Bull. Aust. Math. Soc. **95** (2017), 291–298
doi:10.1017/S0004972716000927

MAPPINGS OF CONSERVATIVE DISTANCES IN p-NORMED SPACES ($0 < p \leq 1$)

XUJIAN HUANG and DONGNI TAN✉

(Received 18 July 2016; accepted 13 August 2016; first published online 2 November 2016)

Abstract

We show that any mapping between two real p-normed spaces, which preserves the unit distance and the midpoint of segments with distance 2^p, is an isometry. Making use of it, we provide an alternative proof of some known results on the Aleksandrov question in normed spaces and also generalise these known results to p-normed spaces.

2010 *Mathematics subject classification*: primary 46B04; secondary 46B20.

Keywords and phrases: Aleksandrov problem, p-strictly convex, isometry, p-normed space.

1. Introduction

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if it satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. For some $r > 0$, we say that f *preserves distance* r if $d_X(x, y) = r$ yields $d_Y(f(x), f(y)) = r$. In particular, we say that f has the *distance one preserving property (DOPP)* if it preserves unit distance. We say that f is a *1-Lipschitz mapping* if $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$.

The classical Mazur–Ulam theorem [12] states that every surjective isometry between two real normed spaces is affine. Baker [2] showed that every isometry of a real normed linear space into a strictly convex real normed linear space is affine without the onto assumption. An old question of Aleksandrov [1] asked under what conditions a mapping on a metric space which preserves unit distance must necessarily be an isometry. Beckman and Quarles proved in [3] that any mapping of $\mathbb{R}^n$ with $n \geq 2$ preserving unit distance is an affine isometry. As far as we know, this problem is still far from being solved. It was solved only for a few concrete two-dimensional normed spaces (see [7] concerning strictly convex normed spaces and [10] for a nonstrictly convex normed space). For modified versions of the Aleksandrov question, there are two known results. Benz [4] (see also [5]) showed that every mapping of a real normed space into a strictly convex real normed space which preserves two distances with an

The authors are supported by the Natural Science Foundation of China (grant nos. 11201337, 11201338, 11371201 and 11301384).

integer ratio is an affine isometry. Rassias and Šemrl [13] proved the following result for mappings having the strong distance one preserving property (SDOPP), that is, for all $x, y \in X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$ and conversely.

Theorem 1.1 [13]. *Let X and Y be two real normed spaces such that one of them has a dimension greater than one, and let $f : X \to Y$ be a surjective mapping satisfying SDOPP. If f is a Lipschitz mapping with Lipschitz constant $K \le 1$ or one of X and Y is strictly convex, then f is an affine isometry.*

The goal of this paper is to consider the Aleksandrov question in the case of p-normed spaces ($0 < p \le 1$). We first prove that any mapping which preserves the unit distance and the midpoint of segments with distance 2^p is an isometry. This result is applied to show that Benz's theorem and the Rassias–Šemrl theorem hold in p-normed spaces. In fact, we provide an alternative proof of these theorems on the Aleksandrov question in normed spaces. We also show that a collineation between two p-normed spaces preserving unit distance is an affine isometry, and that a mapping between two p-strictly convex spaces from high dimensions to two dimensions, which preserves the unit distance, is an affine isometry. These results are new even in normed spaces.

2. Isometries in p-normed spaces ($0 < p \le 1$)

All vector spaces mentioned in this article are assumed to be real.

Definition 2.1. Let X be a vector space, $0 < p \le 1$ and $\|\cdot\|$ be a real-valued function on X, satisfying the following conditions:

(a) $\|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = 0$;
(b) $\|\lambda x\| = \|\lambda|^p\|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in X$;
(c) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$.

The function $\|\cdot\|$ is called a *p-norm* on X and the pair $(X, \|\cdot\|)$ is called a *p-normed space*.

Let X be a p-normed space. Then X equipped with the translation-invariant metric $d(x, y) = \|x - y\|$ for every $x, y \in X$ is a metric linear space. For the case $p = 1$, X is a normed space. The spaces l_p and $L_p[0, 1], 0 < p \le 1$, are p-normed spaces. If $(X, \|\cdot\|)$ is a normed space, we can define a p-norm on X by $\|\cdot\|^p$. In this case, we let X_p denote the p-normed space $(X, \|\cdot\|^p)$. In fact, every Hausdorff locally bounded topological linear space induces a p-normed space with $0 < p \le 1$ (see [9, page 161]).

