NONIDENTIFIABILITY OF THE TWO-STATE MARKOVIAN ARRIVAL PROCESS

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Abstract

In this paper we consider the problem of identifiability for the two-state Markovian arrival process (MAP$_2$). In particular, we show that the MAP$_2$ is not identifiable, providing the conditions under which two different sets of parameters induce identical stationary laws for the observable process.

Keywords: Markovian arrival process; Markov-modulated Poisson process; phase-type renewal process; identifiability problem

2010 Mathematics Subject Classification: Primary 60G55
Secondary 60J05

1. Introduction

The *Markovian arrival process* (MAP) was defined in [16] and [17] as a generalization of the Poisson arrival process allowing for both dependence between arrivals and nonexponentially distributed interarrival times. The MAP is defined by two Markov processes: the first counts the number of arrivals and the second, an underlying Markov process, governs the state changes. At the end of a transition in a MAP an arrival may or may not occur and, although the transition holding times are exponentially distributed, the interarrival times do not, in general, follow an exponential distribution. Special cases of the MAP are phase-type renewal processes (which include both the Erlang and hyperexponential renewal processes) and nonrenewal processes such as the Markov-modulated Poisson process (MMPP). Stationary MAPs are dense in the family of all stationary point processes; see [1]. Another important property of MAPs is that the superposition of independent MAPs is again a MAP.

The MAP is a challenging process from both theoretical and applied points of view. From a theoretical perspective, the queueing system where the MAP governs the arrival process has been widely studied in the literature (see, for example, [15] and [19]). On the other hand, the MAP has been proposed in the literature as a suitable process for modeling teletraffic data; see, for example, [10], [11], [12], [18], and [25]. In this case, the MAP is used to fit data where only the interarrival times are observed and neither the underlying Markov chain nor the individual exponential holding times are available, and, thus, the observed arrival process is a hidden Markov process.

When dealing with inference for hidden Markov processes, it is very common to encounter *identifiability* problems which imply that the likelihood function does not possess a unique maximum. Identifiability conditions for general hidden Markov processes are studied in [13], [14], and [21]. Several works have considered the identifiability issue with regards to some types of MAP. For example, identifiability of the MMPP and phase-type distributions was undertaken in [23], with results derived from a *uniformization* technique in combination with the findings of [13]. Unlike the MMPP, which can be identified (up to permutations of states), it is known that the phase-type distributions are not identifiable since, for example, any two-state phase-type distribution can be represented as a two-state Coxian distribution, which is not identifiable. Also, Green [6] and Bean and Green [3] investigated when a MAP is a Poisson process. He and Zhang [7], [8], [9] used the so-called spectral polynomial algorithm to construct Coxian representations for phase-type distributions whose generators have only real eigenvalues. Finally, a new parameterization which provides an identifiable Coxian model can be found in [2]. In this paper we address the problem of identifiability of the general two-state MAP, or MAP$_2$. We prove the lack of identifiability of the MAP$_2$ using an approach that allows us to construct an equivalent MAP$_2$ for any given MAP$_2$.

The paper is organized as follows. In Section 2 we introduce the MAP and its main properties. In Section 3 we study when two MAP$_2$s have the same interarrival time distributions, a necessary condition for nonidentifiability. We call this property *weak equivalence*. In Section 4 we consider the joint distribution of a sequence of interarrival times generated from the MAP$_2$ and show that there are at least two different parameterizations of the MAP$_2$ giving rise to the same joint distribution, thus proving the nonidentifiability of the MAP$_2$. Finally, in Section 5 we provide conclusions and some possible extensions of this work.

2. The MAP and its main properties

Consider an irreducible, continuous, Markov process $J(t)$ with state space $\mathcal{S} = \{1, \ldots, m\}$ and generator matrix $\boldsymbol{D}$. Let $N(t)$ represent the cumulative number of arrivals in $(0, t]$. A MAP, represented by $\{J(t), N(t)\}$, behaves as follows: the initial state $i_0 \in \mathcal{S}$ is generated according to an initial probability vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ and at the end of an exponentially distributed sojourn time in state i, with mean $1/\lambda_i$, two possible state transitions can occur. Firstly, with probability $0 \leq p_{ij1} \leq 1$, a single arrival occurs and the MAP enters a state $j \in \mathcal{S}$, which may be the same as ($j = i$) or different from ($j \neq i$) the previous state. Secondly, with probability $0 \leq p_{ij0} \leq 1$, no arrival occurs and the MAP enters a different state, $j \neq i$. Given that from all states a transition must occur to a different state without an arrival or to any state with an arrival, then, for $1 \leq i \leq m$,

$$\sum_{j=1, \, j \neq i}^{m} p_{ij0} + \sum_{j=1}^{m} p_{ij1} = 1.$$

When $m = 2$, we have a two-state MAP, denoted by MAP$_2$. Figure 1 illustrates the different transitions that can occur in this process by means of a transition diagram.

Define the matrices $\boldsymbol{P}_0 = (p_{ij0})_{i,j \in \mathcal{S}}$ and $\boldsymbol{P}_1 = (p_{ij1})_{i,j \in \mathcal{S}}$, where $p_{ii0} = 0$. Then the MAP is defined by the set $\{\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{P}_0, \boldsymbol{P}_1\}$, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$. Alternatively, the MAP can be characterized by the rate matrices, $\boldsymbol{D}_0 = (d_{ij0})_{i,j \in \mathcal{S}}$ and $\boldsymbol{D}_1 = (d_{ij1})_{i,j \in \mathcal{S}}$, where $d_{ii0} = -\lambda_i$, $d_{ij0} = \lambda_i p_{ij0}$ for $j \neq i$, and $d_{ij1} = \lambda_i p_{ij1}$ for $1 \leq i, j \leq m$. This definition implies that $\boldsymbol{D} \equiv \boldsymbol{D}_0 + \boldsymbol{D}_1$ is the infinitesimal generator of the underlying Markov process. Intuitively, $\boldsymbol{D}_0$ can be thought of as governing transitions that do not generate arrivals and $\boldsymbol{D}_1$ can be thought
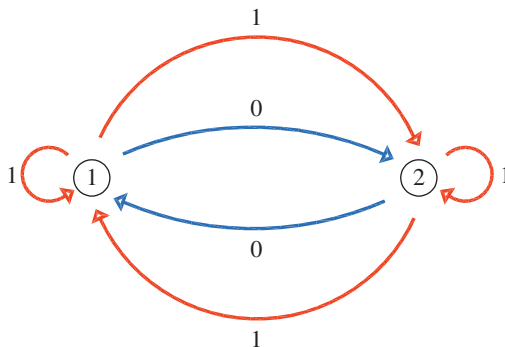
FIGURE 1: Transition diagram for the MAP$_2$. Here 0 and 1 illustrate moves without and with arrivals, respectively.

of as governing transitions that do generate arrivals. The stationary probability vector of the Markov process with generator $\boldsymbol{D}$ is $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$, which satisfies

$$\boldsymbol{\pi} \boldsymbol{D} = \boldsymbol{0}, \qquad \boldsymbol{\pi} \boldsymbol{e} = 1,$$

where $\boldsymbol{e}$ is an $m \times 1$ column vector of 1s. Thus, π_j represents the stationary probability that the process is in state j, for $j = 1, \ldots, m$. The MMPP can be defined as a simplified MAP where the matrix $\boldsymbol{D}_1$ (and thus, $\boldsymbol{P}_1$) is diagonal (see [15]). This implies that arrivals can only occur in transitions to the same state.

Some important properties of the MAP are as follows. Firstly, it is known (see [4]) that the MAP can be regarded as a Markov renewal process. Let X_n be the state of the MAP at the time of the nth arrival, and let T_n be the time between the $(n-1)$th and nth arrival. Then $\{X_{n-1}, T_n\}_{n=1}^{\infty}$ is a Markov renewal process with semi-Markovian kernel given by

$$\int_0^t e^{\boldsymbol{D}_0 t} \boldsymbol{D}_1 \, dt = (\boldsymbol{I} - e^{\boldsymbol{D}_0 t})(-\boldsymbol{D}_0)^{-1} \boldsymbol{D}_1. \tag{2.1}$$

Therefore, $\{X_n\}_{n=1}^{\infty}$ is a Markov chain, where from (2.1) the transition matrix can be computed as

$$\boldsymbol{P}^\star = (\boldsymbol{I} - \boldsymbol{P}_0)^{-1} \boldsymbol{P}_1. \tag{2.2}$$

Also, it can be shown that the stationary distribution, $\boldsymbol{\phi}$, is given by

$$\boldsymbol{\phi} = (\boldsymbol{\pi} \boldsymbol{D}_1 \boldsymbol{e})^{-1} \boldsymbol{\pi} \boldsymbol{D}_1. \tag{2.3}$$

See Appendix A for the proof.

