**H-Finite Irreducible Representations of Simple Lie Algebras**

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Let $L$ denote a simple Lie algebra over the complex number field $\mathbb{C}$ with $H$ a fixed Cartan subalgebra and $C(L)$ the centralizer of $H$ in the universal enveloping algebra $U$ of $L$. It is known [cf. 2, 5] that one can construct from each algebra homomorphism $\phi: C(L) \to \mathbb{C}$ a unique algebraically irreducible representation of $L$ which admits a weight space decomposition relative to $H$ in which the weight space corresponding to $\phi \downarrow H \in H^*$ is one-dimensional. Conversely, if $(\rho, V)$ is an algebraically irreducible representation of $L$ admitting a one-dimensional weight space $V_\lambda$ for some $\lambda \in H^*$, then there exists a unique algebra homomorphism $\phi: C(L) \to \mathbb{C}$ which extends $\lambda$ such that $(\rho, V)$ is equivalent to the representation constructed from $\phi$. Any such representation will be said to be *pointed*. The collection of all pointed representations clearly includes all dominated irreducible representations and is included in the family of all Harish-Chandra modules which are $H$-finite [cf. 2, 3].

In this paper we present a detailed study of the family of pointed representations—in particular, we shall provide a complete description, up to equivalence, of all pointed representations of the simple Lie algebras $sl(n, \mathbb{C})$ for $n = 2, 3$ and 4. Our approach will be to label the equivalence classes of pointed representations of $L$ by elements from the family of algebra homomorphisms $\phi: C(L) \to \mathbb{C}$ in analogy to the technique of labelling the dominant irreducible representations by their “highest weight function”.

**Section 1. Aut ($L: H$).** In order to simplify our study of the family $F_L$ of all algebra homomorphisms $\phi: C(L) \to \mathbb{C}$ and their associated pointed representations we shall introduce an equivalence relation on $F_L$. Let $\text{Aut}(L: H)$ denote the group of all automorphisms $\sigma$ of $L$ such that $\sigma(H) \subseteq H$. If one considers the weight space decomposition of $U$ relation to $H$, viewed as an $L$-module under the adjoint representation, we have

$$U = \sum_{\lambda \in H^*} U_\lambda.$$

Then for any $\sigma \in \text{Aut} (L: H)$ we have $\sigma(U_{\bar{\lambda}}) \subseteq U_{\bar{\sigma} \lambda}$ where $\bar{\sigma} \equiv \sigma \downarrow H$. In particular $U_0 = C(L)$ and $\sigma(U_0) = U_{\sigma}$; i.e. if $\phi \in F_L$ then $\phi \circ \sigma \downarrow C \in F_L$ for all $\sigma \in \text{Aut}(L: H)$. (Note that we also denote by $\sigma$ the natural extension of $\sigma$ to an automorphism of $U$).

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Definition. If \( \phi_1, \phi_2 \in F_L \) we say that \( \phi_1 \) is weakly equivalent to \( \phi_2 \) if and only if there exists \( \sigma \in \text{Aut}(L:H) \) such that \( \phi_1 = \phi_2 \circ \sigma \). This is clearly an equivalence relation on \( F_L \).

Let \( M_\phi \) denote the unique maximal left ideal of \( U \) containing \( \ker \phi \) for \( \phi \in F_L \). Then [cf. 1] the left regular representation of \( L \) on \( U/M_\phi \) is the pointed representation constructed from \( \phi \). If \( \phi_1, \phi_2 \in F_L \) are weakly equivalent then their associated pointed representations are related in the following way:

**Proposition 1.** Let \( \phi_1, \phi_2 \in F_L \) with \( \phi_1 = \phi_2 \circ \sigma \) for some \( \sigma \in \text{Aut}(L:H) \); then there exists a linear space isomorphism \( \sigma : U/M_{\phi_1} \rightarrow U/M_{\phi_2} \) which preserves weight spaces in the sense that

\[
\sigma((U/M_{\phi_1})_\lambda) = (U/M_{\phi_2})_{\lambda \circ \sigma^{-1}}.
\]

**Proof.** Recall that for any \( \phi \in F_L \) we have

\[
M_{\phi} = \sum_{\xi \in H} (U_\xi \cap M_{\phi}) \quad \text{and} \quad u \in U_\xi \cap M_{\phi} \text{ if and only if } U_{-\xi} u \subseteq \ker \phi.
\]

Now we observe that \( \sigma(M_{\phi_1}) \subseteq M_{\phi_2} \). This follows since for any \( u \in U_\xi \cap M_{\phi_1}, \sigma(u) \in U_{\xi \sigma^{-1}} \) and

\[
\phi_2(U_{-\xi \sigma^{-1}} \sigma(u)) = \phi_2(\sigma(U_{-\xi}) \sigma(u)) = \phi_2 \circ \sigma(U_{-\xi} u) = \phi_1(U_{-\xi} u) = 0.
\]

Thus we can define a map \( \tilde{\sigma} : U/M_{\phi_1} \rightarrow U/M_{\phi_2} \) by setting

\[
\sigma(u + M_{\phi_1}) = \sigma(u) + M_{\phi_2}.
\]

Since \( \sigma(M_{\phi_1}) = M_{\phi_2} \) and \( \sigma \) is an automorphism of \( U \), \( \tilde{\sigma} \) is a well-defined, linear isomorphism from \( U/M_{\phi_1} \) onto \( U/M_{\phi_2} \).

Finally, if \( u + M_{\phi_1} \in (U/M_{\phi_1})_\lambda \) then for each \( h \in H \)

\[
h(\sigma(u) + M_{\phi_1}) = \tilde{\sigma}(\sigma^{-1}(h) u + M_{\phi_1}) = \tilde{\sigma}(\lambda \circ \sigma^{-1}(h) u + M_{\phi_1}) = \lambda \circ \sigma^{-1}(h) \tilde{\sigma}(u + M_{\phi_1}) = \lambda \circ \sigma^{-1}(h)(\sigma(u) + M_{\phi_1}).
\]

That is,

\[
\tilde{\sigma}((U/M_{\phi_1})_\lambda) = (U/M_{\phi_2})_{\lambda \circ \sigma^{-1}}.
\]

**Remark.** It should be emphasized that the representations of \( L \) on \( U/M_{\phi_1} \) and \( U/M_{\phi_2} \) are not, in general, equivalent. However, we do have the following result:

**Proposition 2.** If \( \phi_1, \phi_2 \in F_L \) with \( U/M_{\phi_1} \cong U/M_{\phi_2} \) then for any \( \sigma \in \text{Aut}(L:H) \) we have \( U/M_{\phi_1 \sigma} \cong U/M_{\phi_2 \sigma} \).

**Proof.** As an intermediate step we first show that \( U/M_{\phi_1} \cong U/M_{\phi_2} \) if and only if for \( \xi = (\phi_1 - \phi_2) \downarrow H \) there exists \( u_0 \in U_\xi \setminus M_{\phi_1} \) such that \( \phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0) \) for all \( c \in C(L) \) and all \( w \in U_{-\xi} \).
In fact if $U/M_\phi \cong U/M_{\phi_2}$ then there exists an $L$-module homomorphism $\psi : U/M_\phi \to U/M_{\phi_2}$. If $\psi(1 + M_\phi) = u_0 + M_{\phi_2}$ then clearly $u_0 \in U_1 \setminus M_{\phi_2}$ and for $w \in U_{-1}, c \in C(L)$ we have

$$
\psi(wc + M_\phi) = wcu_0 + M_{\phi_2} = \phi_2(wcu_0)(1 + M_{\phi_2})
$$

and also

$$
\psi(wc + M_\phi) = \psi(\phi_1(c)(w + M_\phi)) = \phi_1(c)\psi(w + M_\phi) = \phi_1(c)(wu_0 + M_{\phi_2}) = \phi_1(c)\phi_2(wu_0)(1 + M_{\phi_2}).
$$

Comparing, we have $\phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0)$.

Conversely if $\phi_1, \phi_2 \in F$ and there exists $u_0 \in U_1 \setminus M_{\phi_2}$ such that $\phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0)$ for all $c \in C(L)$ and all $w \in U_{-1}$ we claim $U/M_\phi \cong U/M_{\phi_2}$. Let

$$
M = \text{Ann} (u_0 + M_{\phi_2}) = \{u \in U | uu_0 \in M_{\phi_2}\}.
$$

Clearly $M$ is a maximal left ideal of $U$ and $U/M \cong U/M_{\phi_2}$. It remains only to show that $M = M_{\phi_2}$. Since $M_{\phi_2}$ is the unique maximal left ideal of $U$ containing $\ker \phi_1$ it suffices to show that $\ker \phi_1 \subset M$. Take $c \in C(L)$ with $\phi_1(c) = 0$. Then we have that $\phi_2(wcu_0) = 0$ for all $w \in U_{-1}$. This implies that $cu_0 \in M_{\phi_2}$. That is, $c \in M$ as required.

Returning now to the proposition we assume $U/M_{\phi_1} \cong U/M_{\phi_2}$ and fix $u_0 \in U_1$ with properties as noted above. Then for any $\sigma \in \text{Aut} \((L:H)\)$ we have

$$
\phi_1 \circ \sigma(\sigma^{-1}(c))\phi_2 \circ \sigma(\sigma^{-1}(w)\sigma^{-1}(u_0)) = \sigma_2 \circ \sigma(\sigma^{-1}(w)\sigma^{-1}(c)\sigma^{-1}(u_0)).
$$

But $\sigma^{-1}(C(L)) = C(L)$, $\sigma^{-1}(u_0) \in U_{10} \setminus M_{\phi_2}$ and $\sigma^{-1}(U_{-1}) = U_{-10}$. Therefore for $\phi_1 \circ \sigma$, $\phi_2 \circ \sigma \in F_L$ where $\phi_1 \circ \sigma - \phi_2 \circ \sigma = \xi \circ \sigma$ there exists an element $\sigma^{-1}(u_0) \in U_{10} \setminus M_{\phi_2}$ such that for all $c' \in C(L)$ and all $w' \in U_{-10}$ we have

$$
\phi_1 \circ \sigma(c')\phi_2 \circ \sigma(w'\sigma^{-1}(u_0)) = \phi_2 \circ \sigma(w'\sigma^{-1}(u_0))
$$

which implies that $U/M_{\phi_1} \cong M_{\phi_2}$. 

