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THE POSET OF PERFECT IRREDUCIBLE IMAGES OF A SPACE

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1. Introduction. We begin by briefly summarizing the contents of this paper; details, and some definitions of terminology, appear in subsequent sections. All hypothesized topological spaces are assumed to be Hausdorff. The reader is referred to [**13**] for undefined notation and terminology.

A perfect irreducible continuous surjection is called a *covering map*. Let X be a space, let f and g be two such functions with domain X, and let Rf denote the range of f (i.e., the set $f[X]$). Then f and g are said to be *equivalent* (denoted $f \approx g$) if there is a homeomorphism $h : Rf \longrightarrow Rg$ such that $h \circ f = g$. We identify equivalent covering maps with domain X, and then denote by $IP(X)$ the set of such covering maps. (Note that $|IP(X)| \leqq 2^{|X|}$.) A partial order $\leqq$ can be defined on $IP(X)$ as follows: $g \leqq f$ if there exists a continuous function $h : Rf \longrightarrow Rg$ such that $h \circ f = g$. (The antisymmetry of $\leqq$ follows from the fact that we have identified equivalent covering maps.) It turns out that $(IP(X), \leqq)$ is a complete upper semilattice. Our principal result is the following:

THEOREM 1.1. *Let X and Y be k-spaces without isolated points. Then $(IP(X), \leqq)$ and $(IP(Y), \leqq)$ are order-isomorphic if and only if X and Y are homeomorphic.*

In fact we prove a generalization of this (Theorem 3.10) that is not as succinctly expressed. This generalization has Magill's theorem (see 1.4) as a corollary.

The remainder of the paper (Section 5) contains partial results concerning when $IP(X)$ is a lattice. It is already known that $IP(X)$ is a complete lattice if and only if the set of non-isolated points of X is compact and nowhere dense (see 5.1). We show that if X is not countably compact, or is a compact metric space without isolated points, then $IP(X)$ is not a lattice.

Evidently 1.1 describes a situation in which the topology of a space is determined by the order structure of an associated family of mappings. Theorems of this sort are not new; for twenty years topologists have been studying the order structure of families of extensions of a space, and obtaining theorems like 1.1. Our investigations were motivated by a desire to see if similar results could be obtained by considering a naturally occurring, but quite different, poset associated with a topological space. To set the stage for what follows, we briefy summarize the theory of extensions and quote some of the above-mentioned theorems. See 4.1 of [**13**] for more details.

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Definition 1.2. (a) An *extension* of a space X is a pair (i, S) where S is a space and $i : X \longrightarrow S$ is a dense embedding.

(b) Two extensions (i, S) and (j, T) of X are *equivalent* if there is a homeomorphism $h : S \longrightarrow T$ for which $h \circ i = j$.

(Note that if X is a dense subspace of S and of T, and if i and j are inclusion maps, then (b) above reduces to requiring that $h|X$ be the identity on X.)

Henceforth we identify equivalent extensions of X. Let $\mathcal{E}(X)$ denote the set of extensions of X. (Note that since equivalent extensions are identified, and since we consider only Hausdorff spaces, we have $|\mathcal{E}(X)| \leqq \exp(\exp(\exp|X|))$.) A partial order $\leqq$ can be defined on $\mathcal{E}(X)$ as follows: $(j, T) \leqq (i, S)$ if there is a continuous function $h : S \longrightarrow T$ such that $h \circ i = j$. (The identification of equivalent extensions of X enables us to prove that $\leqq$ is antisymmetric.) The following results are well-known (see 5.3(c) of [**13**], for example; (b) is essentially due to Herrlich and van der Slot [**8**].

THEOREM 1.3. *Let X be a space. Then:*

(a) $(\mathcal{E}(X), \leqq)$ *is a complete upper semilattice.*

(b) *If* $\mathcal{P}$ *is a closed-hereditary, productive topological property, and if* $\mathcal{E}_{\mathcal{P}}(X)$ *is defined to be*

$$\{(i, S) \in \mathcal{E}(X) : S \text{ has } \mathcal{P}\},$$

then $(\mathcal{E}_{\mathcal{P}}(X), \leqq)$ *is also a complete upper semilattice (provided it is non-empty).*

Suppose that $\mathcal{P}$ is as in 1.3(b) above, and let $(i, \gamma_{\mathcal{P}} X)$ denote the largest member of $\mathcal{E}_{\mathcal{P}}(X)$. Many authors have investigated the relationship between the order structure of $\mathcal{E}_{\mathcal{P}}(X)$ and the topological structure of $\gamma_{\mathcal{P}} X \backslash X$. The earliest and best theorem of this sort is due to Magill [**10**], as follows: βX denotes the Stone-Čech compactification of K. Let $\mathcal{K}$ denote the property of being compact.