Secondly, let the random variable T_1 denote the time to the first arrival in a MAP. Then, from [4], the cumulative distribution function (CDF) of T_1 is given by

$$F_{T_1}(t) = \boldsymbol{\theta}(\boldsymbol{I} - e^{\boldsymbol{D}_0 t})(-\boldsymbol{D}_0)^{-1} \boldsymbol{D}_1 \boldsymbol{e} \quad \text{for } t \geq 0.$$

If T represents the stationary interarrival time distribution, it can be shown that

$$F_T(t) = \boldsymbol{\phi}(\boldsymbol{I} - e^{\boldsymbol{D}_0 t})(-\boldsymbol{D}_0)^{-1} \boldsymbol{D}_1 \boldsymbol{e} \quad \text{for } t \geq 0.$$

Finally, the Laplace–Stieltjes transform of the first n interarrival times of a stationary MAP is given by

$$f^*_{T;\boldsymbol{D}_0,\boldsymbol{D}_1}(s_1,\ldots,s_n) = \boldsymbol{\phi}(s_1\boldsymbol{I} - \boldsymbol{D}_0)^{-1}\boldsymbol{D}_1\cdots(s_n\boldsymbol{I} - \boldsymbol{D}_0)^{-1}\boldsymbol{D}_1\boldsymbol{e},$$

or, equivalently,

$$f^*_{T;\boldsymbol{D}_0,\boldsymbol{D}_1}(s_1,\ldots,s_n) = \boldsymbol{\phi}\prod_{i=1}^{n}\boldsymbol{\Delta}(s_i)\boldsymbol{e}, \tag{2.4}$$

where

$$\boldsymbol{\Delta}(s) = (s\boldsymbol{I} - \boldsymbol{D}_0)^{-1}\boldsymbol{D}_1. \tag{2.5}$$

For a more detailed description and further properties of the MAP (and batch MAP (BMAP)), see, for example, [4] or [15].

3. Weak equivalence

There have been a number of examples of fitting MMPPs to Internet data traces. In most applications, the two-state case has been considered (see, for example, [5], [22], or [24]). The MMPPs, despite being simplified MAPs (the matrix $\boldsymbol{P}_1$ is diagonal, and, thus, they are characterized by two fewer parameters), are complex processes and usually two states, at most three, are enough to capture the data behavior.

From now on we consider the two-state MAP, or MAP$_2$, characterized by $\mathcal{M} \equiv \{\boldsymbol{\theta}, \boldsymbol{D}_0, \boldsymbol{D}_1\}$, where

$$\boldsymbol{\theta} = (\theta, 1 - \theta), \qquad \boldsymbol{D}_0 = \begin{pmatrix} x & y \\ z & u \end{pmatrix}, \qquad \boldsymbol{D}_1 = \begin{pmatrix} w & -x - y - w \\ v & -z - u - v \end{pmatrix}, \tag{3.1}$$

and

$$\begin{aligned} x &= -\lambda_1, & y &= \lambda_1 p_{120}, & w &= \lambda_1 p_{111}, \\ z &= \lambda_2 p_{210}, & u &= -\lambda_2, & v &= \lambda_2 p_{211}. \end{aligned}$$

The stationary probability distribution is $\boldsymbol{\phi} = (\phi, 1 - \phi)$, where

$$\phi = \frac{wz - vx}{wz - vx - zy - vy + xu + wu}. \tag{3.2}$$

When modeling real data, usually only the times between arrivals are observed, and, thus, the interest when making an inference for the MAP is focused on the embedded Markov renewal process $\{X_{n-1}, T_n\}_{n=1}^{\infty}$. As a preliminary step to studying the identifiability of the MAP$_2$, we study the conditions under which two MAP$_2$s possess the same marginal interarrival time distributions. For two such MAP$_2$s, we will say that they are weakly equivalent.

Definition 3.1. Let $\mathcal{M}$ represent a MAP$_2$ with parameters $\{\boldsymbol{\theta}, \boldsymbol{D}_0, \boldsymbol{D}_1\}$ as in (3.1). Then we say that another MAP$_2$, $\tilde{\mathcal{M}} \equiv \{\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{D}}_0, \tilde{\boldsymbol{D}}_1\}$ is *weakly equivalent* to $\mathcal{M}$ if and only if

$$T_n \overset{\mathrm{D}}{=} \tilde{T}_n \quad \text{for all } n \geq 1, \tag{3.3}$$

where T_n and $\tilde{T}_n$ represent the times between the $(n-1)$th and nth arrivals under both models, and '$\overset{\mathrm{D}}{=}$' denotes equality in distribution.

The term *weak* is employed because equivalence is expressed in a marginal sense. Since the interarrival times in a MAP$_2$ are not independent, condition (3.3) is necessary but insufficient for nonidentifiability. A general condition looking at the joint density of the sequence of interarrival times is analyzed in Section 4.

Given a known MAP$_2$, $\mathcal{M}$ as in (3.1), define the constant

$$c = z + u - x - y. \tag{3.4}$$

Note that if $c = 0$ then the rate until an arrival occurs from state 1 coincides with that of state 2. This implies that the observable process (that is, where arrivals occur) behaves like a Poisson process, with a single arrival rate. Thus, we will assume that $c \neq 0$. In addition, define the matrix $\boldsymbol{\Phi}$ as that whose rows are composed of the stationary vector $\boldsymbol{\phi}$. Suppose that $\boldsymbol{P}^\star = \boldsymbol{\Phi}$. Then it is immediately clear that there are many weakly equivalent MAP$_2$s, for example, any $\tilde{\mathcal{M}} = \{\tilde{\boldsymbol{\theta}}, \boldsymbol{D}_0, \boldsymbol{D}_1\}$ is equivalent to $\mathcal{M} = \{\boldsymbol{\theta}, \boldsymbol{D}_0, \boldsymbol{D}_1\}$. We can thus also assume that $\boldsymbol{P}^\star \neq \boldsymbol{\Phi}$.

Theorem 3.1 gives the conditions for $\tilde{\mathcal{M}}$ to be weakly equivalent to $\mathcal{M}$.

Theorem 3.1. *Let $\mathcal{M}$ be a MAP$_2$ as in (3.1) with stationary distribution $\boldsymbol{\phi}$, and let $\tilde{\mathcal{M}}$ be another MAP$_2$ with stationary distribution $\tilde{\boldsymbol{\phi}}$. Let T and $\tilde{T}$ represent the interarrival times in the stationary versions of $\mathcal{M}$ and $\tilde{\mathcal{M}}$. Assume that*

(A1) $c \neq 0$ *and* $\tilde{c} \neq 0$;

(A2) $\boldsymbol{P}^\star \neq \boldsymbol{\Phi}$ *or* $\tilde{\boldsymbol{P}}^\star \neq \tilde{\boldsymbol{\Phi}}$.

Then, $\tilde{\mathcal{M}}$ is weakly equivalent to $\mathcal{M}$ if and only if

(C1) $T \overset{\mathrm{D}}{=} \tilde{T}$; *and*

(C2) $(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}) = (\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}})$.

As will be shown in the proof of Theorem 3.1 (see Appendix B), (C1) is equivalent to

$$\frac{\alpha s + \gamma}{s^2 + \beta s + \gamma} = \frac{\tilde{\alpha} s + \tilde{\gamma}}{s^2 + \tilde{\beta} s + \tilde{\gamma}} \quad \text{for all } s, \tag{3.5}$$

where the coefficients α, β, and γ are given by

$$\alpha = \theta(z + u - x - y) - (z + u), \qquad \beta = -x - u, \qquad \gamma = xu - yz;$$

similarly define $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$.

Theorem 3.1 thus provides a straightforward way to check if two given MAP$_2$s share the same interarrival distribution.

Example 3.1. As an example, let us consider the MAP$_2$ defined by

$$\boldsymbol{D}_0 = \begin{pmatrix} -20 & 6 \\ 0.15 & -0.5 \end{pmatrix}, \qquad \boldsymbol{D}_1 = \begin{pmatrix} 12.228 & 1.772 \\ 0.0426 & 0.3074 \end{pmatrix},$$

or, alternatively,

$$\boldsymbol{P}_0 = \begin{pmatrix} 0 & 0.3 \\ 0.3 & 0 \end{pmatrix}, \qquad \boldsymbol{P}_1 = \begin{pmatrix} 0.6114 & 0.0886 \\ 0.0852 & 0.6148 \end{pmatrix},$$

and exponential rates $(\lambda_1, \lambda_2) = (20, 0.5)$. Suppose that the initial probability vector is equal to the stationary distribution, $\boldsymbol{\theta} = \boldsymbol{\phi} = (0.496, 0.504)$. The transition probability matrix, $\boldsymbol{P}^\star$, is computed from (2.2):

$$\boldsymbol{P}^\star = \begin{pmatrix} 0.7 & 0.3 \\ 0.2952 & 0.7048 \end{pmatrix} \neq \boldsymbol{\Phi}.$$

Consider another MAP$_2$ with parameters

$$\tilde{\boldsymbol{D}}_0 = \begin{pmatrix} -19.7 & 10.835 \\ 0.6146 & -0.8 \end{pmatrix}, \qquad \tilde{\boldsymbol{D}}_1 = \begin{pmatrix} 7.1452 & 1.7198 \\ 0.1443 & 0.0411 \end{pmatrix},$$

or, alternatively,

$$\tilde{\boldsymbol{P}}_0 = \begin{pmatrix} 0 & 0.5500 \\ 0.7682 & 0 \end{pmatrix}, \qquad \tilde{\boldsymbol{P}}_1 = \begin{pmatrix} 0.3627 & 0.0873 \\ 0.1804 & 0.0514 \end{pmatrix},$$

and exponential rates $(\tilde{\lambda}_1, \tilde{\lambda}_2) = (19.7, 0.8)$. Assume that $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\phi}} = (0.799, 0.201)$. The transition probability matrix, $\tilde{\boldsymbol{P}}^\star$, is

$$\tilde{\boldsymbol{P}}^\star = \begin{pmatrix} 0.8 & 0.2 \\ 0.79 & 0.21 \end{pmatrix} \neq \boldsymbol{\Phi}.$$

It can be seen that $c = 13.65 \neq 0$ and $\tilde{c} = -8.6796 \neq 0$, and that (3.5) holds. Therefore, from Theorem 3.1, the processes are weakly equivalent, as shown in Figure 2, which depicts the CDF of the time between two arrivals in the stationary version for both MAP$_2$s.