Definition 2.2. A p-normed space X is called *p-strictly convex* if, for each pair x, y of nonzero elements in X such that $\|x + y\|^{1/p} = \|x\|^{1/p} + \|y\|^{1/p}$, it follows that $x = ty$ for some $t > 0$.

It follows from [11, Theorem 2.5] that $(X, \|\cdot\|)$ is a strictly convex normed space if and only if $(X, \|\cdot\|^p)$ is a p-strictly convex space. We see that if X is a p-strictly convex space, then the equations $\|x - z\| = \|y - z\| = \|(x - y)/2\|$ have a unique solution $z = (x + y)/2$ for every $x, y \in X$.

Definition 2.3. Let X and Y be p-normed spaces. A mapping $f : X \to Y$ is said to *preserve the midpoint of segments with distance* r if, for any $x, y \in X$ with $\|x - y\| = r$,

$$f\Big(\frac{x+y}{2}\Big) = \frac{f(x) + f(y)}{2}.$$

Lemma 2.4. *Suppose that X is a p-normed space with $\dim X \geq 2$ and $x, y \in X$ with $c := \|x - y\| > 0$. If $|c^{1/p} - a^{1/p}|^p \leq b \leq |c^{1/p} + a^{1/p}|^p$ for some $a, b > 0$, then there exists a vector z in X such that $\|x - z\| = a$ and $\|y - z\| = b$.*

Proof. Let S_X be the unit sphere of X. Consider the continuous function given by

$$\phi : a^{1/p} S_X + x \to \mathbb{R}, \quad \phi(z) = \|z - y\|.$$

It is easy to check that

$$\|\phi(x + (a/c)^{1/p}(y - x))\| = |c^{1/p} - a^{1/p}|^p \leq b,$$
$$\|\phi(x + (a/c)^{1/p}(x - y))\| = |c^{1/p} + a^{1/p}|^p \geq b.$$

We thus obtain the existence of a vector $z \in X$ such that $\|x - z\| = a$ and $\|y - z\| = b$. □

The following result plays an important role and is used frequently in this paper; we call it the main lemma.

Lemma 2.5 (Main Lemma). *Let X and Y be p-normed spaces with $\dim X \geq 2$ and $f : X \to Y$ a mapping which preserves distance r for some $r > 0$. If f preserves the midpoint of segments with distance $2^p r$, then f is an isometry.*

Proof. (a) We first prove that f preserves distance $n^p r$ for every positive integer n. Let x and y be vectors in X such that $\|x - y\| = n^p r$. Set $z_i := x + (i/n)(y - x)$ for each $i \in \mathbb{N} \cup \{0\}$. Then $z_0 = x$, $z_n = y$ and $\|z_{i-1} - z_{i+1}\| = 2^p r$ for all $i \in \mathbb{N}$. Since f preserves the midpoint of segments with distance $2^p r$, we have $f(z_{i+1}) - f(z_i) = f(z_i) - f(z_{i-1})$ for all $i \in \mathbb{N}$. This implies that $f(y) - f(x) = f(z_n) - f(z_0) = n(f(z_1) - f(z_0))$ and

$$\|f(y) - f(x)\| = n^p \|f(z_1) - f(z_0)\| = n^p \|z_1 - z_0\| = n^p r.$$

(b) Next, we prove that f preserves distance r/k^p for every positive integer k. Let $x, y \in X$ with $\|x - y\| = r/k^p$. By Lemma 2.4, we can find a vector $z \in X$ such that

$$\|x - z\| = \|y - z\| = r.$$

Set $u_k := z + k(x - z)$ and $v_k := z + k(y - z)$. Obviously, we have $\|u_k - v_k\| = r$ and $\|u_k - z\| = \|v_k - z\| = k^p r$. By a similar method as in the proof of (a),

$$f(u_k) - f(x) = (k - 1)(f(x) - f(z)),$$
$$f(v_k) - f(y) = (k - 1)(f(y) - f(z))$$

and it follows that

$$\|f(x) - f(y)\| = \Big\|\frac{1}{k}(f(u_k) - f(v_k))\Big\| = \frac{1}{k^p}\|u_k - v_k\| = \frac{r}{k^p}.$$

