S, A^*)).$$

THEOREM 5. *Let $F, G \in C_0^{**}(S, A)$. Then $F \times G = FG$, where FG is the left Arens product in $C_0^{**}(S, A)$.*

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PROOF. For $\mu \in MW(S, A^*)$, $(F \times G)(\mu) = F(G \times \mu)$ and $(FG)(\mu) = F(G\mu)$. For any $h \in C_0(S, A)$,

$$(G \times \mu)(h) = \mu^{**}(hg_\mu) = (hg_\mu)(\mu) = h(g_\mu \mu) = (g_\mu \mu)(h),$$

and

$$\begin{aligned}(G\mu)(h) = G(\mu h) = Tg(\mu h) &= \int g_{\mu h}\, d(\mu h) \\ &= \int g_\mu\, d(\mu h) \qquad (|\mu h| \ll |\mu|) \\ &= (\mu h)^{**}(g_\mu) = g_\mu(\mu h) = (g_\mu \mu)(h). \qquad \text{(Remark 4(2))}\end{aligned}$$

Thus $F \times G = FG$.

Similarly, for $\mu \in MW(S, A^*)$ define $\mu \otimes F \in C_0^{***}(S, A)$ by

$$(h)(\mu \otimes F) = \mu^{**}(f_\mu h), \qquad (h \in C_0^{**}(S, A)).$$

Then $\mu \otimes F \in C_0^*(S, A)$ in particular. Define $F \otimes G \in C_0^{**}(S, A)$ by $(\mu)(F \otimes G) = G(\mu \otimes F)$. Then $F \otimes G = F \cdot G$, the right Arens product in $C_0^{**}(S, A)$.

THEOREM 6. *$C_0(S, A)$ is Arens regular if and only if A is Arens regular.*

PROOF. We shall show that $F \times G = F \otimes G$ for $F, G \in MW(S, A^*)^*$. Let $f, g \in GL(S, A^{**})$ be such that $Tf = F$, $Tg = G$. For $\mu \in MW(S, A^*)$,

$$\begin{aligned}(Tf \times Tg)(\mu) = Tf(Tg \times \mu) &= \int f_{Tg \times \mu}\, d(Tg \times \mu) \\ &= \int f_\mu\, d(Tg \times \mu) \qquad (|Tg \times \mu| \ll \|\mu|) \\ &= (Tg \times \mu)^{**}(f_\mu) \qquad \text{(Remark 4(2))} \\ &= (Tg \times \mu)(f_\mu) = \mu^{**}(f_\mu g_\mu).\end{aligned}$$

On the other hand, $T(fg)(\mu) = \mu^{**}(f_\mu g_\mu)$, by Remark 4(2). Thus $Tf \times Tg = T(fg)$.

Now we shall show that $Tf \otimes Tg = T(f \cdot g)$,

$$\begin{aligned}(\mu)(Tf \otimes Tg) = Tg(\mu \otimes Tf) &= \int g_{\mu \otimes Tf}\, d(\mu \otimes Tf) \\ &= \int g_\mu\, d(\mu \otimes Tf) \qquad (|\mu \otimes Tf| \ll |\mu|) \\ &= (g_\mu)(\mu \otimes Tf) = \mu^{**}(f_\mu g_\mu).\end{aligned}$$

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On the other hand, $T(f \cdot g)(\mu) = \mu^{**}(f_\mu \cdot g_\mu)$ by Remark 4(2). Thus $Tf \otimes Tg = T(f \cdot g)$.

Since A is Arens regular and $f \cdot g$ take values in A^{**}, $(f \cdot g)(x) = (fg)(x)$. We conclude that $C_0(S, A)$ is Arens regular if A is. The rest of the proof of the theorem is clear.

4. Representation theorems

It is easy to verify that if $F_1, F_2 \in C_0^{**}(S, A)$ agree on $L_1(|\mu|, A^*) \subset C_0^*(S, A)$, then the left and right Arens products satisfy $F_1\mu = F_2\mu$, $\mu F_1 = \mu F_2$ respectively. Thus if $h \in L_\infty(|\mu|, A^{**})$ and we consider it as an element in $C_0^{**}(S, A)$ by Lemma 3, then the left and right Arens products, $h\mu$ and μh, are well-defined.