THEOREM 1.4. *Let X and Y be locally compact spaces. Then* $\mathcal{E}_{\mathcal{K}}(X)$ *and* $\mathcal{E}_{\mathcal{K}}(Y)$ *are order-isomorphic if and only if* $\beta X \backslash X$ *and* $\beta Y \backslash Y$ *are homeomorphic.*

Other results in a similar vein appear in [**11**], [**14**], and [**16**]. The results we will obtain in Section 3 below bear a strong resemblance to these results.

2. The poset of covering maps with fixed domain. In this section we assemble some preliminary results on $IP(X)$. Our new results appear in Sections 3 to 5. Recall (see [**13**] or [**17**]) that a function $f : X \longrightarrow Y$ is *perfect* if it is closed and point-inverses are compact. A perfect function $f : X \longrightarrow Y$ is *irreducible* if $f[X] = Y$ but $f[A] \neq Y$ if A is closed in X and $X \backslash A \neq \emptyset$. *Covering maps* are perfect continuous irreducible surjections. (Covering maps are important in the study of absolutes and their generalizations; see chapter 6 of [**13**], [**15**], [**2**], and [**17**]). In Section 1 we introduced the notion of equivalent covering maps (with common domain) and defined $IP(X)$ to be the set of covering maps with domain X (with equivalent maps identified). We defined a relation $\leqq$ on $IP(X)$

and asserted that it is a partial order with respect to which $IP(X)$ is a complete upper semilattice. We now investigate these claims in more detail.

THEOREM 2.1. *Let $g : X \longrightarrow Z$ and $h : Z \longrightarrow Y$ be continuous surjections. Then:*

(a) *$h \circ g : X \longrightarrow Z$ is perfect if and only if h and g are perfect.*

(b) *$h \circ g$ is a covering map if and only if h and g are.*

(Part (a) is 3.7.3 and 3.7.10 of [**3**], and (b) is a straightforward consequence of (a)).

COROLLARY 2.2. *Let X be a space and $f, g \in IP(X)$. Then $f \leqq g$ if and only if there exists a covering map $h : Rf \longrightarrow Rg$ such that $h \circ f = g$.*

Proof. This follows immediately from the definition of $\leqq$ on $IP(X)$ (see Section 1) and from 2.1(b).

Note that precisely one member of $IP(X)$ is a homeomorphism onto its range; this is the largest member of $(IP(X), \leqq)$. We will assume that this member is the identity function id_X on X.

THEOREM 2.3. *Let X be a space. Then $(IP(X), \leqq)$ is a complete upper semilattice.*

Sketch of proof. First verify that $\leqq$ is a partial order. Reflexivity is trivial. Transitivity is an immediate consequence of 2.1(b). To prove antisymmetry suppose $f, g \in IP(X)$, $f \leqq g$, and $g \leqq f$. By 2.2 there exist covering maps $h : Rf \longrightarrow Rg$ and $k : Rg \longrightarrow Rf$ such that $h \circ f = g$ and $k \circ g = f$. Thus $h \circ k \circ g = g$ and as g is surjective, $h \circ k = \mathrm{id}_{Rg}$. Similarly $k \circ h = \mathrm{id}_{Rf}$, and so k and h are homeomorphisms. As equivalent covering maps are identified, $f = g$.

Now we must show that each non-empty subset of $(IP(X), \leqq)$ has a least upper bound. This is essentially the contents of 3.3 of [**7**]; also see 8.4(f) of [**13**].

There is an order isomorphism from $IP(X)$ onto a certain set of partitions of X, and this correspondence will be useful in what follows.