(c) Finally, we show that f is an isometry. Let $x, y \in X$ be two arbitrary different vectors and set $c := \|x - y\|$. For every ε with $0 < \varepsilon < c/3$, choose $a, b \in \mathbb{Q}$ such that $0 < a^p r < \varepsilon$ and

$$0 < (c^{1/p} - ar^{1/p})^p < b^p r < (c^{1/p} + ar^{1/p})^p. \tag{2.1}$$

By Lemma 2.4, there exists a vector $z \in X$ such that $\|x - z\| = a^p r$ and $\|y - z\| = b^p r$. By the above (a) and (b), f preserves the two distances $a^p r$ and $b^p r$. It follows that $\|f(x) - f(z)\| = a^p r$ and $\|f(y) - f(z)\| = b^p r$. By the triangle inequality and (2.1),

$$c - 2\varepsilon < b^p r - a^p r \le \|f(x) - f(y)\| \le b^p r + a^p r < c + 2\varepsilon.$$

Since this holds for every ε with $0 < \varepsilon < c/3$, we conclude that $\|f(x) - f(y)\| = \|x - y\|$. This completes the proof. □

The following theorems are simple applications of Lemma 2.5. We shall prove that Benz's theorem and the Rassias–Šemrl theorem hold in p-strictly convex spaces.

Theorem 2.6. *Let X and Y be p-normed spaces with $\dim X \ge 2$ and Y p-strictly convex. If $f : X \to Y$ preserves two distances r and $N^p r$ for some $r > 0$ and some integer $N > 1$, then f is an affine isometry.*

Proof. Let x and y be vectors in X such that $\|x - y\| = 2^p r$. Set $z := (x + y)/2$ and $z_i := x + i(z - x)$ for $i = 0, 1, \dots, N$. Then, clearly, $\|z_N - z_0\| = N^p r$ and $\|z_i - z_{i-1}\| = r$ for $i = 1, 2, \dots, N$. Since f preserves the two distances r and $N^p r$,

$$\begin{aligned} Nr^{1/p} &= \|f(z_N) - f(z_0)\|^{1/p} \le \|f(z_N) - f(y)\|^{1/p} + \|f(y) - f(x)\|^{1/p} \\ &\le \sum_{i=3}^{N} \|f(z_i) - f(z_{i-1})\|^{1/p} + \|f(y) - f(z)\|^{1/p} + \|f(z) - f(x)\|^{1/p} = Nr^{1/p}. \end{aligned}$$

Thus,

$$\|f(y) - f(x)\|^{1/p} = \|f(y) - f(z)\|^{1/p} + \|f(z) - f(x)\|^{1/p}.$$

Since $(Y, \|\cdot\|^{1/p})$ is a strictly convex normed space, $f(z) = (f(x) + f(y))/2$. This implies that f preserves the midpoint of segments with distance $2^p r$. By Lemma 2.5, we see that f is an isometry. Thus, f preserves the midpoint of segments with any distance. Since continuity is implied by isometry, the mapping f is affine. □

Theorem 2.7. *Let X and Y be p-normed spaces with $\dim X \ge 2$ and X p-strictly convex. If $f : X \to Y$ is a surjective mapping satisfying SDOPP, then f is an affine isometry.*

Proof. We first prove that f is injective. Suppose on the contrary that there are $x, y \in X, x \ne y$ such that $f(x) = f(y)$. Choose $z \in X$ such that $\|z - x\| = 1$ and $\|z - y\| \ne 1$. Then, clearly, $\|f(z) - f(x)\| = \|f(z) - f(y)\| = 1$. This implies that $\|z - y\| = 1$, which is a contradiction.

We will show that f preserves the midpoint of segments with distance 2^p. This implies that f is an isometry, and therefore affine by Theorem 2.6. Take $x, y \in X$ such

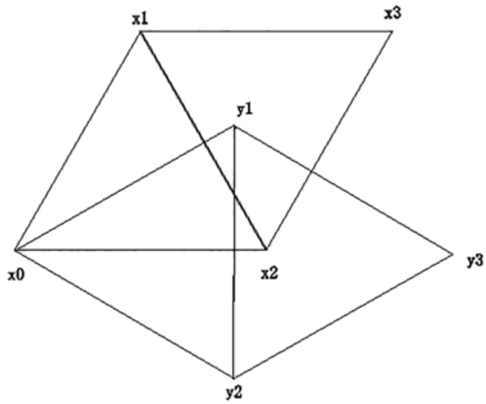

FIGURE 1. An r-probe.