Similarly, if $G_1, G_2 \in C_0^{**}(S, A)$ agree on $L_1(|\mu|, A^*)$ then $G_1F_1(\mu) = G_2F_2(\mu)$ and $(\mu)G_1 \cdot F_1 = (\mu)G_2 \cdot F_2$, respectively. Thus if $f, g \in L_\infty(|\mu|, A^{**})$, the values of the Arens products $(\mu)f \cdot g$ and $fg(\mu)$ are uniquely determined.

DEFINITION. For $\mu \in MW(S, A^*)$, $h \in L_\infty(|\mu|, A^{**})$, define $\mu_h \in MW(S, A^*)$ by

$$\int g\, d\mu_h = \int hg\, d\mu \qquad (g \in C_0(S, A))$$

by the Riesz Representation Theorem.

THEOREM 7. *Let $h \in L_\infty(|\mu|, A^{**})$. Then $\mu_h = \mu h$.*

PROOF. Let $g \in C_0(S, A)$ and let π be the canonical embedding of $C_0(S, A)$ into $C_0^{**}(S, A)$. Then

$$(g)\mu h = (g\mu)h = h(\pi(g)\mu) = h\pi(g)(\mu) = \mu^{**}(h\pi(g)),$$

since $g\mu = \pi(g)\mu$.

On the other hand, we see from Remark 4(2) that $\mu_h(g) = \int hg\, d\mu = \mu^{**}(hg)$. Thus $\mu_h = \mu h$.

THEOREM 8. *For $g, h \in L_\infty(|\mu|, A^{**})$, $\int g\, d(\mu h) = \int h \cdot g\, d\mu$, where $h \cdot g(s)$ is defined by the right Arens product in A^{**}, that is , $h \cdot g(s) = h(s).g(s)$ for $s \in S$.*

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PROOF. We see that from Remark 4(2) that

$$\int g\,d(\mu h) = (\mu h)^{**}(g)$$
$$= (\mu h)g = (\mu)\,h \cdot g$$
$$= \mu^{**}(h \cdot g) = \int h \cdot g\,d\mu.$$

Similarly, let ${}_h\mu \in MW(S, A^*)$ be defined by

$$\int g\,d({}_h\mu) = \int gh\,d\mu \qquad (g \in C_0(S, A)).$$

Then we have

THEOREM 9. *Let $\mu \in MW(S, A^*)$ and let $g, h \in L_\infty(|\mu|, A^{**})$. Then ${}_h\mu = h\mu$ and*

$$\int g\,d(h\mu) = \int gh\,d\mu,$$

where $gh(s)$ is defined by the left Arens product in A^{**}.

REMARKS 10.

(1) Since the A^{**}-valued simple functions are in $L_\infty(|\mu|, A^{**})$ it is not difficult to verify that Theorems 8 and 9 hold for $g \in L_\infty(|\mu|, A^{**})$, $h \in L_1(|\mu|, A^{**})$ or vice versa.

(2) If $h \in C_0^{**}(S, A)$ and $f \in GL(S, A^{**})$ are such that $h = Tf$, then $h = f_\mu$ on $L_1(|\mu|, A^*)$ by Remark 4(2). Thus $\mu h = \mu f_\mu$ and $h\mu = f_\mu \mu$ respectively. Hence Theorems 8 and 9 are valid for $h \in C_0^{**}(S, A)$, that is,

$$\int g\,d(\mu h) = \int f_\mu \cdot g\,d\mu, \qquad \int g\,d(h\mu) = \int g f_\mu\,d\mu.$$

(3) Theorems 8 and 9 are not valid for general $g, h \in L_1(|\mu|, A^{**})$, since hg need not be in $L_1(|\mu|, A^{**})$ even for $A = \mathbb{C}$.

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