More results similar to Theorem 3.1, when assumptions (A1) and (A2) are relaxed (and, thus, the number of weakly equivalent MAP$_2$s to a fixed MAP$_2$ increases), and extensions to the three-state case MAP can be found in [20].
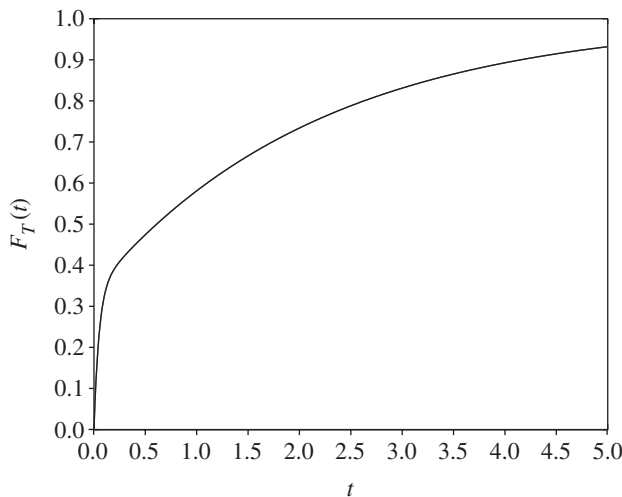

FIGURE 2: CDF of T, the time until the next arrival in the stationary version, in Example 1. As $\boldsymbol{\theta} = \boldsymbol{\phi}$, then $T \overset{\mathrm{D}}{=} T_1$ (similarly, $\tilde{T} \overset{\mathrm{D}}{=} \tilde{T}_1$), and, thus, $T \overset{\mathrm{D}}{=} T_1 \overset{\mathrm{D}}{=} \tilde{T} \overset{\mathrm{D}}{=} \tilde{T}_1$.

4. Nonidentifiability of the MAP$_2$

In this section we prove that the MAP$_2$ is a nonidentifiable process. Following Theorem 3.1, all MAP$_2$s will be assumed to be stationary, and $\mathcal{M}$ and $\tilde{\mathcal{M}}$ will denote the sets $\{\boldsymbol{\phi}, \boldsymbol{D}_0, \boldsymbol{D}_1\}$ and $\{\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{D}}_0, \tilde{\boldsymbol{D}}_1\}$, respectively, from now on. Our definition of nonidentifiability follows [23].

Definition 4.1. The MAP$_2$ is a nonidentifiable process if, for any fixed MAP$_2$, $\mathcal{M}$, there exists another MAP$_2$, $\tilde{\mathcal{M}}$, such that

$$(T_1, \ldots, T_n) \stackrel{\mathrm{D}}{=} (\tilde{T}_1, \ldots, \tilde{T}_n) \quad \text{for all } n \geq 1. \tag{4.1}$$

Note that condition (4.1) is equivalent to the equality of the Laplace transforms,

$$f^*_{T; \boldsymbol{D}_0, \boldsymbol{D}_1}(s_1, \ldots, s_n) = f^*_{\tilde{T}; \tilde{\boldsymbol{D}}_0, \tilde{\boldsymbol{D}}_1}(s_1, \ldots, s_n) \tag{4.2}$$

for all n, s. We will show that, given a MAP$_2$, $\mathcal{M}$, as in (3.1), there always exists a differently parameterized $\tilde{\mathcal{M}}$ such that (4.2) holds for all n, s. Indeed, we will prove that if (4.2) holds for $n = 1, 2$ then it will hold for all n.

Let us first consider the following result that gives the conditions under which (4.2) holds for $n = 1, 2$.

Proposition 4.1. *Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be two MAP$_2$s. Let α, β, γ, δ_1, δ_2 be defined as*

$\alpha = \phi(z + u - x - y) - (z + u),$

$\beta = -x - u,$

$\gamma = xu - yz,$

$\delta_1 = \phi((z + u - x - y)(w - v) + (x + y)(z + u) - (z + u)^2) + (z + u - x - y)v$
        $+ (z + u)^2,$

$\delta_2 = \phi(x + y - z - u)(uw - yv - xv + zw) + (x + y - z - u)(xv - zw) - (u + z)\gamma;$

similarly define $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}_1$, and $\tilde{\delta}_2$. Then, if

$$\tilde{\alpha} = \alpha, \qquad \tilde{\beta} = \beta, \qquad \tilde{\gamma} = \gamma, \qquad \tilde{\delta}_1 = \delta_1, \qquad \tilde{\delta}_2 = \delta_2, \tag{4.3}$$

the equality of Laplace transforms, (4.2), holds for all s and $n = 1, 2$.

Proof. See Appendix C.

The following result gives the solutions to (4.3).

Proposition 4.2. *Consider a MAP$_2$ as in (3.1). For all $\tilde{u} < 0$ and all $\tilde{z} > 0$, let $\tilde{x}(\tilde{u}, \tilde{z})$, $\tilde{y}(\tilde{u}, \tilde{z})$, $\tilde{v}(\tilde{u}, \tilde{z})$, and $\tilde{w}(\tilde{u}, \tilde{z})$ be defined as*

$$\tilde{x}(\tilde{u}, \tilde{z}) = -\tilde{u} + x + u, \tag{4.4}$$

$$\tilde{y}(\tilde{u}, \tilde{z}) = -\frac{1}{\tilde{z}}(\tilde{u}^2 - \tilde{u}x - \tilde{u}u + xu - zy), \tag{4.5}$$

$$\tilde{v}(\tilde{u}, \tilde{z}) = \frac{-\tilde{z}(vx + vy - wz - wu + w\tilde{z} - z\tilde{z} - z\tilde{u})}{-\tilde{u}u - u\tilde{z} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{z} + \tilde{z}^2 - \tilde{u}x - \tilde{z}x - zy}$$
$$+ \frac{-\tilde{z}(-\tilde{u}u + \tilde{z}^2 + 2\tilde{u}\tilde{z} - u\tilde{z} - v\tilde{z} - \tilde{u}v + w\tilde{u} + \tilde{u}^2)}{-\tilde{u}u - u\tilde{z} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{z} + \tilde{z}^2 - \tilde{u}x - \tilde{z}x - zy}, \tag{4.6}$$

$$\tilde{w}(\tilde{u}, \tilde{z}) = \frac{z\tilde{u}x + 2\tilde{z}xu - \tilde{z}zy - xu^2 + zx\tilde{z} + zu\tilde{z} + \tilde{u}u^2 + uzy}{-\tilde{u}u - u\tilde{z} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{z} + \tilde{z}^2 - \tilde{u}x - \tilde{z}x - zy}$$
$$+ \frac{-zxu - 2\tilde{z}\tilde{u}x + z^2y + u\tilde{z}v - wzy + zvy - \tilde{z}^2u - 3\tilde{u}\tilde{z}u - \tilde{u}\tilde{z}v}{-\tilde{u}u - u\tilde{z} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{z} + \tilde{z}^2 - \tilde{u}x - \tilde{z}x - zy}$$
$$+ \frac{v\tilde{u}x - \tilde{z}vy + 2\tilde{u}xu + v\tilde{u}u - w\tilde{u}x - wu\tilde{u} + 2\tilde{u}^2\tilde{z} + z\tilde{u}u + wux}{-\tilde{u}u - u\tilde{z} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{z} + \tilde{z}^2 - \tilde{u}x - \tilde{z}x - zy}$$
$$+ \frac{\tilde{z}^2\tilde{u} - v\tilde{u}^2 - \tilde{u}^2x + \tilde{z}wz + \tilde{u}\tilde{z}w - \tilde{u}\tilde{z}z - vxu + w\tilde{u}^2 - z\tilde{u}^2 + \tilde{u}^3}{-\tilde{u}u - u\tilde{z} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{z} + \tilde{z}^2 - \tilde{u}x - \tilde{z}x - zy}$$
$$+ \frac{-zy\tilde{u} + \tilde{z}u^2 - \tilde{z}^2x - 2\tilde{u}^2u - \tilde{z}wx}{-\tilde{u}u - u\tilde{z} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{z} + \tilde{z}^2 - \tilde{u}x - \tilde{z}x - zy}. \tag{4.7}$$

Then, the set $\{\tilde{u}, \tilde{z}, \tilde{x}(\tilde{u}, \tilde{z}), \tilde{y}(\tilde{u}, \tilde{z}), \tilde{v}(\tilde{u}, \tilde{z}), \tilde{w}(\tilde{u}, \tilde{z})\}$ solves the system of equations given by (4.3).