We now single out a finite subgroup of $\text{Aut} \((L:H)\)$ which will be of importance in this paper. Calling liberally on the results of chapters 14 and 25 of [4] we let $\Delta \subset H^*$ be a root system of $L$ with basis $\Delta_{++}$ and select a Chevalley basis

$$
\{X_\beta, h_\alpha | B \in \Delta, \alpha \in \Delta_{++}\}
$$

of $L$. To each $\alpha \in \Delta_{++}$ we define a map $S_\alpha : H^* \to H^*$ by setting

$$
S_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha
$$

where $(\cdot, \cdot)$ denotes the symmetric, non-degenerate Killing form on $H^*$. The maps $S_\alpha$ are automorphisms sending $\Delta$ into itself and one can induce, via the Killing form, an automorphism (again denoted by $S_\alpha$) of the Cartan subalgebra $H$. By Theorem 14.2 [4] there exists a unique automorphism, denoted by $\sigma_\alpha$,
of \( L \) such that \( \sigma_a \) extends \( S_a \) and
\[
\sigma_a(s_{\alpha'}) = X_{\sigma_a(\alpha')}
\]
for all \( \alpha' \in \Delta_+ \). Let \( A(L) \) denote the subgroup of \( \text{Aut}(L:H) \) generated by \( \{ \sigma_{\alpha} | \alpha \in \Delta_+ \} \). From the definition of the maps \( \sigma_a \) we can show that
\[
\sigma_a(X_{\gamma}) = \pm X_{\sigma_a(\gamma)}
\]
for all \( \gamma \in \Delta \). Since \( \{ \sigma_a \downarrow H | \alpha \in \Delta_+ \} \) generates a group isomorphic to the Weyl group we can conclude that \( A(L) \) is a finite group. In the particular case of \( L = A_n \) the group \( A(L) \) is isomorphic to the Weyl group of \( A_n \).

**Section 2. The family \( F_L \).** By combining the results of two previous papers [6, 7] we construct a family of algebra homomorphisms \( \phi : C(L) \rightarrow C \) as follows. In [7] we constructed for each fixed \( s \in C \) and each fixed linear functional \( \lambda \) in the dual of the Cartan subalgebra of \( A_n \) an explicit representation \( (\rho, V_{s, \lambda}) \) of \( A_n \). The representation space \( V_{s, \lambda} \) is the complex linear space having basis
\[
\{ v(k) | k = (k_1, \ldots, k_n) \in \mathbb{Z} \times \ldots \times \mathbb{Z} \}
\]
and the representatives of elements \( x_{\alpha_i} = e_{i,i+1} \) and \( y_{\alpha_i} = e_{i+1,i} \) in \( A_n \) are given by the formulas
\[
\rho(x_{\alpha_i})v(k) = (s - \lambda(h_1 + \ldots + h_{i-1}) - k_{i-1} + k_i)v(k + \xi_i)
\]
\[
\rho(y_{\alpha_i})v(k) = (s - \lambda(h_1 + \ldots + h_i) - k_i + k_{i+1})v(k - \xi_i)
\]
where \( \xi_i \) is the \( n \)-tuple having 1 in its \( i^{th} \) component and zeroes elsewhere. By convention \( h_0 = 0 \) and \( k_0 = k_{n+1} = 0 \). Since \( \{x_{\alpha_i}, y_{\alpha_i} | i = 1, 2, \ldots, n \} \) generates \( A_n \) these formulas completely specify the representation \( (\rho, V_{s, \lambda}) \). For any such representation we obtain an algebra homomorphism \( \phi : C(A_n) \rightarrow C \) by setting
\[
\phi(c)v(0) = \rho(c)v(0) \quad (\forall c \in C(A_n)).
\]
Any algebra homomorphism defined as above will be called standard. As is easily checked for \( n \geq 2 \) the parameters \( s \) and \( \lambda \) of a standard algebra homomorphism are uniquely determined.

To construct algebra homomorphisms \( \phi : C(L) \rightarrow C \) for an arbitrary simple Lie algebra \( L \) we first require some notation. Let \( \Delta \subseteq H^* \) be the root system of \( L \) with basis \( \Delta_+ \) and set \( \Delta_+ \) as the positive roots of \( L \) relative to \( \Delta_+ \). Let \( \{ \Gamma_i | i = 1, 2, \ldots, \ell \} \) be a collection of disconnected complete subsets of \( \Delta \) relative to \( \Delta_+ \). Recall [cf. 6] that this means:
1) \( -\Gamma_i \subseteq \Gamma_i \quad (\forall i) \)
2) \( \alpha, \beta \in \Gamma_i, \quad \alpha + B \in \Delta \Rightarrow \alpha + \beta \in \Gamma_i \quad (\forall i) \)
3) \( \alpha, \beta \in \Delta_+, \quad \alpha + \beta \in \Gamma_i \Rightarrow \alpha, \beta \in \Gamma_i \quad (\forall i) \)
4) \( \Delta_+ \cap \Gamma_i \) is a basis of \( \Gamma_i \) \( (\forall i) \)
5) \( \alpha \in \Gamma_i, \beta \in \Gamma_j, \quad i \neq j \Rightarrow \alpha + \beta \notin \Delta \).
Note that such a collection can be constructed by selecting any subset of $\Delta_{++}$ and forming the closure in $\Delta$ of this set under $\pm$.

Select a Chevalley basis of $L$ say $\{y_\beta, x_\beta, h_\alpha|\beta \in \Delta_+, \alpha \in \Delta_+\}$ and apply the Poincaré-Birkhoff-Witt Theorem to obtain a linear basis of $U(L)$ consisting of all monomials

$$\prod_{\beta \in \Delta_+} y_\beta \prod_{\beta \in \Delta_+} x_\beta \prod_{\alpha \in \Delta_+} h_\alpha$$

where the exponents are non-negative integers and each product preserves a fixed order. A linear basis of $C(L)$ then consists of all monomials of the form (*) where

$$\sum_{\beta \in \Delta_+} (r_\beta - t_\beta) \beta = 0.$$

Denote by $C(\bigcup_i \Gamma_i)$ (resp. $C(\Gamma_i)$) the linear subspace of $C(L)$ generated by all basis elements of $C(L)$ for which $t_\beta = r_\beta = 0$ for all $\beta \in \Delta_+ \setminus \bigcup_i \Gamma_i$ (resp. $\beta \in \Delta_+ \setminus \Gamma_i$). Also set $\overline{C}(\bigcup_i \Gamma_i)$ (resp. $\overline{C}(\Gamma_i)$) equal to the linear subspace of $C(L)$ generated by all basis elements of $C(L)$ not in $C(\bigcup_i \Gamma_i)$ (resp. $C(\Gamma_i)$). By the properties of the $\Gamma_i$'s one can readily see that $C(\bigcup_i \Gamma_i)$ and $C(\Gamma_i)$ are subalgebras of $C(L)$ and $\overline{C}(\bigcup_i \Gamma_i)$ and $\overline{C}(\Gamma_i)$ are two-sided ideals of $C(L)$ with

$$C(L) = C(\bigcup_i \Gamma_i) \oplus \overline{C}(\bigcup_i \Gamma_i) = C(\Gamma_i) \oplus \overline{C}(\Gamma_i)$$

as linear spaces.

From now on we assume that the $\Gamma_i$'s are isomorphic to root systems of algebras $A_{n_i}$ (for positive integers $n_i$). Then the subalgebra $U(\Gamma_i)$ of $U$ generated by

$$\{1, h_\alpha, x_\beta, y_\beta|\alpha \in \Delta_+ \cap \Gamma_i, \beta \in \Delta_+ \cap \Gamma_i\}$$

is isomorphic to the universal enveloping algebra of $A_{n_i}$ and $C(L) \cap U(\Gamma_i) \cong C(A_{n_i})$. Identifying $C(A_{n_i})$ with $C(L) \cap U(\Gamma_i)$ and observing that

$$C(\Gamma_i) = \{C(L) \cap U(\Gamma_i)\} \cdot U(H),$$

any algebra homomorphism $\phi : C(A_{n_i}) \to C$ can be extended to an algebra homomorphism $\overline{\phi} : C(\Gamma_i) \to C$ by setting $\overline{\phi}(h_\alpha)$ to an arbitrary value for $\alpha \in \Delta_+ \setminus \Gamma_i$.

Finally if $\overline{\phi}_i : C(\Gamma_i) \to C$ are constructed as above starting from standard algebra homomorphisms $\phi_i : C(A_{n_i}) \to C$ such that $\overline{\phi}_i \downarrow U(H) = \overline{\phi}_j \downarrow U(H)$ for all $i, j$ then by Theorem 6 [6] there exists an algebra homomorphism $\phi : C(L) \to C$ such that

1) $\phi \downarrow C(\Gamma_i) = \overline{\phi}_i$ for all $i$ and
2) $\phi \downarrow \overline{C}(\bigcup_i \Gamma_i) = 0$.

Any such algebra homomorphism will be called a generalized (or g-) standard algebra homomorphism relative to $\bigcup_i \Gamma_i$. 

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Conjecture 1. Every algebra homomorphism $\phi: C(L) \to G$ is weakly equivalent to a $g$-standard one. More precisely, there exists $\sigma \in A(L)$ such that $\phi \circ \sigma$ is $g$-standard.

We now proceed to verify this conjecture for the algebras $A_1, A_2$ and $A_3$.

Case 1. The Algebra $A_1 = \text{sl}(2, \mathbb{C})$. A Chevalley basis of $A_1$ is given by $h = e_{11} - e_{22}, x = e_{12}$ and $y = e_{21}$ (where $e_{ij}$ denotes the $2 \times 2$ matrix with $(i,j)^{th}$ component 1 and zero elsewhere). Fix $G \cdot h$ as the Cartan subalgebra and observe that $C(A_1)$ is generated, as an algebra, by $\{1, h, xy\}$. Clearly $C(A_1)$ is commutative and has a linear basis given by

$$\{(yx)^{q_1} h^{q_2} | q_1, q_2 \in \mathbb{Z}^+\}.$$ 

Any algebra homomorphism $\phi \in F_{A_1}$ is then completely determined by specifying arbitrary values for $\phi(h)$ and $\phi(yx)$ and extending. In particular, we may select arbitrary scalars $s, \lambda \in \mathbb{C}$ and set $\phi(h) = \lambda$ and $\phi(yx) = s(s - \lambda - 1)$. Hence any algebra homomorphism $\phi \in F_{A_1}$ is standard.