that $\|x - y\| = 2^p$ and set $z := (x + y)/2$. We claim that $f(x), f(y), f(z)$ are collinear. Otherwise, we can find $w \in X$ with $w \neq z$ such that

$$\|f(w) - f(x)\| = \|f(w) - f(y)\| = 1.$$

This means that $\|w - x\| = \|w - y\| = 1$. As X is p-strictly convex, we have $w = z$, which is a contradiction. Consequently, there exists a real number t such that

$$f(y) - f(z) = t(f(x) - f(z)).$$

It is clear that

$$\|f(x) - f(z)\| = \|f(y) - f(z)\| = 1.$$

Hence, $t = -1$, and so

$$f\Big(\frac{x+y}{2}\Big) = f(z) = \frac{f(x) + f(y)}{2}.$$ □

For the next main result, we need some new notation. Let X be a p-normed space. We call the 3-tuple $(x, y, z) \in X^3$ an r-equilateral triangle if

$$r = \|x - y\| = \|x - z\| = \|y - z\|.$$

We call the 7-tuple $(x_0, x_1, x_2, x_3, y_1, y_2, y_3) \in X^7$ an r-probe if $\{x_3, y_1, y_2, y_3\} \subset \mathrm{aff}(x_0, x_1, x_2)$ and

$$\begin{aligned} r &= \|x_0 - x_1\| = \|x_0 - x_2\| = \|x_1 - x_2\| = \|x_1 - x_3\| = \|x_2 - x_3\| \\ &= \|x_0 - y_1\| = \|x_0 - y_2\| = \|y_1 - y_2\| = \|y_1 - y_3\| = \|y_2 - y_3\| = \|x_3 - y_3\| \end{aligned}$$

(see [7, 8]; see also Figure 1). It is known [14] that a real normed linear space X is strictly convex if and only if any two-dimensional subspace of X has the following property.

https://doi.org/10.1017/S0004972716000927 Published online by Cambridge University Press

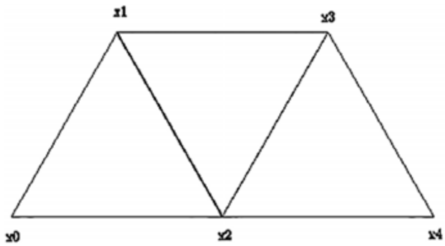

FIGURE 2. Diagram for the proof of Theorem 2.8.

(∗) For any $a \neq b$ on the line L and any c, d on the same side of L, if $\|a - c\| = \|a - d\|$ and $\|b - c\| = \|b - d\|$, then $c = d$.

By this property and [7, Lemma 4], any r-equilateral triangle can be extended to an r-probe in strictly convex spaces. Obviously, this result holds in p-strictly convex spaces.

THEOREM 2.8. *Let X and Y be p-strictly convex spaces with $\dim X \geq \dim Y = 2$, and $f : X \to Y$ a mapping which preserves distance r for some $r > 0$. Then f is an affine isometry.*

PROOF. It is sufficient to prove that f preserves the midpoint of segments with distance $2^p r$. Let x_0, x_4 be in X such that $\|x_0 - x_4\| = 2^p r$. Using the notation of Figure 2,

$$x_2 := \frac{x_0 + x_4}{2}, \quad \|x_1 - x_0\| = \|x_1 - x_2\| = \|x_0 - x_2\| = r, \quad x_3 := x_1 + (x_2 - x_0).$$

Since Y is p-strictly convex and $\|f(x_3) - f(x_1)\| = \|f(x_3) - f(x_2)\| = r$, by the property (∗) we see that

$$f(x_3) \in \{f(x_0), f(x_1) + f(x_2) - f(x_0)\}.$$

We shall prove that $f(x_3) \neq f(x_0)$. Suppose, on the contrary, that $f(x_3) = f(x_0)$. Now consider the r-probe $(x_0, x_1, x_2, x_3, y_1, y_2, y_3) \in X^7$ (see Figure 1). Then the 7-tuple $(f(x_0), f(x_1), f(x_2), f(x_3), f(y_1), f(y_2), f(y_3)) \in Y^7$ is also an r-probe and the four points $f(x_0), f(y_1), f(y_2), f(y_3)$ have distance r from each other. Two of the points are on the same side of the line passing through the other two points. According to the property (∗), this is impossible. It follows that $f(x_3) = f(x_1) + f(x_2) - f(x_0)$, and similarly $f(x_4) = f(x_2) + f(x_3) - f(x_1)$. Hence,

$$f\left(\frac{x_0 + x_4}{2}\right) = f(x_2) = \frac{f(x_0) + f(x_4)}{2}. \qquad \square$$