The proof of Proposition 4.2 is tedious but straightforward, involving solving the system of equations (4.3) by conventional methods, and substituting ϕ and $\tilde{\phi}$ from their definitions (see (3.2)). Although Proposition 4.2 gives an infinite number of solutions to the system of equations (4.3), a priori, the values of $\tilde{x}$, $\tilde{y}$, $\tilde{v}$, and $\tilde{w}$ may not define a MAP$_2$. The following theorem details how to select feasible values of $\tilde{u}$ and $\tilde{z}$ in the sense that $\tilde{x} < 0$, $\tilde{y} > 0$, $\tilde{v} > 0$, $\tilde{w} > 0$, $-\tilde{x} - \tilde{y} - \tilde{w} > 0$, $-\tilde{z} - \tilde{u} - \tilde{v} > 0$, and $\tilde{\phi} \in [0, 1]$, that is, it provides a way to choose $\tilde{u}$ and $\tilde{z}$ so that $\tilde{\mathcal{M}}$ is equivalent to $\mathcal{M}$.

Theorem 4.1. *Consider a MAP$_2$, $\mathcal{M}$, as in (3.1), and define*

$$\varepsilon_1 = -x,$$
$$\varepsilon_2 = \frac{u - x}{2},$$
$$\varepsilon_3 = \frac{z(1 - \phi)}{\phi},$$
$$\varepsilon_4 = \frac{(u - x) + \sqrt{(x - u)^2 + 4zy}}{2},$$
$$\varepsilon_5 = -\frac{z}{v}(z + u + v),$$
$$\varepsilon_6 = -\frac{z}{2v}[(u + v + z + w) - \sqrt{(u + v + z + w)^2 + 4v(-w - y - x)}].$$

Let ε be chosen from

$$0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} \quad \text{if } x < u, \tag{4.8}$$
$$0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} \quad \text{if } x = u, \tag{4.9}$$

and set $\tilde{u} \equiv u - \varepsilon$ and $\tilde{z} \equiv z + \varepsilon$. Then there exist an infinite number of MAP$_2$s, $\tilde{\mathcal{M}}$, given by $\mathcal{F} = \{\tilde{u}, \tilde{z}, \tilde{x}(\tilde{u}, \tilde{z}), \tilde{y}(\tilde{u}, \tilde{z}), \tilde{v}(\tilde{u}, \tilde{z}), \tilde{w}(\tilde{u}, \tilde{z})\}$, where $\tilde{x}(\tilde{u}, \tilde{z})$, $\tilde{y}(\tilde{u}, \tilde{z})$, $\tilde{v}(\tilde{u}, \tilde{z})$, and $\tilde{w}(\tilde{u}, \tilde{z})$ are defined by (4.4)–(4.7), such that (4.3) holds.

Note that the case $x > u$ has not been considered in Theorem 4.1, since a MAP$_2$ defined by $\{x, y, z, u, w, v\}$ with $x > u$ is equivalent to the MAP$_2$ $\{x', y', z', u', w', v'\} = \{u, z, y, x, -z - u - v, -x - y - w\}$ with $x' < u'$, or, equivalently, the MAP$_2$ can be parametrized by replacing state 1 by state 2.

Proof of Theorem 4.1. See Appendix D.

As a consequence of Theorem 4.1, the following two corollaries can be derived.

Corollary 4.1. *Consider a MAP$_2$, $\mathcal{M}$, as in (3.1), and values $\tilde{u}$, $\tilde{z}$, $\tilde{x}(\tilde{u}, \tilde{z})$, $\tilde{y}(\tilde{u}, \tilde{z})$, $\tilde{v}(\tilde{u}, \tilde{z})$, and $\tilde{w}(\tilde{u}, \tilde{z})$ as in Proposition 4.2 characterizing another MAP$_2$, $\tilde{\mathcal{M}}$. Let $\mathbf{\Delta}(s)$ be defined as in (2.5) (define $\tilde{\mathbf{\Delta}}(s)$ similarly). Then,*

$$\boldsymbol{\phi}\mathbf{\Delta}(s_1)\boldsymbol{e} = \tilde{\boldsymbol{\phi}}\tilde{\mathbf{\Delta}}(s_1)\boldsymbol{e}, \tag{4.10}$$

$$\boldsymbol{\phi}\mathbf{\Delta}(s_1)\mathbf{\Delta}(s_2)\boldsymbol{e} = \tilde{\boldsymbol{\phi}}\tilde{\mathbf{\Delta}}(s_1)\tilde{\mathbf{\Delta}}(s_2)\boldsymbol{e}, \tag{4.11}$$

$$(0, 1)\mathbf{\Delta}(s_1)\boldsymbol{e} = (0, 1)\tilde{\mathbf{\Delta}}(s_1)\boldsymbol{e}, \tag{4.12}$$

$$(0, 1)\mathbf{\Delta}(s_1)\mathbf{\Delta}(s_2)\boldsymbol{e} = (0, 1)\tilde{\mathbf{\Delta}}(s_1)\tilde{\mathbf{\Delta}}(s_2)\boldsymbol{e}, \tag{4.13}$$

for all s_1, s_2.

From (2.4), expressions (4.10)–(4.11) are an alternative way to state that (4.2) holds for $n = 1$ and $n = 2$. By substituting the values of $\mathcal{F}$ found in Theorem 4.1 in the expression for $\tilde{\mathbf{\Delta}}(s)$ (see (2.5)), routine but tedious calculations yield (4.12)–(4.13).

Corollary 4.2. *Consider a MAP$_2$, $\mathcal{M}$, as in (3.1), and values $\tilde{u}$, $\tilde{z}$, $\tilde{x}(\tilde{u}, \tilde{z})$, $\tilde{y}(\tilde{u}, \tilde{z})$, $\tilde{v}(\tilde{u}, \tilde{z})$, and $\tilde{w}(\tilde{u}, \tilde{z})$ as in Proposition 4.2 characterizing another MAP$_2$, $\tilde{\mathcal{M}}$. Let $\mathbf{\Delta}(s)$ be defined as in (2.5) (define $\tilde{\mathbf{\Delta}}(s)$ similarly). If $\mathbf{\Delta}(s)$ is given by*

$$\mathbf{\Delta}(s) = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}$$

then the solution to (4.10)–(4.13) is

$$\tilde{\mathbf{\Delta}}(s) = \begin{pmatrix} \tilde{a}(s) & \tilde{b}(s) \\ \tilde{c}(s) & \tilde{d}(s) \end{pmatrix}, \tag{4.14}$$

where

$$\tilde{a}(s) = \frac{\phi(a(s) - c(s)) + \tilde{\phi}c(s)}{\phi}, \tag{4.15}$$

$$\tilde{b}(s) = \frac{\phi\tilde{\phi}(d(s) + 2c(s) - a(s)) + \phi^2(a(s) - d(s) + b(s) - c(s)) - \tilde{\phi}^2 c(s)}{\phi\tilde{\phi}}, \tag{4.16}$$

$$\tilde{c}(s) = \frac{\tilde{\phi}c(s)}{\phi}, \tag{4.17}$$

$$\tilde{d}(s) = \frac{\phi(c(s) + d(s)) - \tilde{\phi}c(s)}{\phi}. \tag{4.18}$$

Equations (4.10)–(4.13) form a system with four equations, where the unknowns are the elements of (4.14). A trivial verification shows that (4.15)–(4.18) solves the system. The previous corollary motivates the following definition.

Definition 4.2. Let $\boldsymbol{G}$ and $\tilde{\boldsymbol{G}}$ be 2×2 matrices, where

$$\boldsymbol{G} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{G}} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}.$$

It will be said that $\tilde{G}$ is related to G, given the values ϕ and $\tilde{\phi}$, if and only if

$$\tilde{a} = \frac{\phi(a - c) + \tilde{\phi}c}{\phi}, \tag{4.19}$$

$$\tilde{b} = \frac{\phi\tilde{\phi}(d + 2c - a) + \phi^2(a - d + b - c) - \tilde{\phi}^2 c}{\phi\tilde{\phi}}, \tag{4.20}$$

$$\tilde{c} = \frac{\tilde{\phi}c}{\phi}, \tag{4.21}$$

$$\tilde{d} = \frac{\phi(c + d) - \tilde{\phi}c}{\phi}. \tag{4.22}$$

This relation will be denoted by $G \overset{\phi,\tilde{\phi}}{\sim} \tilde{G}$.

The following result is a direct consequence of the definition of $\tilde{G}$, (4.19)–(4.22).

Proposition 4.3. *If* $G \overset{\phi,\tilde{\phi}}{\sim} \tilde{G}$ *then*

$$\boldsymbol{\phi} G e = \tilde{\boldsymbol{\phi}} \tilde{G} e,$$

where $\boldsymbol{\phi} = (\phi, 1 - \phi)$ *and* $\tilde{\boldsymbol{\phi}} = (\tilde{\phi}, 1 - \tilde{\phi})$.

The proof follows straightforwardly from the definition of $\tilde{G}$, (4.19)–(4.22).

The next result, whose proof can be found in Appendix E, is crucial for proving the nonidentifiability of the MAP$_2$.