Case 2. The algebra $A_2 = \text{sl}(3, \mathbb{C})$. A Chevalley basis for $A_2$ is given by the elements

$$\{h_a = e_{11} - e_{22}, h_\beta = e_{22} - e_{33}, x_a = e_{12}, x_\beta = e_{23}, x_{a+\beta} = e_{13},$$

$$y_a = e_{21}, y_\beta = e_{32}, y_{a+\beta} = e_{31}\}$$

where $e_{ij}$ denotes the $3 \times 3$ matrix with 1 in the $(i,j)^{th}$ component and zeroes elsewhere. Let $H = Ch_a + Ch_\beta$ be the fixed Cartan subalgebra. As in [1] we observe that $C(A_2)$ is generated, as an algebra, by

$$\{1, h_a, h_\beta, c_1 = y_a x_a, c_2 = y_\beta x_\beta, c_3 = y_{a+\beta} x_{a+\beta}, c_4 = y_a y_\beta x_{a+\beta}, c_5 = y_\beta y_a x_{a+\beta}\}$$

and has a linear basis given by

$$\{(c_5 \text{ or } c_4)^{q_1} c_3^{q_2} c_2^{q_3} c_1^{q_4} h_a^{q_5} h_\beta^{q_6} | q_i \text{ are non-negative integers}\}.$$ 

If one sets $\phi(h_a) = \lambda_1, \phi(h_\beta) = \lambda_2$ and $\phi(c_i) = z_i$ for $i = 1, 2, \ldots, 6$ then $\phi$ can be extended to a linear map on $C(A_2)$ using the above linear basis. This linear map $\phi$ is an algebra homomorphisms if and only if $\phi$ preserves the multiplication of the generators. This gives rise to the following four equations:

1. Since $c_1 c_2 = c_2 c_1 + c_5 - c_4$ we must have $z_4 = z_5$. 
2. Since $c_1 c_4 = c_4 c_1 + c_3 c_1 - c_2 c_1 + c_5 - c_3 - (c_1 - c_3)(h_a + 1)$ we must have $\lambda_1(z_4 - z_3) = z_1(z_3 - z_2)$. 
3. Since $c_2 c_4 = c_4 c_2 + c_5 c_1 + c_5 - c_3 c_2 - c_1 h_\beta - c_4$ we must have $\lambda_2 z_4 = z_2(z_1 - z_3)$. 

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4. Since $c_4 c_5 = c_3 c_2 h_a + c_3 c_1 c_5 + c_2 c_3 h_a + c_2 c_5 + 2 c_3 c_1 + 2 c_3 h + 2 c_4 - 2 c_3 c_2 - c_5 h_a - c_3 h_a - 2 c_5 + c_5 c_2 - c_5 c_1 - c_5 h_a$ we must have

\[(z_4 - z_3) (z_2 - z_1 - \lambda_1 - z_4) + z_3 (z_2 + \lambda_2) (z_1 + \lambda_1) = 0.\]

The conditions imposed by multiplication of all other pairs of generators yield equations which are dependent on those above. Provided $z_4 \neq 0, -\lambda_i$ for $i = 1, 2$ any solution of this system of equations is also a solution of the following system:

1. $z_4 = z_3$
2. $N z_1 = (z_1 + \lambda_1 - z_2) z_1 z_2$
3. $N z_3 = (\lambda_1 + \lambda_2) z_1 z_2$
4. $N (\lambda_1 + \lambda_2) = (z_2 - z_1 + \lambda_2) (z_2 - z_1 - \lambda_1)$

where $N = z_1 \lambda_2 + z_2 \lambda_1 + \lambda_1 \lambda_2$. This latter system of equations has been solved by Bouwer [1] under the tacit assumption that $\lambda_1 + \lambda_2 \neq 0$. Since every such solution of $1' - 4'$ is also a solution of $1 - 4$ in order to determine all solutions of $1 - 4$ it remains only to solve this system under each of the above mentioned restrictions separately. Solving we obtain the following complete list of solutions to $1 - 4$ and hence all algebra homomorphisms $\phi : C(A_2) \to \mathbb{C}$.

<table>
<thead>
<tr>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T_4$</th>
<th>$T_5$</th>
<th>$T_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_a$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
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<td>$\lambda_1$</td>
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</tr>
<tr>
<td>$h_\beta$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$c_1$</td>
<td>$s(s - \lambda_1 - 1)$</td>
<td>$p$</td>
<td>$0$</td>
<td>$-\lambda_1$</td>
<td>$0$</td>
<td>$-\lambda_1$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$(s - \lambda_1)(s - \lambda_1 - \lambda_2 - 1)$</td>
<td>$0$</td>
<td>$-\lambda_2$</td>
<td>$p$</td>
<td>$p$</td>
<td>$0$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$s(s - \lambda_1 - \lambda_2 - 1)$</td>
<td>$0$</td>
<td>$p$</td>
<td>$-\lambda_1 - \lambda_2$</td>
<td>$0$</td>
<td>$p$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$s(s - \lambda_1)(s - \lambda_1 - \lambda_2 - 1)$</td>
<td>$0$</td>
<td>$p$</td>
<td>$p$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$s(s - \lambda_1)(s - \lambda_1 - \lambda_2 - 1)$</td>
<td>$0$</td>
<td>$p$</td>
<td>$p$</td>
<td>$0$</td>
<td>$-\lambda_1 - \lambda_2 - p$</td>
</tr>
</tbody>
</table>

(The symbols $\lambda_1, \lambda_2, s$ and $p$ denote fixed but arbitrary complex numbers).

Note that the solutions of type $T_0$, $T_1$ and $T_4$ are $g$-standard algebra homomorphisms relative to $\Delta, \{ \pm \alpha \}$ and $\{ \pm \beta \}$ respectively. We claim that the other solutions are weakly equivalent to $T_1$ or $T_4$. In fact recall that $A(A_2)$ is generated by the two elements $\sigma_\alpha$ and $\sigma_\beta$ where the explicit definition of these automorphisms is given by

| $\sigma_\alpha$ | $-h_a$ | $h_a + h_\beta$ | $x_\alpha$ | $x_\beta$ | $x_{\alpha + \beta}$ | $y_\alpha$ | $y_\beta$ | $y_{\alpha + \beta}$ |
| $\sigma_\beta$ | $h_a + h_\beta$ | $-h_\beta$ | $x_{\alpha + \beta}$ | $y_\alpha$ | $x_{\alpha + \beta}$ | $y_\beta$ | $x_{\alpha + \beta}$ | $y_{\alpha + \beta}$ |
Extending these maps to automorphisms of $C(A_2)$ a direct computation verifies that if $\phi$ is a solution of type $T_3$ then $\phi \circ \sigma_3$ is a solution of type $T_1$ and if $\phi$ is of type $T_3$ then $\phi \circ \sigma_3 \circ \sigma_3$ is of type $T_1$. In addition if $\phi$ is a solution of type $T_3$ (resp. type $T_4$) then $\phi \circ \sigma_3$ (resp. $\sigma_3 \circ \sigma_3 \circ \sigma_3$) is a solution of type $T_4$. Thus we have shown that conjecture $I$ is valid for the algebra $A_2$.

Remark. Solutions of type $T_1$ and $T_4$ are also weakly equivalent using the automorphism $\Phi$ defined by $\Phi(h_a) = h_3, \Phi(h_3) = h_a, \Phi(x_3) = -x_3$ and $\Phi(x_3) = x_3$. Note however that $\Phi \not\in A(A_2)$.

Case 3. The algebras $A_n = sl(n + 1, \mathbb{C})$ for $n \geq 3$. A Chevalley basis for $A_n$ is given by the following set of elements:

$$
\begin{align*}
\hat{h}_{a_i} &= e_{i,i} - e_{i+1,i+1} \quad \text{for} \quad i = 1, 2, \ldots, n \\
\hat{x}_{a_1 + a_2 + \ldots + a_j} &= e_{i,i+1} \quad \text{for} \quad 1 \leq i \leq j \leq n \\
\hat{y}_{a_1 + a_2 + \ldots + a_j} &= e_{j+1,i} \quad \text{for} \quad 1 \leq i \leq j \leq n
\end{align*}
$$

where $e_{i,j}$ denotes an $(n + 1) \times (n + 1)$ matrix with 1 in the $(i,j)^{th}$ component and zeroes elsewhere. We fix $H = \sum_{i=1}^n C\hat{h}_{a_i}$ as a Cartan subalgebra. By the Poincaré-Birkhoff-Witt Theorem there exists a linear basis of $U(A_n)$ given by

$$
\prod_{1 \leq i \leq j \leq n} \hat{y}_{a_1 + \ldots + a_j} \prod_{1 \leq i \leq j \leq n} \hat{x}_{a_1 + \ldots + a_j} \prod_{i=1}^n \hat{h}_{a_i}^i
$$

where the products preserve a fixed order on the basis elements of $A_n$ and the exponents are non-negative integers. By the degree of any such monomial we mean

$$
\sum_{1 \leq i \leq j \leq n} (l_{i,i+1} + r_{i,j+1}) + \sum_{i=1}^n l_i.
$$

**Proposition 3.** The algebra $C(A_n)$ is generated by the set

$$
\{1, \hat{h}_{a_1}, \ldots, \hat{h}_{a_n}\} \cup \left\{ C(M) = \prod_{1 \leq i \leq j \leq n} \hat{y}_{a_1 + \ldots + a_j} \prod_{1 \leq i \leq j \leq n} \hat{x}_{a_1 + \ldots + a_j} \right\}
$$

where $M = (m_{i,j}) \neq 0$ is an $(n + 1) \times (n + 1)$ matrix of 0’s and 1’s with $m_{i,i} = 0$ and

$$
\sum_{i=1}^{n+1} m_{i,k} = \sum_{k=1}^{n+1} m_{k,i} = 0 \quad \text{or} \quad 1 \quad \text{for each} \quad k
$$

and $M$ cannot be expressed as a nontrivial sum of two such matrices.

**Proof.** The automorphisms $\sigma_{a_i} \in A(A_n)$ can be realized by setting $\sigma_{a_i}(x) = P_{i-1} \times P_i$ for all $x \in A_n$ where $P_i$ is the permutation matrix of the transposition $(i, i + 1)$.

To prove this proposition it suffices to show that every basis monomial $c \in C(A_n)$ can be expressed as a linear combination of products of the given
generators. We assume inductively that the theorem is true for \(A_{n-1}\) and that the above statement is valid for basis monomials of \(C(A_n)\) of degree \(< k\). Now if \(c \in C(A_n)\) is a basis monomial of degree \(k\) and contains some \(h_a\) as a factor then we can express \(c\) as a product of two basis monomials of \(C(A_n)\) of degree strictly less than \(k\) and then the result follows from the inductive hypothesis.

Thus without loss of generality we assume \(c \in C(A_n)\) is a basis monomial of degree \(k\) where
\[
c = \prod_{1 \leq i \leq s} y_{a_i + \ldots + a_j} \prod_{1 \leq i \leq s} x_{a_i + \ldots + a_j}
\]
and we associate with \(c\) the matrix \(\Lambda = (l_{ij})\) where \(l_{ii} = 0\). If \(\Lambda\) is one of the matrices described in the statement of the proposition then \(c\) itself is a generator and we are finished. If not, we note that since \(c \in C(A_n)\) we have
\[
\sum_{i=1}^{n+1} l_{i,k} = \sum_{i=1}^{n+1} l_{k,i}
\]
for all \(k\) and hence we must have for some \(k\)
\[
\sum_{i=1}^{n+1} l_{i,k} = \sum_{i=1}^{n+1} l_{k,i} \geq 2.
\]
In fact we may assume that this is true for \(k = n + 1\). (This follows since we have \(\sigma_1(c) = c' + \text{terms of degree } < k\) where \(c'\) is a basis monomial of \(C(A_n)\) with associated matrix \(P_{i^{-1}}AP_i\)).