Next, we return to the Aleksandrov question on general p-normed spaces. For a real vector space X, we denote the line joining two different points $x, y \in X$ by $\overline{xy}$ and the affine subspace generated by $M \subset X$ by $\mathrm{Aff}(M)$. Let X and Y be real vector spaces. A mapping $f : X \to Y$ is called a *collineation* if it maps any three collinear points into collinear points. It is straightforward to check that if f is a collineation, then we have $f(\mathrm{Aff}(x, y, z)) \subset \mathrm{Aff}(f(x), f(y), f(z))$ for any $x, y, z \in X$.

https://doi.org/10.1017/S0004972716000927 Published online by Cambridge University Press

THEOREM 2.9. *Let X and Y be p-normed spaces with $\dim X \geq 2$, and $f : X \to Y$ be a collineation satisfying DOPP. Then f is an affine isometry.*

PROOF. Let $x, y \in X$ with $\|x - y\| = 2^p$ and set $z := \frac{1}{2}(x + y)$. We first prove that $f(x) \neq f(y)$. Assume on the contrary that $f(x) = f(y)$. Choose $w \in X$ such that

$$\|w - x\| = \|w - z\| = 1.$$

Then, clearly, $\|f(x) - f(z)\| = \|f(w) - f(x)\| = \|f(w) - f(z)\| = 1$. This implies that $f(x), f(z), f(w)$ are not collinear. However,

$$f(z) \in f(\mathrm{Aff}(x, y, w)) \subset \overline{f(x)f(w)},$$

which is a contradiction. It is trivial to check that

$$\|f(x) - f(z)\| = \|f(y) - f(z)\| = 1.$$

Since $f(x), f(y), f(z)$ are collinear, there exists a real number t such that

$$f(y) - f(z) = t(f(x) - f(z)).$$

We conclude that $t = -1$, and thus

$$f\Big(\frac{x + y}{2}\Big) = f(z) = \frac{f(x) + f(y)}{2}.$$

We have proved that f preserves the midpoint of segments with distance 2^p and this means that f is an isometry by Lemma 2.5. The same reasoning as in Theorem 2.6 proves that f is affine. □

To begin the discussion of the next main result, we introduce some more notation. Let X be a real p-normed space. For any $x, y \in X$, set

$$H_1(x, y) = \Big\{u \in X : \|x - u\| = \|y - u\| = \Big\|\frac{x - y}{2}\Big\|\Big\},$$

$$H_n(x, y) = \Big\{u \in H_{n-1}(x, y) : \|u - v\| \leq \frac{\delta(H_{n-1}(x, y))}{2^p}, v \in H_{n-1}(x, y)\Big\}, \quad n = 2, 3, \ldots.$$

Here $\delta(H_n(x, y))$ denotes the diameter of $H_n(x, y)$, which is the supremum of distances between pairs of its elements. Clearly, $\delta(H_n(x, y)) \leq 2(1/2^p)^n\|x - y\|$. It follows from the proof of [6, Lemma 1.3.1] that the intersection of these sets $H_n(x, y)$ consists of a single point $(x + y)/2$ called the *metric centre* of x and y. If $x \in X$ is a vector in X, we denote by $B(x, 1)$ the set of vectors $u \in X$ such that $\|u - x\| \leq 1$.