Proposition 4.4. *If* $G \overset{\phi,\tilde{\phi}}{\sim} \tilde{G}$ *and* $H \overset{\phi,\tilde{\phi}}{\sim} \tilde{H}$, *then*

$$GH \overset{\phi,\tilde{\phi}}{\sim} \tilde{G}\tilde{H}.$$

Finally, we can prove the general theorem.

Theorem 4.2. *The MAP$_2$ is not an identifiable process.*

Proof. The proof is based on the fact that, given a MAP$_2$, $\mathcal{M}$, as in (3.1), any other MAP$_2$, $\tilde{\mathcal{M}}$, chosen from the set $\mathcal{F}$ (see Theorem 4.1) satisfies equality (4.2) for all n.

If $\tilde{\mathcal{M}}$ is selected from $\mathcal{F}$ then, from Corollary 4.1,

$$\boldsymbol{\Delta}(s_1) \overset{\phi,\tilde{\phi}}{\sim} \tilde{\boldsymbol{\Delta}}(s_1) \quad \text{and} \quad \boldsymbol{\Delta}(s_1)\boldsymbol{\Delta}(s_2) \overset{\phi,\tilde{\phi}}{\sim} \tilde{\boldsymbol{\Delta}}(s_1)\tilde{\boldsymbol{\Delta}}(s_2) \quad \text{for all } s_1, s_2.$$

We conclude from Proposition 4.4 that

$$\boldsymbol{\Delta}(s_1)\boldsymbol{\Delta}(s_2)\boldsymbol{\Delta}(s_3) \overset{\phi,\tilde{\phi}}{\sim} \tilde{\boldsymbol{\Delta}}(s_1)\tilde{\boldsymbol{\Delta}}(s_2)\tilde{\boldsymbol{\Delta}}(s_3) \quad \text{for all } s_1, s_2, s_3,$$

and, finally,

$$\prod_{i=1}^{n} \boldsymbol{\Delta}(s_i) \overset{\phi,\tilde{\phi}}{\sim} \prod_{i=1}^{n} \tilde{\boldsymbol{\Delta}}(s_i),$$

which, by Proposition 4.3, is equivalent to (4.2) for all $n \geq 1$.

To illustrate Theorem 4.2, we provide an example showing two MAP$_2$s that verify (4.2) for all $n \geq 1$.

Example 4.1. Consider the MAP$_2$ defined by

$$D_0 = \begin{pmatrix} -10 & 2.5 \\ 0.6 & -3 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 4 & 3.5 \\ 1.35 & 1.05 \end{pmatrix},$$

or, alternatively,

$$P_0 = \begin{pmatrix} 0 & 0.25 \\ 0.2 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0.4 & 0.35 \\ 0.45 & 0.35 \end{pmatrix},$$

with exponential rates $(\lambda_1, \lambda_2) = (10, 3)$ and stationary probability distribution

$$\phi = (0.5474, 0.4526).$$

It can be seen that ε, defined in (4.8), has to be chosen from

$$0 < \varepsilon < \min\{10, 3.5, 0.4961, 7.2081, 0.4667, 0.5120\}.$$

Let $\varepsilon = 0.3$. Then, according to the equations given in Theorem 4.1,

$$\tilde{D}_0 = \begin{pmatrix} -9.7 & 3.9 \\ 0.9 & -3.3 \end{pmatrix}, \qquad \tilde{D}_1 = \begin{pmatrix} 4.675 & 1.1250 \\ 2.025 & 0.375 \end{pmatrix},$$

or

$$\tilde{P}_0 = \begin{pmatrix} 0 & 0.4020 \\ 0.2727 & 0 \end{pmatrix}, \qquad \tilde{P}_1 = \begin{pmatrix} 0.4820 & 0.1160 \\ 0.6137 & 0.1136 \end{pmatrix},$$

with exponential rates $(\tilde{\lambda}_1, \tilde{\lambda}_2) = (9.7, 3.3)$ and stationary distribution $\tilde{\phi} = (0.8217, 0.1783)$. Because of the proof of Theorem 4.2, (4.2) holds for all s and n, that is, both MAP$_2$s will possess the same joint interarrival time distribution.

Example 4.2. We consider in this example a MAP$_2$ such that $x = u$, defined by

$$D_0 = \begin{pmatrix} -3 & 0.9 \\ 0.6 & -3 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 0.9 & 1.2 \\ 1.5 & 0.9 \end{pmatrix},$$

or, alternatively,

$$P_0 = \begin{pmatrix} 0 & 0.3 \\ 0.2 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0.3 & 0.4 \\ 0.5 & 0.3 \end{pmatrix},$$

with exponential rates $(\lambda_1, \lambda_2) = (3, 3)$ and stationary probability distribution

$$\phi = (0.5333, 0.4667).$$

In this case ε, defined in (4.9), has to be chosen from

$$0 < \varepsilon < \min\{3, 0.5250, 0.7348, 0.36, 0.5367\}.$$

Set, for example, $\varepsilon = 0.15$. Then, according to the equations given in Theorem 4.1,

$$\tilde{D}_0 = \begin{pmatrix} -2.85 & 0.69 \\ 0.75 & -3.15 \end{pmatrix}, \qquad \tilde{D}_1 = \begin{pmatrix} 1.275 & 0.885 \\ 1.875 & 0.5250 \end{pmatrix},$$

or

$$\tilde{P}_0 = \begin{pmatrix} 0 & 0.2421 \\ 0.2381 & 0 \end{pmatrix}, \qquad \tilde{P}_1 = \begin{pmatrix} 0.4474 & 0.3105 \\ 0.5952 & 0.1667 \end{pmatrix},$$

with exponential rates $(\tilde{\lambda}_1, \tilde{\lambda}_2) = (3.15, 2.85)$ and stationary distribution

$$\tilde{\phi} = (0.6667, 0.3333),$$

is equivalent to $\{D_0, D_1\}$.

Some remarks need to be made here. First, let us point out that, as has been described in the proof of Theorem 4.2, given a fixed MAP$_2$, $\mathcal{M}$, any other MAP$_2$, $\tilde{\mathcal{M}}$, chosen from the set $\mathcal{F}$ will verify (4.2) for all s and n, and, thus, both MAP$_2$s will have the same joint interarrival time distribution.

Our second remark is connected with the MMPP. Its definition implies that $w \equiv -x - y$ and $v \equiv 0$. As Rydén [23] showed, the MMPP is an identifiable process up to permutations of the states, or, equivalently, for the two-states case, if the set $\{x, y, z, u, w = -x - y, v = 0\}$ defines an MMPP$_2$ then the only MMPP$_2$ with the same likelihood will be given by

$$\{\tilde{x} = u, \ \tilde{y} = z, \ \tilde{z} = y, \ \tilde{u} = x, \ \tilde{w} = -z - u, \ \tilde{v} = 0\}. \tag{4.23}$$

It can be verified that, when $\tilde{v} = v = 0$, $\tilde{w} = -\tilde{x} - \tilde{y}$, and $w = -x - y$, Proposition 4.2 provides just two solutions, the original MMPP$_2$ and its permuted version, given by (4.23). Thus, our results are equivalent to Rydén's.

From the nonidentifiability of the MAP$_2$ and the identifiability of the MMPP$_2$, one could wonder if, given a MAP$_2$, there exists an equivalent MMPP$_2$. This has been tested from the equations given in Proposition 4.2, and the answer is that this is not true, in general. The following example illustrates this fact.

Example 4.3. Let us consider the same MAP$_2$ defined in Example 1:

$$\boldsymbol{D}_0 = \begin{pmatrix} -10 & 2.5 \\ 0.6 & -3 \end{pmatrix}, \qquad \boldsymbol{D}_1 = \begin{pmatrix} 4 & 3.5 \\ 1.35 & 1.05 \end{pmatrix}.$$

Then, according to the solutions given in Proposition 4.2 (having previously fixed $w = -x - y$ and $v = 0$), the only equivalent MMPP$_2$ is given by

$$\tilde{\boldsymbol{D}}_0 = \begin{pmatrix} -7.4231 & 2.2712 \\ 5.6788 & -5.5769 \end{pmatrix}, \qquad \tilde{\boldsymbol{D}}_1 = \begin{pmatrix} 5.1519 & 0 \\ 0 & -0.1019 \end{pmatrix},$$

which, since $d_{221} < 0$, does not define a real MMPP$_2$. Thus, in this case, there does not exist an MMPP$_2$ equivalent to the given MAP$_2$. However, there do exist MAP$_2$s that can be reduced to MMPP$_2$s. For example,

$$\boldsymbol{D}_0 = \begin{pmatrix} -20 & 8 \\ 3.5 & -5 \end{pmatrix}, \qquad \boldsymbol{D}_1 = \begin{pmatrix} 11.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

define a MAP$_2$ that is equivalent to the MMPP$_2$ defined by

$$\tilde{\boldsymbol{D}}_0 = \begin{pmatrix} -19.4211 & 7.8973 \\ 4.6027 & -5.5789 \end{pmatrix}, \qquad \tilde{\boldsymbol{D}}_1 = \begin{pmatrix} 11.5238 & 0 \\ 0 & 0.9762 \end{pmatrix}.$$