We now factor \(c\) into generating elements of \(C(A_{n-1}+\alpha_1, \ldots, \alpha_{n-1})\) by suppressing the index \(\alpha_n\), say \(c = c_1 c_2 \ldots c_p + \text{terms of lower degree}\). (Note that this factorization is not unique and whenever \(y_{a_n}\) or \(x_{a_n}\) occur as factors in \(c\) they are treated as separate factors in this product). Since each factor \(c_i\) is a generating element of \(C(A_{n-1}+\alpha_1, \ldots, \alpha_{n-1})\) or one of the terms \(y_{a_n}\) or \(x_{a_n}\) we have that it can contain at most one factor of the form \(y_{a_1+\ldots+a_n}\). Thus for each \(i, c_i \in C(A_n)\) or \(U(A_n)_{\pm a_n}\). By assumption
\[
\sum_{i=1}^{n+1} l_{i,n+1} = \sum_{i=1}^{n+1} l_{n+1,i} \geq 2
\]
and hence the above factorization must contain at least two factors. If there are exactly two factors then each factor must contain exactly one term of the form \(y_{a_1+\ldots+a_n}\) and one term of the form \(x_{a_1+\ldots+a_n}\) and hence both factors are in \(C(A_n)\) and we may apply out inductive hypothesis on each factor. If there are more than two factors, then either all are in \(C(A_n)\) in which case we are finished or at least one, say \(c_1\), is in \(U(A_n)_{\pm a_n}\) and at least one, say \(c_t\), is in \(U(A_n)_{-a_n}\). Then \(e = (c_1 c_t) (c_2 \ldots) + \text{terms of lower degree and } c_t c_i, c_2 \ldots \in C(A_n)\) and again we may apply our inductive hypothesis to complete the proof.

We now return to the problem of constructing the family of algebra homomorphisms \(F_{\alpha}\) and prove the following reduction:

**Proposition 4.** Any algebra homomorphism \(\phi: C(A_n) \to C\) is completely determined by its values on the generators of \(C(A_n)\) of degree \(\leq 3\). In particular, \(\phi\) is trivial on \(C(A_n)\) if \(\phi = 0\) on all generators of degrees 1 and 2.
Proof. We proceed by induction on $n$, noting that the cases $n = 1$ and 2 are trivially true. For the inductive step we observe that every generator of $C(A_n)$ of degree $\leq n$ is contained in a subalgebra isomorphic to $C(A_{n-1})$. Thus it suffices to verify that the value of $\phi$ on the generators of degree $n + 1$ are determined by the values of $\phi$ on the generators of degree $\leq n$.

The problem is further reduced by observing that $\phi$ is completely determined on all generators of degree $n + 1$ provided $\phi$ is known on all generators of degree $\leq n$ and one generator of degree $n + 1$. In fact consider the following identities in $C(A_n)$:

a) \[ [y_{a_0}x_{a_1}, y_{a_1} + \ldots + a_{n-1}, x_{a_1}x_{a_2} \ldots x_{a_{n-1}}] = y_{a_1} + \ldots + a_{n-1}, x_{a_1} - y_{a_0}y_{a_1} + \ldots + a_{n-1}, x_{a_1} \ldots x_{a_{n-1}} + a_{n} \]

b) \[ [y_{a_1}x_{a_1}, y_{a_2} + \ldots + a_{n-1}, x_{a_1}x_{a_2} \ldots x_{a_{n-1}}] = y_{a_2} + \ldots + a_{n-1}, x_{a_1}x_{a_2} \ldots x_{a_{n-1}} - y_{a_1} + \ldots + a_{n-1}, x_{a_1} \ldots x_{a_{n-1}} \]

c) \[ [y_{a_1}x_{a_1}, y_{a_1} + \ldots + a_{n-1}, x_{a_1}x_{a_2} \ldots x_{a_{n-1}} - x_{a_1}, x_{a_1} + a_{n+1}, x_{a_2} \ldots x_{a_{n-1}} = y_{a_1} + \ldots + a_{n-1}, x_{a_1} \ldots x_{a_{n-1}} - x_{a_1}, x_{a_1} + a_{n+1}, x_{a_2} \ldots x_{a_{n-1}} \]

Setting $M_0 = c_{n+1, 1} + \sum_{i=1}^{n} c_{i, i+1}$ and applying the algebra homomorphism $\phi$ to the above identities we have

a) and b) \[ \Rightarrow \phi(c(M_0)) = \phi(c(P_n^{-1}M_0P_n)) = \phi(c(P_{i-1}M_0P_i)) \]

c) \[ \Rightarrow \phi(c(M_0)) = \phi(c(P_{i-1}M_0P_i)) + \phi(\text{a degree } n \text{ term}) \]

for $i = 2, 3, \ldots, n - 1$.

If $c(M)$ is an arbitrary degree $n + 1$ generator of $C(A_n)$, we have $M = P_n^{-1}M_0P$ where $P$ is a product of transposition matrices $P_i$. By sequentially applying the corresponding product of automorphisms $\sigma_{a_i} \in A(A_n)$ to the above identities we may conclude that

$$\phi(c(M)) = \phi(c(M_0)) + \phi(\text{terms of degree } \leq n).$$

Thus $\phi$ is completely determined if one knows the image of $\phi$ on all generators of degree $\leq n$ and on one generator of degree $n + 1$.

Assume now that $\phi$ is zero on all generators (\(\neq 1\)) of degree $\leq n$. Considering the identity

$$\left( y_{a_0}y_{a_1} + \ldots + a_{n-1}, x_{a_1} \ldots x_{a_{n-1}} + a_{n} \right) \left( y_{a_1} + \ldots + a_{n-1}, x_{a_1} \ldots x_{a_{n-1}} + a_{n} \right) \left( y_{a_2} + \ldots + a_{n-1}, x_{a_2} \ldots x_{a_{n-1}} + a_{n} \right) \cdots \left( y_{a_n} + a_{n} \right)$$

and applying the map $\phi$ we obtain $\phi(c(P_n^{-1}M_0P_n))\phi(c(M_0)) = 0$. But by a) above this implies $\phi(c(M_0))^2 = 0$; i.e. $\phi(c(M_0)) = 0$. Thus $\phi$ is identically
zero on all degree $n + 1$ generators. From Table I we note that any algebra homomorphism $\phi : C(A_n) \to C$ for which $\phi = 0$ on degree 1 and 2 generators is also zero on all degree 3 generators and hence the second statement of the proposition is verified.

We may now assume that $\phi$ is non-zero on some generator of degree $\leq 2$; in fact, without loss of generality we may assume that $\phi \circ \sigma(y_{a_1}, x_{a_1}) \neq 0$ for some $\sigma \in A(A_n)$. Now consider the identity

$$(y_{a_1} + y_{a_2} + \ldots + y_{a_n})(y_{a_1} + y_{a_2} + \ldots + y_{a_n}) = (y_{a_1}y_{a_1}) + (y_{a_2} + y_{a_3} + \ldots + y_{a_n})(y_{a_1} + y_{a_2} + \ldots + y_{a_n}).$$

Applying the homomorphism $\phi \circ \sigma$ to this identity we have that the value of $\phi$ on one generator of degree $n + 1$, namely the degree $n + 1$ generator associated with

$$\sigma(y_{a_2} + y_{a_1} + y_{a_1} + y_{a_2} + \ldots + y_{a_n}) = \phi(y_{a_1} + y_{a_2} + \ldots + y_{a_n}),$$

can be expressed as a rational function of the values of $\phi$ on generators of degree $\leq n$.

We now particularize these results to the case of $n = 3$ where we construct, up to weak equivalence, all members of $F_{A_3}$. Take an arbitrary algebra homomorphism $\phi \in F_{A_3}$ and assume first that $\phi$, restricted to one of the four naturally embedded copies of $C(A_2)$, is of type $T_i$ for $i = 1, 2, \ldots, 6$ (cf. Table I). Applying an appropriate automorphism from $\text{Aut}(A_3)$ we may assume that $\phi$ restricted to $C(A_2, \alpha, \beta, \gamma)$ is of Type $T_1$. This places restrictions on the other values of $\phi$ as shown in the following table:

<table>
<thead>
<tr>
<th>Table II</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>$\phi(h_2)$</td>
</tr>
<tr>
<td>$\phi(h_3)$</td>
</tr>
<tr>
<td>$\phi(h_4)$</td>
</tr>
<tr>
<td>$\phi(c_1)$</td>
</tr>
<tr>
<td>$\phi(c_2)$</td>
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<tr>
<td>$\phi(c_3)$</td>
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<tr>
<td>$\phi(c_4)$</td>
</tr>
<tr>
<td>$\phi(c_5)$</td>
</tr>
<tr>
<td>$\phi(c_6)$</td>
</tr>
<tr>
<td>$\phi(c_7) = \phi(c_8)$</td>
</tr>
<tr>
<td>$\phi(c_9) = \phi(c_{10})$</td>
</tr>
<tr>
<td>$\phi(c_{11}) = \phi(c_{12})$</td>
</tr>
<tr>
<td>$\phi(c_{12}) = \ldots = \phi(c_{20})$</td>
</tr>
</tbody>
</table>
Remarks. 1. For convenience we have labelled the generators of $C(A_3)$ by setting

- $c_1 = y_\alpha x_\alpha; \ c_2 = y_\beta x_\beta; \ c_3 = y_\gamma x_\gamma; \ c_4 = y_{\alpha+\beta} x_{\alpha+\beta}; \ c_5 = y_{\beta+\gamma} x_{\beta+\gamma};$
- $c_6 = y_{\alpha+\beta+\gamma} x_{\alpha+\beta+\gamma}; \ c_7 = y_{\alpha+\beta} x_{\alpha+\beta}; \ c_8 = y_{\beta+\gamma} x_{\beta+\gamma}; \ c_9 = y_{\gamma} x_{\gamma};$
- $c_{10} = y_{\alpha+\beta+\gamma} x_{\alpha+\beta+\gamma}; \ c_{11} = y_{\alpha+\beta} x_{\alpha+\beta}; \ c_{12} = y_{\alpha+\beta+\gamma} x_{\alpha+\beta+\gamma};$
- $c_{13} = y_{\alpha+\beta+\gamma} x_{\alpha+\beta+\gamma}; \ c_{14} = y_{\beta+\gamma} x_{\beta+\gamma}; \ c_{15} = y_{\alpha+\beta+\gamma} x_{\beta+\gamma};$
- $c_{16} = y_{\beta+\gamma} x_{\beta+\gamma}; \ c_{17} = y_{\alpha+\beta+\gamma} x_{\alpha+\beta+\gamma}; \ c_{18} = y_{\alpha+\beta+\gamma} x_{\alpha+\beta+\gamma}; \ c_{19} = y_{\beta+\gamma} x_{\beta+\gamma}; \ c_{20} = y_{\beta+\gamma} x_{\beta+\gamma}.$

2. The values in column 1 result from the assumption that $\phi \downarrow C(A_2[\alpha, \beta + \gamma])$ is of type $T_1$.