THEOREM 2.10. *Let X and Y_p be p-normed spaces with $\dim X \geq 2$, where Y is a normed space. Suppose that $f : X \to Y_p$ is a 1-Lipschitz mapping from $B(x, 1)$ onto $B(f(x), 1)$ for all $x \in X$. If f satisfies DOPP, then f is an affine isometry of X onto Y_p.*

PROOF. Let $x, y \in X$ with $0 < \|x - y\| \leq 1$ and set $z := x + (y - x)/\|y - x\|^{1/p}$. Then, clearly, $\|z - x\| = 1$ and $\|z - y\| = (1 - \|x - y\|^{1/p})^p$. Note that $(Y, \|\cdot\|)$ is a normed space and $f : X \to Y_p$ is a 1-Lipschitz mapping. It follows that

$$1 = \|f(z) - f(x)\| \leq \|f(z) - f(y)\| + \|f(y) - f(x)\| \leq \|z - y\|^{1/p} + \|y - x\|^{1/p} = 1.$$

This implies that $\|f(x) - f(y)\|^p = \|x - y\|$ for all $x, y \in X$ with $\|x - y\| \leq 1$.

https://doi.org/10.1017/S0004972716000927 Published online by Cambridge University Press

Let $x, y \in X$ be two points with $\|x - y\| = 1$ and set $z := (x + y)/2$. By induction, $f(H_n(x, y)) = H_n(f(x), f(y))$ for every positive integer n (see [6, Lemma 1.3.2]). It follows that $f(z)$ is the unique element of $f(\bigcap_{n=1}^{\infty} H_n(x, y)) = \bigcap_{n=1}^{\infty} H_n(f(x), f(y))$, which is $\frac{1}{2}(f(x) + f(y))$. By Lemma 2.5, the mapping f is indeed an isometry. By the same reasoning as before, f maps the midpoint of the line segment joining x and y onto the midpoint of the line segment joining $f(x)$ and $f(y)$ for all x and y in X. Since isometries are continuous, f is affine. This completes the proof. □

Remark 2.11. Let X and Y be real normed spaces with $\dim X \geq 2$ and $f : X \to Y$ a surjective mapping satisfying SDOPP. Rassias and Šemrl [13, Theorem 1] showed that f is a mapping from $B(x, n)$ onto $B(f(x), n)$ and preserves distance n in both directions for any positive integer n. But we do not know whether these results hold in p-normed space. However, by applying Theorem 2.10, we have an alternative proof of the Rassias–Šemrl theorem on real normed space.

References

[1] A. D. Aleksandrov, 'Mappings of families of sets', *Soviet Math. Dokl.* **11** (1970), 116–120.
[2] J. A. Baker, 'Isometries in normed spaces', *Amer. Math. Monthly* **78** (1971), 655–658.
[3] F. S. Beckman and D. A. Quarles, 'On isometries of Euclidean spaces', *Proc. Amer. Math. Soc.* **4** (1953), 810–815.
[4] W. Benz, 'Isometrien in normierten Räumen', *Aequationes Math.* **29** (1985), 204–209.
[5] W. Benz and H. Berens, 'A contribution to a theorem of Ulam and Mazur', *Aequationes Math.* **34** (1987), 61–63.
[6] R. J. Fleming and J. E. Jameson, *Isometries on Banach Spaces: Function Spaces* (Chapman and Hall/CRC, Boca Raton, FL, 2003).
[7] G. P. Gehér, 'A contribution to the Aleksandrov conservative distance problem in two dimensions', *Linear Algebra Appl.* **481** (2015), 280–287.
[8] R. Juhász, 'Another proof of the Beckman–Quarles theorem', *Adv. Geom.* **15**(4) (2015), 519–521.
[9] G. Köthe, *Topological Vector Spaces I* (Springer, Berlin, 1969).
[10] J. M. Ling, 'A normed space with the Beckman–Quarles property', *Contrib. Algebra Geom.* **48**(1) (2007), 131–139.
[11] Y. Ma, 'The Aleksandrov problem for unit distance preserving mapping', *Acta Math. Sci.* **20** (2000), 359–364.
[12] S. Mazur and S. Ulam, 'Sur les transformations isométriques d'espaces vectoriels normés', *C. R. Math. Acad. Sci. Paris* **194** (1932), 946–948.
[13] Th. M. Rassias and P. Šemrl, 'On the Mazur–Ulam problem and the Aleksandrov problem for unit distance preserving mappings', *Proc. Amer. Math. Soc.* **118** (1993), 919–925.
[14] J. E. Valentine, 'Some implications of Euclid's Proposition 7', *Math. Japon.* **28** (1983), 421–425.

XUJIAN HUANG, Department of Mathematics,
Tianjin University of Technology, 300384 Tianjin, China
e-mail: huangxujian86@sina.cn

DONGNI TAN, Department of Mathematics,
Tianjin University of Technology, 300384 Tianjin, China
e-mail: tandongni0608@sina.cn