Finally, we discuss the case of phase-type (PH) renewal processes, a special type of MAP. It is known (see, for example, [15]) that a PH renewal process with representation $(\boldsymbol{\eta}, \boldsymbol{T})$ is a MAP, where $\boldsymbol{D}_0 = \boldsymbol{T}$ and $\boldsymbol{D}_1 = -\boldsymbol{T}\boldsymbol{e}\boldsymbol{\eta}$. If we focus on two-state MAPs, defined by (3.1), this implies that

$$\eta = -\frac{w}{x+y} = -\frac{v}{u+z}, \tag{4.24}$$

where $\boldsymbol{\eta} = (\eta, 1 - \eta)$, and, thus, for a PH renewal process,

$$v = \frac{w(z+u)}{x+y}. \tag{4.25}$$

Example 4.4. As an example, consider the PH process given by

$$T = \begin{pmatrix} -10 & 2.5 \\ 0.6 & -3 \end{pmatrix}, \qquad \eta = (0.4, 0.6). \tag{4.26}$$

Then, the process defined by (4.26) can be also expressed as a MAP_2 as

$$D_0 = \begin{pmatrix} -10 & 2.5 \\ 0.6 & -3 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 3 & 4.5 \\ 0.96 & 1.44 \end{pmatrix},$$

where it can be easily seen that (4.25) holds. Conversely, given the MAP_2

$$D_0 = \begin{pmatrix} -5 & 1.4 \\ 0.25 & -0.5 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 2.52 & 1.08 \\ 0.175 & 0.075 \end{pmatrix}, \tag{4.27}$$

it can be checked that (4.25) holds, and, therefore, (4.27) admits a PH representation (η, T), where $T = D_0$ and η is computed from (4.24) as $\eta = (0.7, 0.3)$.

In this setting, given a fixed two-state PH renewal process, it is of interest to know the set of equivalent MAP_2s, which implies deriving expressions (4.4)–(4.7), assuming that $v = w(z + u)/(x + y)$. It follows immediately that if $v = w(z + u)/(x + y)$ then

$$\{\tilde{u}, \tilde{z}, \tilde{x}(\tilde{u}, \tilde{z}), \tilde{y}(\tilde{u}, \tilde{z}), \tilde{v}(\tilde{u}, \tilde{z}), \tilde{w}(\tilde{u}, \tilde{z})\},$$

computed according to Proposition 4.2, also verifies that $\tilde{v} = \tilde{w}(\tilde{z}+\tilde{u})/(\tilde{x}+\tilde{y})$. This implies that the MAP_2s equivalent to a given two-state PH renewal process are also PH renewal processes. Another interesting fact with regards to two-state PH renewal processes is that, since $v = w(z + u)/(x + y)$, it can be verified that $\varepsilon_3 = \varepsilon_5 = \varepsilon_6$ (see Theorem 4.1). In the following example, some MAP_2s, equivalent to a given two-state PH distribution, are computed.

Example 4.5. Consider the two-state PH renewal process defined by (4.27). Then, in order to find the equivalent MAP_2s, the value of ε in Theorem 4.1 has to be chosen from

$$0 < \varepsilon < \{5, 2.25, 0.1071, 4.5765, 0.1071, 0.1071\}.$$

Note that, indeed, $\varepsilon_3 = \varepsilon_5 = \varepsilon_6$. Let $\varepsilon = 0.05$. Then, an equivalent MAP_2 is obtained as

$$\tilde{D}_0 = \begin{pmatrix} -4.95 & 1.9083 \\ 0.3 & -0.55 \end{pmatrix}, \qquad \tilde{D}_1 = \begin{pmatrix} 2.555 & 0.4867 \\ 0.21 & 0.04 \end{pmatrix},$$

where it can be easily seen that (4.25) holds, and, thus, the generated equivalent MAP_2 is a PH renewal process given by $(\tilde{\eta}, \tilde{T})$, where $\tilde{T} = \tilde{D}_0$ and $\tilde{\eta} = (0.84, 0.16)$. If instead $\varepsilon = 0.09$ is chosen, the obtained equivalent MAP_2 is

$$\tilde{D}_0 = \begin{pmatrix} -4.9 & 2.2571 \\ 0.35 & -0.6 \end{pmatrix}, \qquad \tilde{D}_1 = \begin{pmatrix} 2.59 & 0.0529 \\ 0.245 & 0.05 \end{pmatrix},$$

with PH representation $(\tilde{\eta}, \tilde{T})$, where $\tilde{T} = \tilde{D}_0$ and $\tilde{\eta} = (0.98, 0.02)$.

5. Conclusions and extensions

In spite of the good properties that the MAPs present, which make them very suitable processes for modeling nonexponential events, there exist few works dealing with the identifiability of the MAP, which is of crucial importance when inference for the process is considered.

In this work we have provided a procedure that shows, for any fixed MAP$_2$, how to build another, equivalent MAP$_2$. We have also shown that if two MAP$_2$s have equal Laplace–Stieltjes transform (LST) for one and two data, then their LST will be equal for any set of points $(s_1, \ldots, s_n)$. Calculations have been carried out using MATLAB®, and all code utilized in the examples is available from the authors on request.

A number of extensions are possible. Firstly, we could consider the batch MAP$_2$, where batch arrivals are permitted. Furthermore, we could extend this analysis to MAPs or BMAPs with more than two states. Finally, it is of practical interest to consider what happens when there is missing data, that is, when a full sequence of interarrival times is not considered. Work on these problems is underway.

Appendix A. Proof of (2.3)

As $\boldsymbol{\phi}$ is the unique solution to $\boldsymbol{\phi} P^\star = \boldsymbol{\phi}$, we need to show that

$$(\boldsymbol{\pi} D_1 e)^{-1} \boldsymbol{\pi} D_1 P^\star = (\boldsymbol{\pi} D_1 e)^{-1} \boldsymbol{\pi} D_1.$$

Define $\boldsymbol{\Lambda} = \text{diag}\{\lambda_1, \ldots, \lambda_m\}$, that is, a diagonal $m \times m$ matrix with diagonal equal to $(\lambda_1, \ldots, \lambda_m)$. Then $D_1 = \boldsymbol{\Lambda} P_1$ and $D_0 = \boldsymbol{\Lambda}(P_0 - I)$. From (2.2), $P^\star = (-\boldsymbol{\Lambda}^{-1} D_0)^{-1} \boldsymbol{\Lambda}^{-1} D_1$, and, thus,

$$
\begin{aligned}
(\boldsymbol{\pi} D_1 e)^{-1} \boldsymbol{\pi} D_1 P^\star &= (\boldsymbol{\pi} D_1 e)^{-1} \boldsymbol{\pi} D_1 (-\boldsymbol{\Lambda}^{-1} D_0)^{-1} \boldsymbol{\Lambda}^{-1} D_1 \\
&= -(\boldsymbol{\pi} D_1 e)^{-1} \boldsymbol{\pi} D_1 D_0^{-1} D_1 \\
&= -(\boldsymbol{\pi} D_1 e)^{-1} \boldsymbol{\pi} (D - D_0) D_0^{-1} D_1 \\
&= -(\boldsymbol{\pi} D_1 e)^{-1} (\boldsymbol{\pi} D - \boldsymbol{\pi} D_0) D_0^{-1} D_1 \\
&= (\boldsymbol{\pi} D_1 e)^{-1} \boldsymbol{\pi} D_1.
\end{aligned}
$$

Appendix B. Proof of Theorem 3.1

Firstly, we provide some lemmas that will be necessary for the proof.

Lemma B.1. *Let T_n and $\tilde{T}_n$ denote the times between the $(n-1)$th and nth arrival in two MAP$_2$s, $\mathcal{M}$ and $\tilde{\mathcal{M}}$. Then,*

$$T_n \overset{\mathrm{D}}{=} \tilde{T}_n \quad \text{for all } n \geq 1$$

if and only if

$$\boldsymbol{\theta}(P^\star)^{(n-1)}(sI - D_0)^{-1} D_1 e = \tilde{\boldsymbol{\theta}}(\tilde{P}^\star)^{(n-1)}(sI - \tilde{D}_0)^{-1} \tilde{D}_1 e \qquad (\text{B.1})$$

for all $n \geq 1$ and all s.

Proof. The variables T_n and $\tilde{T}_n$ are equally distributed if and only if their Laplace transforms are the same. These are given by (B.1), where $\boldsymbol{\theta}(P^\star)^{(n-1)}$ and $\tilde{\boldsymbol{\theta}}(\tilde{P}^\star)^{(n-1)}$ represent the 'initial' probabilities after $n-1$ arrivals.

A similar result to Lemma B.1, which provides a different characterization of condition (C1) in Theorem 3.1, is shown next.

Lemma B.2. *Let T and $\tilde{T}$ denote the interarrival times of two stationary MAP$_2$s, $\mathcal{M}$ and $\tilde{\mathcal{M}}$, with stationary probabilities $\boldsymbol{\phi}$ and $\tilde{\boldsymbol{\phi}}$. Then,*

$$T \overset{\text{D}}{=} \tilde{T}$$

if and only if

$$\boldsymbol{\phi}(s\boldsymbol{I} - \boldsymbol{D}_0)^{-1}\boldsymbol{D}_1\boldsymbol{e} = \tilde{\boldsymbol{\phi}}(s\boldsymbol{I} - \tilde{\boldsymbol{D}}_0)^{-1}\tilde{\boldsymbol{D}}_1\boldsymbol{e}. \tag{B.2}$$

Proof. We proceed as in the proof of Lemma B.1, but taking into account the fact that

$$\lim_{n\to\infty}(\boldsymbol{P}^{\star})^n = \boldsymbol{\Phi}, \qquad \lim_{n\to\infty}(\tilde{\boldsymbol{P}}^{\star})^n = \tilde{\boldsymbol{\Phi}},$$

or, equivalently, $\lim_{n\to\infty}\boldsymbol{\theta}(\boldsymbol{P}^{\star})^n = \boldsymbol{\phi}$ and $\lim_{n\to\infty}\tilde{\boldsymbol{\theta}}(\tilde{\boldsymbol{P}}^{\star})^n = \tilde{\boldsymbol{\phi}}$.