3. The values in columns 2a)-d) represent the four possible solutions for $\phi \downarrow C(A_2[\beta, \gamma])$ consistent with $\phi(c_6) = 0$. In columns 2c) and d) we also must have $\phi(h_\beta) + \phi(h_\gamma) = \lambda_2 + \lambda_3 = 0$.

4. The values in columns 3a)-d) represent the four possible solutions for $\phi \downarrow C(A_2[\alpha + \beta, \gamma])$ consistent with $\phi(c_6) = 0$. In columns 3c) and d) we also must have $\phi(h_\alpha + h_\beta) + \phi(h_\gamma) = \lambda_1 + \lambda_2 + \lambda_3 = 0$.

If $\phi$ satisfies conditions 2a) and 3a) then $\phi = 0$ on all generators of $C(A_3)$ in $C[\pm \alpha, \pm \beta, \pm (\alpha + \beta)]$ of degree $\leq 3$. Thus $\phi$ must coincide with the trivial extension of an algebra homomorphism $\phi : C(A_2[\alpha, \beta]) \to C$. By the previous analysis of $F_{A_2}$, there exists $\sigma \in A(A_2[\alpha, \beta])$ such that $\phi \circ \sigma : C(A_2[\alpha, \beta]) \to C$ is $g$-standard. Since any $\sigma \in A(A_2[\alpha, \beta])$ has a natural extension to a map $\tilde{\sigma} \in A(A_3)$ with the property that

$$\tilde{\sigma}(C[\pm \alpha, \pm \beta, \pm (\alpha, \beta)]) \subseteq C[\pm \alpha, \pm \beta, \pm (\alpha, \beta)],$$

we conclude that $\phi \circ \tilde{\sigma}$ agrees with a $g$-standard algebra homomorphism of $F_{A_3}$ on all generators of degree $\leq 3$ and hence by Proposition 3, $\phi \circ \tilde{\sigma}$ is itself $g$-standard.

If $\phi$ satisfies conditions 2a) and 3b) then $\phi = 0$ on all generators of $C(A_3)$ in $C[\pm \alpha, \pm \gamma]$ of degree $\leq 3$. Thus $\phi$ is a trivial extension of algebra homomorphisms $\phi_1 : C(\pm \alpha) \to C$ and $\phi_2 : C(\pm \gamma) \to C$ and hence is $g$-standard relative to $\pm \alpha \cup \pm \gamma$.

In each of the other cases, by using identities from $C(A_3)$, and automorphisms from $A(A_3)$ we can show that $\phi$ is weakly equivalent to a $g$-standard algebra homomorphism.

It remains now to consider those algebra homomorphisms $\phi \in F_{A_3}$ such that the restrictions of $\phi$ to each of the four copies of $C(A_2)$ in $C(A_3)$ are standard; i.e. of type $T_0$ from Table I. We parametrize $\phi$ separately on each restriction as follows:
Table III

<table>
<thead>
<tr>
<th>( C(A_2; \alpha, \beta) )</th>
<th>( C(A_2; \beta, \gamma) )</th>
<th>( C(A_2; \alpha + \beta, \gamma) )</th>
<th>( C(A_2; \alpha, \beta + \gamma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi(\delta_4) )</td>
<td>( \lambda_1 )</td>
<td>( \lambda_1 )</td>
<td>( \lambda_1 )</td>
</tr>
<tr>
<td>( \phi(\delta_2) )</td>
<td>( \lambda_1 )</td>
<td>( \lambda_2 )</td>
<td>( \lambda_2 )</td>
</tr>
<tr>
<td>( \phi(\delta_3) )</td>
<td>( \lambda_1 )</td>
<td>( \lambda_2 )</td>
<td>( \lambda_3 )</td>
</tr>
<tr>
<td>( \phi(\delta) )</td>
<td>( s(s - \lambda_1 - 1) )</td>
<td>( t(t - \lambda_1 - 1) )</td>
<td>( u(u - \lambda_1 - 1) )</td>
</tr>
<tr>
<td>( \phi(c_2) )</td>
<td>( (s - \lambda_1)(s - \lambda_2 - \lambda_3 - 1) )</td>
<td>( (t - \lambda_1)(t - \lambda_2 - \lambda_3 - 1) )</td>
<td>( (s - \lambda_1 - \lambda_2)(s - \lambda_1 - \lambda_2 - \lambda_3 - 1) )</td>
</tr>
<tr>
<td>( \phi(c_3) )</td>
<td>( (s - \lambda_1 - \lambda_2 - 1) )</td>
<td>( t(t - \lambda_2 - \lambda_3 - 1) )</td>
<td>( v(v - \lambda_2 - \lambda_3 - 1) )</td>
</tr>
<tr>
<td>( \phi(c_4) )</td>
<td>( (s - \lambda_1 - \lambda_2 - \lambda_3 - 1) )</td>
<td>( t(t - \lambda_2 - \lambda_3 - 1) )</td>
<td>( u(u - \lambda_1 - \lambda_2 - \lambda_3 - 1) )</td>
</tr>
<tr>
<td>( \phi(c_5) = \phi(c_4) )</td>
<td>( s(s - \lambda_1)(s - \lambda_2 - \lambda_3 - 1) )</td>
<td>( t(t - \lambda_2)(t - \lambda_2 - \lambda_3 - 1) )</td>
<td>( v(v - \lambda_2)(v - \lambda_2 - \lambda_3 - 1) )</td>
</tr>
</tbody>
</table>
In order that \( \phi \) be well-defined we must have certain relations among the parameters; in fact, we must have

1. \( s = u \) or \( s = 1 + \lambda_1 - u \)
2. \( s = t + \lambda_1 \) or \( s = 1 + \lambda_1 + \lambda_2 - t \)
3. \( t = v - \lambda_1 \) or \( t = 1 + \lambda_1 + 2\lambda_2 + \lambda_3 - v \)
4. \( s = v \) or \( s = 1 + \lambda_1 + \lambda_2 - v \)
5. \( t = u - \lambda_1 \) or \( t = 1 + \lambda_1 + \lambda_2 + \lambda_3 - u \)
6. \( v = u \) or \( v = 1 + \lambda_1 + \lambda_2 + \lambda_3 - u \)

By analyzing each of the distinct combinations of relations and applying Proposition 3, we may conclude that either \( \phi \) is a standard algebra homomorphism in \( F_{A_3} \) or \( \phi \) is weakly equivalent under \( A(A_3) \) to one of the previously described algebra homomorphisms. Thus to summarize we have that Conjecture I is valid for the algebra \( A_3 \).

Although we are as yet unable to verify this conjecture for the algebra \( A_n \) with \( n \geq 4 \) we do have the following first step in this direction:

**Proposition 5.** If \( \phi : C(A_n) \to C \) is an algebra homomorphism such that \( \phi \) restricted to each copy of \( C(A_3) \) in \( C(A_n) \) is standard then \( \phi \) itself is standard.

**Proof.** We proceed by induction on \( n \), noting that the case \( n = 3 \) is trivially true. Assume that the proposition is true for \( n - 1 \geq 3 \) and consider \( \phi : C(A_n) \to C \) as given. By our inductive hypothesis \( \phi \) restricted to the subalgebras

\[
C(A_{n-1}) = C(A_{n-1}|\alpha_1, \alpha_2, \ldots, \alpha_{n-1}|), \quad C(A_{n-1}|\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_n|), \ldots,
\]

is standard with parameters \( s_1, s_2, \ldots, s_{n-1} \) respectively. In order that \( \phi \) be well-defined we must have \( s_1 = s_2 = \ldots = s_n = s_{n+1} + \phi(h_{\alpha_1}) \). Since every degree \( \leq 3 \) generator of \( C(A_n) \) is in at least one of these subalgebras we have that \( \phi \) agrees on all generators of degree \( \leq 3 \) with a standard algebra homomorphism of \( F_{A_n} \) paramerized by \( s_1 \) and \( \phi \downarrow H \). By Proposition 4 we have that \( \phi \) itself is then standard.

**Section 3. Pointed representations.** In this section we shall "label" the pointed representations of a simple Lie algebra \( L \) in the following sense. We wish to specify a set \( \hat{F}_L \subseteq F_L \) having the following properties:

1) If \( \phi_1, \phi_2 \in \hat{F}_L \) with \( \phi_1 \neq \phi_2 \) then \( U/M_{\phi_1} \not\cong U/M_{\phi_2} \) as \( L \)-modules.

2) If \( V \) is a pointed representation of \( L \) then there exists \( \phi \in F_L \) such that \( V \cong U/M_{\phi} \) and \( \phi \) is weakly equivalent modulo \( A(L) \) to an element in \( \hat{F}_L \).
Since the group $A(L)$ is finite we would thus associate with each $\phi \in \mathcal{F}_L$ a finite number of non-equivalent pointed representations of $L$.

**Definition.** A standard algebra homomorphism $\phi : C(A_n) \rightarrow \mathbb{C}$ with parameters $s \in \mathbb{C}$ and $\lambda \in H^*$ is said to be complete if and only if

$$s - \phi\left(\sum_{i=0}^{n} h_\alpha \right) \notin \mathbb{Z} \quad \text{for} \quad p = 0, 1, \ldots, n \quad \text{and} \quad 0 \leq \operatorname{Re}(h_\alpha) < 1$$

for $i = 1, 2, \ldots, n$.

where $\{\alpha_i\}$ is the dual basis of $\{\alpha_i\}$ relative to the Killing form.

(Note that if $\phi \downarrow H = \sum_{j=1}^{s} S_\alpha$ then $\phi(h_\alpha) = S_i$).

**Definition.** A $g$-standard algebra homomorphism $\phi : C(L) \rightarrow \mathbb{C}$ defined relative to $\bigcup_{i=1}^{\ell} \Gamma_i$ and $\mathbb{H}^*$ is said to be extreme if and only if $\phi \downarrow \{C(L) \cap U(\Gamma_i)\}$ is complete for each $i$.

**Remark.** In particular any algebra homomorphism $\phi : C(L) \rightarrow \mathbb{C}$ which is identically zero on the ideal $C(\emptyset)$ is an extreme $g$-standard algebra homomorphism.

**Conjecture II.** The family of all extreme $g$-standard algebra homomorphisms $\phi \in \mathcal{F}_L$ labels the pointed representations of $L$.