Lemma B.3. *Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ denote two MAP$_2$s, and let $\boldsymbol{\rho} = (\rho, 1-\rho)$, $\tilde{\boldsymbol{\rho}} = (\tilde{\rho}, 1-\tilde{\rho})$ be any probability vectors. If*

$$\boldsymbol{\rho}(s\boldsymbol{I} - \boldsymbol{D}_0)^{-1}\boldsymbol{D}_1\boldsymbol{e} = \tilde{\boldsymbol{\rho}}(s\boldsymbol{I} - \tilde{\boldsymbol{D}}_0)^{-1}\tilde{\boldsymbol{D}}_1\boldsymbol{e}$$

then

$$c\rho + \tilde{c}\tilde{\rho} + d = 0, \tag{B.3}$$

where c and $\tilde{c}$ are defined as in (3.4), and $d = \tilde{z} + \tilde{u} - z - u$.

Proof. Substituting $\boldsymbol{D}_0$, $\boldsymbol{D}_1$, $\tilde{\boldsymbol{D}}_0$, and $\tilde{\boldsymbol{D}}_1$ expressed as in (3.1), it can be seen that

$$\boldsymbol{\rho}(s\boldsymbol{I} - \boldsymbol{D}_0)^{-1}\boldsymbol{D}_1\boldsymbol{e} = \frac{\alpha s + \gamma}{s^2 + \beta s + \gamma},$$

where the coefficients α, β, and γ are

$$\alpha = \rho(z + u - x - y) - (z + u), \qquad \beta = -x - u, \qquad \gamma = xu - yz.$$

Similarly,

$$\tilde{\boldsymbol{\rho}}(s\boldsymbol{I} - \boldsymbol{D}_0)^{-1}\boldsymbol{D}_1\boldsymbol{e} = \frac{\tilde{\alpha} s + \tilde{\gamma}}{s^2 + \tilde{\beta} s + \tilde{\gamma}},$$

where

$$\tilde{\alpha} = \tilde{\rho}(\tilde{z} + \tilde{u} - \tilde{x} - \tilde{y}) - (\tilde{z} + \tilde{u}), \qquad \tilde{\beta} = -\tilde{x} - \tilde{u}, \qquad \tilde{\gamma} = \tilde{x}\tilde{u} - \tilde{y}\tilde{z}.$$

Next, if

$$\frac{\alpha s + \gamma}{s^2 + \beta s + \gamma} = \frac{\tilde{\alpha} s + \tilde{\gamma}}{s^2 + \tilde{\beta} s + \tilde{\gamma}} \tag{B.4}$$

then it can easily be seen that

$$\alpha = \tilde{\alpha},$$

which is equivalent, given the definitions of α and $\tilde{\alpha}$, to

$$c\rho + \tilde{c}\tilde{\rho} + d = 0$$

with c and $\tilde{c}$ as in (3.4), and $d = \tilde{z} + \tilde{u} - z - u$.

Lemma B.4. *Let $P^\star$ be the transition probability matrix in a MAP$_2$ with vector of stationary probabilities $\boldsymbol{\phi}$. If all the rows of $P^\star$ are equal then $P^\star = \Phi$, the matrix with all rows equal to $\boldsymbol{\phi}$.*

Proof. The proof is straightforward once the equation $\boldsymbol{\phi}P^\star = \boldsymbol{\phi}$ is solved, where it is assumed that all the rows of $P^\star$ are equal.

Finally, if $P^\star$ is a transition matrix with vector of stationary probabilities $\boldsymbol{\phi}$, where

$$P^\star = \begin{pmatrix} p^\star_{11} & p^\star_{12} \\ p^\star_{21} & p^\star_{22} \end{pmatrix},$$

and $\boldsymbol{\phi} = (\phi, 1-\phi)$, then it is straightforward to check that

$$\phi = \frac{p^\star_{21}}{1 - p^\star_{11} + p^\star_{21}}. \tag{B.5}$$

Proof of Theorem 3.1. Conditions (C1) and (C2) imply weak convergence. Let us first assume that both (C1) and (C2) hold. We want to prove equivalence given by (B.1) for all s and $n \geq 1$. By Lemma B.2, condition (C1) is equivalent to (B.2). But, since $(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}) = (\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}})$, $\boldsymbol{\theta}P^\star = \boldsymbol{\phi}P^\star = \boldsymbol{\phi}$, and $\tilde{\boldsymbol{\theta}}\tilde{P}^\star = \tilde{\boldsymbol{\phi}}\tilde{P}^\star = \tilde{\boldsymbol{\phi}}$, then (B.2) becomes (B.1) and, thus, weak equivalence is obtained.

Weak convergence implies conditions (C1) and (C2). If two given MAP$_2$s are weakly equivalent, (B.1) holds for all s and $n \geq 1$. If $n \to \infty$ then, from Lemma B.2, (B.2) holds, and, thus, (C1) holds too. Let us deduce (C2) from weak equivalence; since (B.2) holds, then the pair $(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}})$ verifies (B.3) (where $\rho = \phi$ and $\tilde{\rho} = \tilde{\phi}$), that is,

$$c\phi + \tilde{c}\tilde{\phi} + d = 0.$$

Because of weak equivalence, (B.1) holds for $n = 1$, and, thus, the pair $(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}})$ also satisfies (B.3), i.e.

$$c\theta + \tilde{c}\tilde{\theta} + d = 0.$$

Both equations imply that

$$c\phi + \tilde{c}\tilde{\phi} = c\theta + \tilde{c}\tilde{\theta},$$

or, equivalently, using (B.5),

$$c\frac{p^\star_{21}}{1 - p^\star_{11} + p^\star_{21}} + \tilde{c}\frac{\tilde{p}^\star_{21}}{1 - \tilde{p}^\star_{11} + \tilde{p}^\star_{21}} = c\theta + \tilde{c}\tilde{\theta}.$$

Again, because of weak equivalence, and taking $n = 2$ in condition (B.1), then

$$c\frac{p^\star_{21}}{1 - p^\star_{11} + p^\star_{21}} + \tilde{c}\frac{\tilde{p}^\star_{21}}{1 - \tilde{p}^\star_{11} + \tilde{p}^\star_{21}} = c\theta^{(1)} + \tilde{c}\tilde{\theta}^{(1)},$$

where

$$\boldsymbol{\theta}^{(1)} = \boldsymbol{\theta}P^\star = (\theta^{(1)}, 1 - \theta^{(1)}), \qquad \tilde{\boldsymbol{\theta}}^{(1)} = \tilde{\boldsymbol{\theta}}\tilde{P}^\star = (\tilde{\theta}^{(1)}, 1 - \tilde{\theta}^{(1)}).$$

It can be checked that

$$\theta^{(1)} = \theta(p^\star_{11} - p^\star_{21}) + p^\star_{21}, \qquad \tilde{\theta}^{(1)} = \tilde{\theta}(\tilde{p}^\star_{11} - \tilde{p}^\star_{21}) + \tilde{p}^\star_{21},$$

and, thus, we need to solve for $(\theta, \tilde{\theta})$ in the following system of linear equations:

$$c\theta + \tilde{c}\tilde{\theta} = c\frac{p_{21}^{\star}}{1 - p_{11}^{\star} + p_{21}^{\star}} + \tilde{c}\frac{\tilde{p}_{21}^{\star}}{1 - \tilde{p}_{11}^{\star} + \tilde{p}_{21}^{\star}},$$

$$c(p_{11}^{\star} - p_{21}^{\star})\theta + \tilde{c}(\tilde{p}_{11}^{\star} - \tilde{p}_{21}^{\star})\tilde{\theta} = c\left(\frac{p_{21}^{\star}}{1 - p_{11}^{\star} + p_{21}^{\star}} - p_{21}^{\star}\right) + \tilde{c}\left(\frac{\tilde{p}_{21}^{\star}}{1 - \tilde{p}_{11}^{\star} + \tilde{p}_{21}^{\star}} - \tilde{p}_{21}^{\star}\right),$$

whose coefficient matrix is

$$\boldsymbol{C} = \begin{pmatrix} c & \tilde{c} \\ c(p_{11}^{\star} - p_{21}^{\star}) & \tilde{c}(\tilde{p}_{11}^{\star} - \tilde{p}_{21}^{\star}) \end{pmatrix}.$$

It can be easily seen that $\boldsymbol{\theta} = \boldsymbol{\phi}$, and $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\phi}}$ solves the system. We need to determine the uniqueness of this solution. This comes from (A1), (A2), and Lemma B.4; since $\boldsymbol{P}^{\star} \neq \boldsymbol{\Phi}$ or $\tilde{\boldsymbol{P}}^{\star} \neq \tilde{\boldsymbol{\Phi}}$, then, by Lemma B.4, either the rows of $\boldsymbol{P}^{\star}$ or that of $\tilde{\boldsymbol{P}}^{\star}$ are not equal. This implies that $p_{11}^{\star} - p_{21}^{\star} \neq 0$ or $\tilde{p}_{11}^{\star} - \tilde{p}_{21}^{\star} \neq 0$. In addition, since $c, \tilde{c} \neq 0$, the rank of $\boldsymbol{C}$ is 2, and the solution is unique: $\boldsymbol{\theta} = \boldsymbol{\phi}$ and $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\phi}}$.

Appendix C. Proof of Proposition 4.1

Let us first consider the case when $n = 1$. It is known from the proof of Lemma B.2 that the equality of Laplace transforms $f^{*}_{T; \boldsymbol{D}_0, \boldsymbol{D}_1}(s) = f^{*}_{\tilde{T}; \tilde{\boldsymbol{D}}_0, \tilde{\boldsymbol{D}}_1}(s)$ in the stationary version is equivalent to (B.4), where $\boldsymbol{\rho} = \boldsymbol{\phi}$ and $\tilde{\boldsymbol{\rho}} = \tilde{\boldsymbol{\phi}}$. If $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$, and $\tilde{\gamma} = \gamma$, or, equivalently,

$$\tilde{\phi}(\tilde{z} + \tilde{u} - \tilde{x} - \tilde{y}) - (\tilde{z} + \tilde{u}) = \alpha, \tag{C.1}$$

$$-\tilde{x} - \tilde{u} = \beta, \tag{C.2}$$

$$\tilde{x}\tilde{u} - \tilde{y}\tilde{z} = \gamma, \tag{C.3}$$

then (B.4) holds, and, thus, $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are *weakly* equivalent.

In the two data case, it can be shown that

$$f^{*}_{T; \boldsymbol{D}_0, \boldsymbol{D}_1}(s_1, s_2) = \frac{\delta_1 s_1 s_2 + \delta_2 s_2 + \alpha\gamma s_1 + \gamma^2}{s_1^2 s_2^2 + \gamma s_1^2 + \gamma s_2^2 + \beta s_1^2 s_2 + \beta s_1 s_2^2 + \beta^2 s_1 s_2 + \beta\gamma s_2 + \gamma^2},$$

where

$$\delta_1 = \phi((z + u - x - y)(w - v) + (x + y)(z + u) - (z + u)^2) + (z + u - x - y)v$$
$$+ (z + u)^2,$$
$$\delta_2 = \phi(x + y - z - u)(uw - yv - xv + zw) + (x + y - z - u)(xv - zw) - (u + z)\gamma.$$

If (C.1)–(C.3) are satisfied, and, in addition,

$$\tilde{\phi}((\tilde{z} + \tilde{u} - \tilde{x} - \tilde{y})(\tilde{w} - \tilde{v}) + (\tilde{x} + \tilde{y})(\tilde{z} + \tilde{u}) - (\tilde{z} + \tilde{u})^2) + (\tilde{z} + \tilde{u} - \tilde{x} - \tilde{y})\tilde{v} + (\tilde{z} + \tilde{u})^2$$
$$= \delta_1, \tag{C.4}$$
$$\tilde{\phi}(\tilde{x} + \tilde{y} - \tilde{z} - \tilde{u})(\tilde{u}\tilde{w} - \tilde{y}\tilde{v} - \tilde{x}\tilde{v} + \tilde{z}\tilde{w}) + (\tilde{x} + \tilde{y} - \tilde{z} - \tilde{u})(\tilde{x}\tilde{v} - \tilde{z}\tilde{w}) - (\tilde{u} + \tilde{z})\tilde{\gamma}$$
$$= \delta_2, \tag{C.5}$$

then (4.2) holds for $n = 1$ and $n = 2$.

Appendix D. Proof of Theorem 4.1

The set $\mathcal{F}$ is contained in the set of solutions given by Proposition 4.2. To prove that the set $\mathcal{F}$ provides feasible solutions (real MAP$_2$s), let us first assume that $x < u$. Let ε be defined as in (4.8), that is,

$$0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}.$$

We begin by proving that $\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} > 0$. It is easily checked that

$$\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} > 0.$$

In addition, $\varepsilon_5 = -(z/v)(z + u + v) > 0$, since $z, v > 0$ and $-z - u - v > 0$. Finally, $\varepsilon_6 > 0$ since $-z/2v < 0$ and $-w - y - x > 0$, and, therefore,

$$(u + v + z + w)^2 < (u + v + z + w)^2 + 4v(-w - y - x).$$

Then,

$$\tilde{u} = u - \varepsilon < 0 \quad \text{and} \quad \tilde{z} = z + \varepsilon > 0.$$

Moreover, since $\varepsilon < \varepsilon_2 = (u - x)/2$, this assures that $\tilde{x} < \tilde{u}$, and, thus, the parameterization of $\tilde{\mathcal{M}}$ is different from that of $\mathcal{M}$ with permuted states. Next,

$$\frac{(u - x) - \sqrt{(x - u)^2 + 4zy}}{2} < 0 < \varepsilon < \frac{(u - x) + \sqrt{(x - u)^2 + 4zy}}{2}$$

implies that

$$\tilde{y}(\tilde{u}, \tilde{z}) \equiv \frac{-(\varepsilon^2 + (x - u)\varepsilon - zy)}{z + \varepsilon} > 0.$$

In addition,

$$\tilde{w}(\tilde{u}, \tilde{z}) \equiv \frac{wz + v\varepsilon}{z} > 0, \qquad \tilde{v} \equiv \frac{v(z + \varepsilon)}{z} > 0,$$

and, since $\varepsilon < z(1 - \phi)/\phi$,

$$\tilde{\phi} \equiv \frac{(z + \varepsilon)\phi}{z} \in [0, 1].$$

It remains to prove that $-\tilde{z} - \tilde{u} - \tilde{v} > 0$ and $-\tilde{x} - \tilde{y} - \tilde{w} > 0$. It is easy to check that

$$-\tilde{z} - \tilde{u} - \tilde{v} = -z - u - \frac{v(z + \varepsilon)}{z},$$

which is positive if and only if $\varepsilon < \varepsilon_5 = -(z/v)(z + u + v)$. Finally, an easy computation shows that $-\tilde{x} - \tilde{y} - \tilde{w} > 0$ is equivalent to

$$-\varepsilon - x > \frac{-(\varepsilon^2 + (x - u)\varepsilon - zy)}{z + \varepsilon} + \frac{wz + v\varepsilon}{z},$$

which holds if and only if $\varepsilon \in (r_1, r_2)$, where

$$r_1 = -\frac{z}{2v}[(u + v + z + w) + \sqrt{(u + v + z + w)^2 + 4v(-w - y - x)}] < 0,$$

$$r_2 = -\frac{z}{2v}[(u + v + z + w) - \sqrt{(u + v + z + w)^2 + 4v(-w - y - x)}] = \varepsilon_6 > 0.$$

Now, let us assume that $x = u$. Then, let ε be defined as in (4.9), where in this case, $\varepsilon_3 \equiv \sqrt{zy}$. Then,

$$\tilde{u} = u - \varepsilon < 0,$$
$$\tilde{z} = z + \varepsilon > 0,$$
$$\tilde{x} = x + \varepsilon < 0 \quad (\text{since } \varepsilon < -x),$$
$$\tilde{y} = \frac{zy - \varepsilon^2}{z + \varepsilon} > 0 \quad (\text{since } \varepsilon < \sqrt{zy}),$$
$$\tilde{w} = \frac{wz + v\varepsilon}{v} > 0,$$
$$\tilde{v} = \frac{v(z + \varepsilon)}{z} > 0,$$

and $\tilde{\phi} \in [0, 1]$, $-\tilde{z} - \tilde{u} - \tilde{v} > 0$, and $-\tilde{x} - \tilde{y} - \tilde{w} > 0$ follow from the assumptions $\varepsilon < \varepsilon_4$, $\varepsilon < \varepsilon_5$, and $\varepsilon < \varepsilon_6$, respectively.

Appendix E. Proof of Proposition 4.4

Let us assume that

$$\boldsymbol{G} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \boldsymbol{H} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{and} \quad \boldsymbol{GH} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Now it is straightforward to verify that if $\tilde{\boldsymbol{G}}$ and $\tilde{\boldsymbol{H}}$ are defined by (4.19)–(4.22) (with respect to the elements of $\boldsymbol{G}$ and $\boldsymbol{H}$, respectively), then

$$\tilde{\boldsymbol{G}}\tilde{\boldsymbol{H}} = \begin{pmatrix} \dfrac{\phi(A - C) + \tilde{\phi}C}{\phi} & \dfrac{\phi\tilde{\phi}(D + 2C - A) + \phi^2(A - D + B - C) - \tilde{\phi}^2 C}{\phi\tilde{\phi}} \\[4mm] \dfrac{\tilde{\phi}C}{\phi} & \dfrac{\phi(C + D) - \tilde{\phi}C}{\phi} \end{pmatrix}.$$

Acknowledgements

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