Our aim in this section will be to prove that any two distinct extreme $g$-standard algebra homomorphisms give rise to non-equivalent pointed representations and that if $\phi$ is a $g$-standard algebra homomorphism then there exists an extreme $g$-standard algebra homomorphism $\overline{\phi}$ such that $U/M_{\overline{\phi}} \cong U/M_{\phi}$ for some $\sigma \in A(L)$. This will imply that for any algebra $L$ satisfying Conjecture I, Conjecture II is also valid.

We first give an explicit description for the pointed representations associated with standard and $g$-standard algebra homomorphisms. Let $\phi : C(A_n) \rightarrow \mathbb{C}$ be the standard algebra homomorphism parametrized by $s \in \mathbb{C}$ and $\lambda \in H^*$. For each $u \in U_k$ where $\xi = \sum_{i=1}^{n} k_\alpha (k_i \in \mathbb{Z})$ we define a scalar $\mu(u)$ by setting

$$\rho(u)v(0) = \mu(u)v(k_1, \ldots, k_n).$$

We claim that $\mu(u) = 0$ implies $u \in M_\phi$. In fact, it suffices to show that for any $w \in U_{-\xi}$ we have $\phi(wu) = 0$ and this follows since

$$\phi(wu)v(0) = \rho(wu)v(0) = \rho(w)\mu(u)v(0) = \rho(w)\mu(u)v(k_1, \ldots, k_n) = 0.$$

By construction of $U/M_\phi$, every weight function must be of the form

$$\eta = (\phi + \sum_{i=1}^{n} l_i \alpha_i) \downarrow H$$

where the coefficients $l_i$'s are integers. Setting $\xi = \sum_{i=1}^{n} l_i \alpha_i$, we know that $(U/M_\phi)_\xi \cong U_k/(U_k \cap M_\phi)$ as $H$-modules. Taking $u_1, u_2 \in U_k$ we claim that
the set \( \{u_1 + M_\phi, u_2 + M_\phi\} \) is always linearly dependent. Without loss of
generality we may assume that \( \mu(u_2) \neq 0 \) and hence consider the element
\[
\begin{align*}
    u_1 &= \frac{\mu(u_1)}{\mu(u_2)} u_2.
\end{align*}
\]
For all \( w \in U_{-\xi} \) we have
\[
\phi\left(w\left(u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2\right)\right)v(0) = \rho\left(w\left(u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2\right)\right)v(0)
\]
\[
= \rho(w)\rho\left(u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2\right)v(0)
\]
\[
= \rho(w)\left(\frac{\mu(u_1)}{\mu(u_2)} - \frac{\mu(u_1)}{\mu(u_2)}\right)\mu(u_2)v(l_1, \ldots, l_n)
\]
\[
= 0 \quad \text{or}
\]
\[
\phi\left(U_{-\xi}\left(u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2\right)\right) = 0 \quad \text{and hence } \quad u_2 - \frac{\mu(u_1)}{\mu(u_2)} u_2 \in M_\phi.
\]
Therefore \( \dim(U/M_\phi)_\eta \leq 1 \) for all \( \eta \).

To complete our description of the representation \( U/M_\phi \) it remains only to
indicate which weight spaces are one-dimensional. To this end we set
\[
P_i = \begin{cases} 
    s - \lambda(h_{a_1} + \ldots + h_{a_i}) & \text{if this is a positive integer} \\
    +\infty & \text{otherwise}
\end{cases}
\]
and
\[
q_i = \begin{cases} 
    s - \lambda(h_{a_1} + \ldots + h_{a_i}) & \text{if this is a non-positive integer} \\
    -\infty & \text{otherwise}
\end{cases}
\]
where \( i = 0, 1, 2, \ldots, n \) and by convention \( h_{a_0} = 0 \). Define
\[
D_{s,\lambda} = \{ (l_1, \ldots, l_n) \in \mathbb{Z}^n \mid q_i \leq l_i - l_{i+1} < P_i \quad \text{for all } \quad i = 0, 1, \ldots, n \}
\]
(note that \( l_0 = l_{n+1} = 0 \) by convention). We claim then that the linear func­
tional \( \phi + \sum_{i=1}^n l_i \alpha_i \) is a one-dimensional weight function of \( U/M_\phi \)
if and only if \( (l_1, \ldots, l_n) \in D_{s,\lambda} \). Recall from [7] that if
\[
s - \lambda(h_{a_1} + \ldots + h_{a_i}) = m \in \mathbb{Z}
\]
then the subspace of \( V_{s,\lambda} \) with basis \( \{v(k_1, \ldots, k_n) \mid k_i - k_{i+1} \geq m\} \) is a sub­
representation of \( (\rho, V_{s,\lambda}) \). Suppose now that \( u \in U_\xi \) with \( \xi = \sum_{i=1}^n l_i \alpha_i \) and
\( (l_1, \ldots, l_n) \not\in D_{s,\lambda} \) then for any \( w \in U_{-\xi} \) we must have
\[
\phi(wu)v(0) = \rho(wu)v(0) = \rho(w)\rho(u)v(0) = 0
\]
since there exists a subrepresentation of \( V_{s,\lambda} \) to which only one of the vectors
\( v(0) \) and \( v(l_1, \ldots, l_n) \) belongs. If, on the other hand, \( (l_1, \ldots, l_n) \in D_{s,\lambda} \) then
one can select elements \( u \in U_\xi \) and \( w \in U_{-\xi} \) such that \( \phi(wu) \neq 0 \); i.e. \( u \not\in M_\phi \).

Summarizing we have
Proposition 6. With the notation introduced above, if \( \phi : C(A_\omega) \to C \) is a standard algebra homomorphism parametrized by \( s \in C \) and \( \lambda \in H^* \) then the associated pointed representation of \( A_\omega \) is

\[
U/M_\phi = \sum_{(a_1, \ldots, a_\lambda) \in D_\lambda} (U/M_\phi)_{\phi(a_i)} \otimes (U/M_\phi)_{\phi(a_i)}
\]

where each weight space is one-dimensional.

We now consider a \( g \)-standard algebra homomorphism \( \phi : C(L) \to C \) relative to \( \bigcup \Gamma_i \) and make the following observations:

1) For any \( v = \Delta_+ \bigcup \bigcup \Gamma_i, x_v \in M_\phi \). In fact, if \( w \in U_{-v} \) then \( wx_v \in \bar{C}(\bigcup \Gamma_i) \) and hence \( \phi(wx_v) = 0 \); i.e. \( x_v \in M_\phi \).

2) If \( u \) is a basis element of \( U \) of the form (*) for which \( \exists \beta \in \Delta_+ \bigcup \bigcup \Gamma_i \) with \( r_\beta \neq 0 \) then \( u \in M_\phi \). This follows from 1) using induction on the degree of \( u \).

3) If \( \xi = \sum_{a \in \Delta_+ \bigcap \bigcup \Gamma_i} k_a \cdot \alpha \) where \( (\forall \alpha) k_a \in \mathbb{Z} \) then for any basis element \( u \in U \) we have either \( u \in M_\phi \) or \( u = u_1 u_2 \ldots u_z \) where \( z \in U(H) \) and, if

\[
\xi_u = \sum_{a \in \Delta_+ \bigcap \bigcup \Gamma_i} k_a \cdot \alpha,
\]

\( u \in U(\Gamma_i)_E \).

If \( u \not\in M_\phi \) then by 2) we may assume that \( r_\beta = t_\beta = 0 \) for all \( \beta \in \Delta_+ \bigcup \bigcup \Gamma_i \). Then applying induction on the degree of \( u \), we may reorder the terms of \( u \) into the required form.

4) For each \( i, M_\phi \cap U(\Gamma_i) \) is a maximal left ideal of \( U(\Gamma_i) \).

It is clear that \( M_\phi \cap U(\Gamma_i) \) is a left ideal of \( U(\Gamma_i) \) and since \( \ker \phi \cap U(\Gamma_i) \subset M_\phi \cap U(\Gamma_i) \) it remains only to show that for any \( u \in U(\Gamma_i) \) \( \ker \phi \cap U(\Gamma_i) \) where \( \eta \) is an integral linear combination of roots from \( \Delta_+ \bigcap \Gamma_i \), there exists \( v \in U(\Gamma_i)_{-\eta} \) such that \( \phi(\omega v) \neq 0 \). Since \( M_\phi \) is maximal in \( U \) there exists \( \omega \in U_{-\eta} \) with \( \phi(\omega u) \neq 0 \). If \( \omega_0 \) is a basis element of \( U \) of minimal degree such that \( \phi(\omega_0 u) \neq 0 \) then \( \omega_0 \in U(\Gamma_i)_{-\eta} \). In fact \( \omega_0 \) does not contain any factors of type \( h_a \) since in this case we have \( \omega_0 = w' h_a + \) lower degree terms and hence a contradiction:

\[
0 \neq \phi(\omega_0 u) = \phi(w' h_a u) = \phi(w' u) \phi(h_a) + \eta(h_a) \phi(w' h_a) = 0.
\]

We also know that \( \omega_0 \in U(\bigcup \Gamma_i) \) as otherwise \( w_0 u \in C(\bigcup \Gamma_i) \). Thus by 3) we have \( \omega_0 = \omega v + \) lower degree terms where \( \omega \in C(L) \) and \( v \in U(\Gamma_i)_{-\eta} \). By the minimality of the degree of \( \omega_0 \) we must have \( \omega \) is a non-zero scalar and hence \( \omega_0 \in U(\Gamma_i)_{-\eta} \), as required.

With the help of these observations we can now prove the following result:

Proposition 7. Let

\[
\xi = \sum_{a \in \Delta_+ \bigcup \Gamma_i} k_a \cdot \alpha \quad \text{and} \quad \xi_i = \sum_{a \in \Delta_+ \bigcup \Gamma_i} k_a \cdot \alpha
\]
where $k_a \in \mathbb{Z}$ for all $a$. Then $\dim (U/M_\phi)_\lambda \leq 1$ for $\lambda = (\phi + \xi) \downarrow H$ and moreover $\dim (U/M_\phi)_\lambda = 1$ if and only if

$$\dim (U(\Gamma_i)/(M_\phi \cap U(\Gamma_i)))_{\phi + \xi_i} = 1$$

for all $i = 1, 2, \ldots, l$.

**Proof.** Since for each $i$, $U(\Gamma_i) \cong U(A_n)$ and $\phi \downarrow (C(L) \cap U(\Gamma_i))$ is a standard algebra homomorphism, Proposition 6 implies that

$$\dim (U(\Gamma_i)/(M_\phi \cap U(\Gamma_i)))_{\phi + \xi_i} \leq 1$$

and gives explicit conditions when it is exactly 1.

Assume first that there exists $i_0$ such that

$$\dim (U(\Gamma_{i_0})/(M_\phi \cap U(\Gamma_{i_0})))_{\phi + \xi_{i_0}} = 0.$$ 

This implies that

$$U(\Gamma_{i_0})_{\xi_{i_0}} \subseteq M_\phi.$$ 

We claim that in this case $U_{\xi} \subseteq M_\phi$ and hence $\dim (U/M_\phi)_\lambda = 0$. In fact if $u \in U_\xi$ is a basis element we may assume by remark 3 that $u = u_1u_2 \ldots u_\xi$ where $z \in U(H)$ and $u_i \in U(\Gamma_i)$. Then

$$u = u_1u_2 \ldots u_\xi = u_1 \ldots \hat{u}_{i_0} \ldots u_\xi u_{i_0} + \xi_{i_0}(z)u_1 \ldots \hat{u}_{i_0} \ldots u_\xi u_{i_0} \in M_\phi.$$ 

That is, $U_\xi \subseteq M_\phi$ as required.

Assume now that for all $i = 1, 2, \ldots, l$ we have

$$\dim (U(\Gamma_i)/(M_\phi \cap U(\Gamma_i)))_{\phi + \xi_i} = 1$$

and hence there exists $g_i \in U(\Gamma_i)_{\xi_i}\backslash M_\phi$ such that for any $u_i \in U(\Gamma_i)_{\xi_i}$

$$u_i = g_i \ldots g_i \in U_\xi \backslash M_\phi.$$ 

Since $\{U(\Gamma_i), U(\Gamma_j)\} = \{0\}$ for $i \neq j$ we have that $g_1 \ldots g_i \in U_\xi \backslash M_\phi$ and for any $u \in U_\xi$, $u$ is a scalar multiple of $g_1 \ldots g_i$ modulo $M_\phi$. That is, $\dim (U/M_\phi)_\lambda = 1$.

We now make use of these descriptions of pointed representations to complete our labelling programme.

**Lemma.** If $\phi : C(L) \to C$ is an extreme $g$-standard algebra homomorphism relative to $\bigcup_i \Gamma_i$ then the set of weight functions of $U/M_\phi$ is contained in the set

$$\{\phi + \sum_{a \in \Delta_+} k_a \cdot \alpha | (\forall \alpha) k_\alpha \in \mathbb{Z}; \quad (\forall \alpha \in \Delta_+ \backslash \bigcup_i \Gamma_i) \quad k_\alpha \leq 0\}.$$ 

**Proof.** Set $\lambda = \phi + \sum_{a \in \Delta_+} k_a \cdot \alpha$ and $\xi = \sum k_\alpha \cdot \alpha$ where $(\forall \alpha) k_\alpha \in \mathbb{Z}$ and consider any basis element $u \in U_\phi$. If $k_\beta > 0$ for some $\beta \in \Delta_+ \backslash \bigcup_i \Gamma_i$ then there must exist some $\beta' \in \Delta_+ \cup \bigcup_i \Gamma_i$ such that $r_{\beta'} \neq 0$ in $u$ and hence by remark 2 we have $u \in M_\phi$. That is, $\dim (U/M_\phi)_\lambda = 0$. Thus in order for $\lambda$ to be a weight function of $U/M_\phi$ we must have $k_\alpha \leq 0$ for all $\alpha \in \Delta_+ \backslash \Gamma_i$.

**Proposition 8.** If $\phi_1, \phi_2 : C(L) \to C$ are two distinct extreme $g$-standard algebra homomorphisms then $U/M_{\phi_1} \cong U/M_{\phi_2}$ as $L$-modules.
Proof. Assume that $\phi_1$ and $\phi_2$ are as given and $U/M_{\phi_1} \cong U/M_{\phi_2}$. We claim that $\phi_1 = \phi_2$. Since equivalent representations have the same set of weight functions we must have that $\phi_1 \downarrow H$ is a weight function of $U/M_{\phi_2}$ and hence

$$\phi_1 \downarrow H = \phi_2 \downarrow H + \sum_{\alpha \in \Delta_{++}} l_{\alpha} \cdot \alpha$$

where $(\forall \alpha) l_{\alpha} \in \mathbb{Z}$. We also note that if $\phi_1$ is $g$-standard relative to $\bigcup \Gamma_i^{(1)}$ and $\phi_2$ is $g$-standard relative to $\bigcup \Gamma_i^{(2)}$ then $\bigcup \Gamma_i^{(1)} = \bigcup \Gamma_i^{(2)}$. Indeed if $\beta \in \bigcup \Gamma_i^{(1)}$ and $\beta \notin \bigcup \Gamma_i^{(2)}$ then $\phi_1 \downarrow H + l \cdot \beta$ is a weight function of $U/M_{\phi_1}$ and therefore of $U/M_{\phi_2}$ for all $l \in \mathbb{Z}$. But then

$$\phi_1 \downarrow H + l \beta = \phi_2 \downarrow H + \sum_{\alpha \in \Delta_{++}} l_{\alpha} \cdot \alpha + l \cdot \beta$$

is a weight function of $U/M_{\phi_2}$ for all $l \in \mathbb{Z}$ and since $\beta \notin \bigcup \Gamma_i^{(2)}$ this contradicts the lemma above.

Now fix any $\beta_0 \in \Delta_{++} \setminus \bigcup \Gamma_i^{(1)} = \Delta_{++} \setminus \bigcup \Gamma_i^{(2)}$ and note that

$$\phi_1 \downarrow H = \phi_2 \downarrow H + \sum_{\alpha \in \Delta_{++}} l_{\alpha} \cdot \alpha$$

is a weight function of $U/M_{\phi_2}$. Therefore by the above lemma $l_{\beta_0} \leq 0$. But we also have that

$$\phi_2 \downarrow H = \phi_1 \downarrow H + \sum_{\alpha \in \Delta_{++}} (-l_{\alpha}) \cdot \alpha$$

is a weight function of $U/M_{\phi_1}$ and again applying the lemma we have $-l_{\beta_0} \leq 0$. Therefore we have that $l_{\beta_0} = 0$ for all $\beta_0 \in \Delta_{++} \setminus \bigcup \Gamma_i^{(1)}$.

On the other hand assume $\beta_0 \in \bigcup \Gamma_i^{(1)} = \bigcup \Gamma_i^{(2)}$. Then by definition of extreme $g$-standard we have that $0 \leq \text{Re} \, \phi_i(h_{\beta_0}) < 1$ for $i = 1, 2$. But

$$\phi_1(h_{\beta_0}) = \phi_2(h_{\beta_0}) + l_{\beta_0}$$

and hence $l_{\beta_0} = 0$. Thus $\phi_1 \downarrow H = \phi_2 \downarrow H$ and since $U/M_{\phi_1} \cong U/M_{\phi_2}$ we have $\phi_1 = \phi_2$ as required.

It remains now only to show that for any $g$-standard algebra homomorphism $\phi : C(L) \twoheadrightarrow \mathbb{C}$ there exists an extreme $g$-standard $\tilde{\phi} : C(L) \to \mathbb{C}$ such that $U/M_{\phi} \cong U/M_{\tilde{\phi}}$ for some $\sigma \circ A(L)$. We proceed through a series of lemmas.

**Lemma 9a.** If $\phi : C(A_n) \to \mathbb{C}$ is a standard algebra homomorphism parametrized by $s \in \mathbb{C}$ and $\lambda \in H^*$ then

1) $\phi \circ \sigma_1$ is standard parametrized by $s - \lambda(h_{a_1}) \in \mathbb{C}$ and $\lambda \circ \sigma_1 \in H^*$.

2) $\phi \circ \sigma_i$ is standard parametrized by $s \in \mathbb{C}$ and $\lambda \circ \sigma_i \in H^*$ for $i = 2, 3, \ldots, n$.

3) If $\xi = \sum_{i=1}^n l_i \cdot \alpha_i$ where $l_i \in \mathbb{Z}$ and $(\phi + \xi) \downarrow H$ is a 1-dimensional weight function of $U/M_{\phi}$ then the algebra homomorphism $\phi' : C(A_n) \to \mathbb{C}$ associated with $(\phi + \xi) \downarrow H$ is standard parametrized by $s + l_1 \in \mathbb{C}$ and $(\phi + \xi) \downarrow H \in H^*$.
Proof. 1) Define two representations

\[(\rho, V_{r,\lambda})\] and \[(\rho', V_{r-\lambda(h_{a_1})}, \lambda \sigma_{a_1})\]

as in [7] where the underlying vector space is the same for both. Using the explicit description of these representations one can easily verify that

\[(\rho \cdot \sigma_{a_1}, V_{r,\lambda}) \cong (\rho', V_{r-\lambda(h_{a_1})}, \lambda \sigma_{a_1})\]

where the equivalence map is the identity. Then we have

\[\phi \circ \sigma_{a_1}(c)v(0) = \rho \circ \sigma_{a_1}(c)v(0) = \rho'(c)v(0) \quad (\forall c \in C(A_n)).\]

That is, \(\phi \circ \sigma_{a_1}\) is standard, parametrized by \(s - \lambda(h_{a_1}) \in C\) and \(y \circ \sigma_{a_1} \in H^*\).

2) This follows in the same manner as 1) on noting that for \(i \geq 2\)

\[(\rho \circ \sigma_{a_1}, V_{r,\lambda}) \cong (\rho', V_{r,\lambda \sigma_{a_1}})\]

where the equivalence map is the identity.

3) Recall from [7, Proposition 2] that the representations \((\rho, V_{r,\lambda})\) and \((\rho', V_{r,\lambda'})\) where \(\lambda' - \lambda = \sum_{i=1}^{n} l_i \cdot \alpha_i\) and \(t = s + l_1\) are equivalent and the equivalence map \(\psi : V_{r,\lambda} \rightarrow V_{r,\lambda'}\) is given by

\[\psi(v(k_1, \ldots, k_n)) = v(k_1 - l_1, \ldots, k_n - l_n).\]

By assumption we also have \(U/M_0 \cong U/M'\) and this equivalence can be realized by the map \(\Phi : U/M_0 \rightarrow U/M'\) where \(\Phi(1 + M_0) = u_0 + M'\), with \(u_0 \in U \setminus M_0\) where \(r = \sum_{i=1}^{n} l_i \alpha_i\).

We may also assume that \(u_0\) has been selected in such a way that

\[\rho'(u_0)v(-l_1, -l_2, \ldots, -l_n) = v(0).\]

In fact for any \(u \in U \setminus M_0\) we have

\[\rho'(u)v(-l_1, \ldots, -l_n) = \rho'(u)\psi(v(0)) = \psi(\rho(u)v(0))\]

and \(\rho(u)v(0)\) is a non-zero scalar multiple of \(v(l_1, \ldots, l_n)\) since \(u \notin M_0\). That is,

\[\rho'(u)v(-l_1, \ldots, -l_n) = Kv(0)\]

with \(K \neq 0\) and hence we may select \(u_0 = u/K\). Also since \(u_0 \notin M_0\) we can select an element \(\omega_0 \in U_r\) such that \(\phi(\omega_0u_0) = 1\). Now by Proposition 2 we have \(\phi'(c) = \phi(\omega_0cu_0)\) for all \(c \in C(A_n)\). Finally for all \(c \in C(A_n)\) we have

\[\rho'(c)v(0) = \rho'(c)\rho'(u_0)v(-l_1, \ldots, -l_n) = \rho'(cu_0)\psi(v(0))\]

\[= \psi \circ \rho(cu_0)v(0) = \rho(\omega_0cu_0)v(0) = \phi(\omega_0cu_0)v(0) = \rho'(c)v(0).\]

Thus \(\phi'\) is standard, parametrized by \(s + l_1 \in C\) and \((\phi + \xi) \downarrow H \in H^*\).
Lemma 9b. Assume \( \phi : C(A_n) \to \mathbf{C} \) is a standard algebra homomorphism, parametrized by \( s \in \mathbf{C} \) and \( \lambda \in H^* \) such that for some \( p = 0, 1, \ldots, n \),

\[
s - \lambda \left( \sum_{i=0}^{p} h_{\alpha_i} \right) \in \mathbf{Z}.
\]

Then there exists a \( g \)-standard algebra homomorphism \( \phi' : C(A_n) \to \mathbf{C} \) relative to the complete subset \( \Gamma' \) or \( \Gamma'' \) of \( \Delta \) generated by \( \{\alpha_1, \ldots, \alpha_{n-1}\} \) or \( \{\alpha_2, \ldots, \alpha_n\} \) such that \( U/M_{\phi'} \cong U/M_{\phi} \) for some \( \sigma \in A(A_n) \).

Proof. Let \( m \) denote the minimum integer, by absolute value, among the integers in the set

\[
\{s - \lambda \left( \sum_{i=0}^{p} h_{\alpha_i} \right) | p = 0, 1, \ldots, n\}.
\]

Assume first that

\[
m = s - \lambda \left( \sum_{i=0}^{p} h_{\alpha_i} \right) \leq 0.
\]

If \( r \neq 0 \) (ie. \( s \neq m \)) then applying parts 1) and 2) of Lemma 9a we have that if

\[
\sigma = \sigma_{x_1} \circ \ldots \circ \sigma_{x_l} \in A(A_n)
\]

then \( \phi \circ \sigma \) is a standard algebra homomorphism parametrized by \( s' \in \mathbf{C} \) and \( \lambda' \in H^* \) where

\[
s' = s - \lambda \left( \sum_{i=0}^{p} h_{\alpha_i} \right) = m.
\]

By Proposition 6, \( (\phi \circ \sigma - m \alpha_1) \downarrow H \) is a 1-dimensional weight space of \( U/M_{\phi \circ \sigma} \). Applying part 3) of Lemma 9a, the algebra homomorphism \( \phi' : C(A_n) \to \mathbf{C} \) associated with the 1-dimensional weight function \( (\phi \circ \sigma - m \alpha_1) \downarrow H \) is also standard parametrized by \( s'' \in \mathbf{C} \) and \( \lambda'' \in H^* \) where \( s'' = s' - m_1 = 0 \). It then follows that \( \phi' \downarrow \tilde{C}(\Gamma') \equiv 0 \). That is, \( \phi' \) is \( g \)-standard relative to \( \Gamma' \). Finally we also have \( U/M_{\phi_{\circ \sigma}} \cong U/M_{\phi \circ \sigma} \).

On the other hand, if we assume that \( m > 0 \) by a similar argument we can define an algebra homomorphism \( \phi' : C(A_n) \to \mathbf{C} \) which is \( g \)-standard relative to \( \Gamma'' \) and \( U/M_{\phi'} \cong U/M_{\phi} \) for some \( \sigma \in A(A_n) \).

Lemma 9c. Let \( \phi : C(L) \to \mathbf{C} \) be a \( g \)-standard algebra homomorphism relative to \( \bigcup_{i=1}^{l} \Gamma_i \). Then:

1) For any \( \alpha \in \Delta_{++} \cap \Gamma_{i_0} \) we have \( \phi \circ \sigma_{\alpha} \) is \( g \)-standard relative to \( \bigcup \Gamma_i \). More precisely we have \( \phi \circ \sigma_{\alpha} \equiv \phi \) on \( U(\Gamma_j) \cap C(L) \) for \( j \neq i_0 \) and \( \phi \circ \sigma_{\alpha} \equiv 0 \) on \( C(\bigcup \Gamma_i) \).

2) If

\[
\xi = \sum_{\alpha \in \Delta_{++} \cap \Gamma_{i_0}} l_{\alpha} \cdot \alpha
\]

with \( l_{\alpha} \in \mathbf{Z} \) for all \( \alpha \) such that \( (\phi + \xi) \downarrow H \) is a 1-dimensional weight function of \( U/M_{\phi \circ \sigma_{\alpha}} \) then the algebra homomorphism \( \phi' \) associated with \( (\phi + \xi) \downarrow H \) is \( g \)-standard relative to \( \bigcup \Gamma_i \). More precisely we have \( \phi' \equiv \phi \) on \( U(\Gamma_j) \cap C(L) \) for \( j \neq i_0 \) and \( \phi' \equiv 0 \) on \( C(\bigcup \Gamma_i) \).
Proof. 1) For any $j \neq i_0$ and $\beta \in \Delta \cap \Gamma_j$ we have $\sigma_\beta(\beta) = \beta$. That is, for any $c \in C(L) \cap U(\Gamma_j)$, $\sigma_\alpha(c) = c$. Hence $\phi \circ \sigma_\alpha(c) = \phi(c)$ for all $c \in C(L) \cap U(\Gamma_j)$.

For any $\beta \in \Delta \cup \Gamma_i$, $\sigma_\beta(\beta) \in \Delta \cup \Gamma_i$ and hence for any $c \in \bar{C}(\cup \Gamma_i)$, $\sigma_\alpha(c) \in \bar{C}(\cup \Gamma_i)$. Therefore $\phi \circ \sigma_\alpha(c) = 0$ for all $c \in \bar{C}(\cup \Gamma_i)$.

Finally $\phi \circ \sigma_\alpha \downarrow (C(L) \cap U(\Gamma_{i_0}))$ is standard by Lemma 9c and hence $\phi \circ \sigma_\alpha$ is $\mathfrak{g}$-standard relative to $\bigcup \Gamma_i$.

2) Take $u \in U(\Gamma_{i_0})_\Delta \setminus M_\delta$ and note that

$$\left( \forall c \in C(L) \right) \phi'(c)(u + M_\delta) = c(u + M_\delta)$$

for any $c \in C(L) \cap U(\Gamma_j)$ with $j \neq i_0$ we have

$$\phi'(c)(u + M_\delta) = c(u + M_\delta) = uc + M_\delta = \phi(c)(u + M_\delta).$$

Hence $\phi'(c) = \phi(c)$.

Also for any $c \in \bar{C}((\cup_{i=1}^l \Gamma_i))$ we note that $U_{-1} cu \subseteq \bar{C}(\cup \Gamma_i) \subseteq M_\delta$ and hence $cu \in M_\delta$. Therefore

$$\phi'(c)(u + M_\delta) = cu + M_\delta = 0(u + M_\delta).$$

Thus $\phi'(c) = 0$.

Finally $\phi' \downarrow (C(L) \cap U(\Gamma_{i_0}))$ is standard by Lemma 9a and hence $\phi'$ is $\mathfrak{g}$-standard relative to $\bigcup \Gamma_i$.

Combining these lemmas we now have the main result of this section.

**Proposition 9.** Let $\phi : C(L) \rightarrow \mathbb{C}$ be a $\mathfrak{g}$-standard algebra homomorphism relative to $\bigcup_{i=1}^l \Gamma_i$. Then there exists an extreme $\mathfrak{g}$-standard algebra homomorphism $\tilde{\phi} : C(L) \rightarrow \mathbb{C}$ such that $U/M_\delta \cong U/M_{\sigma_1}$ for some $\sigma_1 \in A(L)$.

**Proof.** We define the order of a $\mathfrak{g}$-standard algebra homomorphism relative to $\bigcup_{i=1}^l \Gamma_i$ to be $\sum_{i=1}^l \#(\Delta_{++} \cap \Gamma_i)$. Every order 0 $\mathfrak{g}$-standard algebra homomorphism is by definition extreme hence we assume inductively that the proposition is true for $\mathfrak{g}$-standard algebra homomorphisms of order $< N$. Then consider a $\mathfrak{g}$-standard algebra homomorphism $\phi : C(L) \rightarrow \mathbb{C}$ of order $N$.

If there exists $i_0 = 1, 2, \ldots, l$ such that $\phi \downarrow (C(L) \cap U(\Gamma_{i_0}))$ satisfies the conditions of Lemma 9b then by Lemmas 9b and 9c there exists $\sigma \in A(L)$ such that $\phi \circ \sigma$ is $\mathfrak{g}$-standard of order $N-1$ and $U/M_\delta \cong U/M_{\sigma\sigma_1}$ for some $\sigma_1 \in A(L)$. By the inductive hypothesis then there exists an extreme $\mathfrak{g}$-standard algebra homomorphism $\tilde{\phi} : C(L) \rightarrow \mathbb{C}$ such that $U/M_{\sigma\sigma_1} \cong U/M_{\tilde{\sigma}\sigma_1}$ for some $\tilde{\sigma} \in A(L)$. Hence by Proposition 2 $U/M_{\tilde{\sigma}\sigma_1} \cong U/M_{\sigma_1}$ as required.

We may therefore assume that

$$(\phi + \sum_{\alpha \in \Delta_{++} \cap (\cup \Gamma_i)} l_a \cdot \alpha) \downarrow H$$

is a 1-dimensional weight function of $U/M_\delta$ for all $l_a \in \mathbb{Z}$. Thus setting $k_\alpha = \lfloor \Re \phi(h_\alpha) \rfloor$ for all $\alpha \in \Delta_{++} \cap (\cup \Gamma_i)$ (where $\lfloor \cdot \rfloor$ denote the greatest integer function),

$$(\phi - \sum_{\alpha \in \Delta_{++} \cap (\cup \Gamma_i)} k_\alpha \cdot \alpha) \downarrow H$$

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is a 1-dimensional weight function of $U/M_\phi$. If $\tilde{\phi}$ is the associated algebra homomorphism then $U/M_\phi \cong U/M_{\tilde{\phi}}$, $\tilde{\phi}$ is $g$-standard by Lemma 9c and is extreme since $0 \leq \text{Re} (\phi(h_a) - k_a) - 1$.

References


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