Definition 2.4. (a) A *covering partition* of a space X is an upper semicontinuous partition $\mathcal{P}$ of X into compact sets such that if V is a non-empty open set of X, then there exists $P \in \mathcal{P}$ such that $P \subseteq V$.

(b) If f is a covering map with domain X, define $\mathcal{P}(f)$ to be

$$\{f^{\leftarrow}(y) : y \in Rf\}$$

and define $\mathcal{P}_2(f)$ to be

$$\{A \in \mathcal{P}(f) : |A| \geqq 2\}.$$

(c) if $\mathcal{A}$ is a covering partition of X, define $\varphi_{\mathcal{A}} : X \longrightarrow \mathcal{A}$ by defining $\varphi_{\mathcal{A}}(x)$ to be the unique member of $\mathcal{A}$ to which x belongs.

THEOREM 2.5. *(a) Let f and g be covering maps with domain X. Then f and g are equivalent (as defined in Section 1) if and only if $\mathcal{P}(f) = \mathcal{P}(g)$.*

(b) *The map $f \longrightarrow \mathcal{P}(f)$ is a bijection from $IP(X)$ onto the set $\mathcal{S}(X)$ of all covering partitions of X. If $\mathcal{S}(X)$ is partially ordered by "is refined by", then this map is an order isomorphism; explicitly, $f \leqq g$ if and only if $\mathcal{P}(f)$ is refined by $\mathcal{P}(g)$.*

Sketch of proof. (a) One direction is obvious. For the other, suppose $\mathcal{P}(f) = \mathcal{P}(g)$. If $x \in Rf$, define $h(x)$ to be the unique point y of Rg for which $f^{\leftarrow}(x) = g^{\leftarrow}(y)$. It is straightforward to prove that h is a homeomorphism for which $h \circ f = g$.

(b) By (a) the map is one-to-one. To show it is onto, let $\mathcal{A}$ be a covering partition of X and give $\mathcal{A}$ the quotient topology induced by $\varphi_{\mathcal{A}}$. Then $\varphi_{\mathcal{A}}$ is easily seen to be a covering map, and hence equivalent to, and hence the same as, a member of $IP(X)$. Obviously $\mathcal{P}(\varphi_{\mathcal{A}}) = \mathcal{A}$, so our map is onto. The preservation of order is easily verified.

Note that an extension of X was defined to be a pair (i, S) consisting of a space S and a dense embedding $i : X \longrightarrow S$. To emphasize further the analogy between $\mathcal{E}(X)$ (defined in Section 1) and $IP(X)$, we could have defined members of $IP(X)$ to be ordered pairs (f, S), where $f : X \longrightarrow S$ is a covering map and S is a space. This seems unnecessarily complicated and hence was not done.

When we generalize 1.1 we will wish to consider certain subsets of $IP(X)$, as follows.

Definition 2.6. Let U be an open subset of a space X. Then $IP(X, U)$ is defined to be

$$\{f \in IP(X) : \forall x \in U, |f^{\leftarrow}(f(x))| = 1\}.$$

Note that $IP(X) = IP(X, \emptyset)$. Furthermore, observe that if $g \in IP(X, U), f \in IP(X)$, and $g \leqq f$, then $f \in IP(X, U)$. Hence we infer:

THEOREM 2.7. *Let U be an open subset of X. If $\phi \neq \mathcal{G} \subseteq IP(X, U)$, then $\vee \mathcal{G}$ (the supremum of $\mathcal{G}$ in $IP(X)$) belongs to $IP(X, U)$. In particular, $IP(X, U)$ is a complete upper semilattice with respect to the order defined on $IP(X)$.*

3. The main results. The topology of a space X obviously determines the order structure of $IP(X)$; i.e., if X and Y are spaces and $h : X \longrightarrow Y$ is a homeomorphism, then there is an order-isomorphism

$$\varphi : IP(X) \longrightarrow IP(Y).$$

We want to know when the converse is true. In other words, suppose X and Y are spaces and

$$\varphi : IP(X) \longrightarrow IP(Y)$$

is an order isomorphism. Does it follow that X and Y are homeomorphic? Theorem 1.1, together with some limiting examples, provides an answer.

Example 3.1. Let $i(X)$ denote the set of isolated points of the space X. It is easily verified that if $x \in i(X)$ and $f \in IP(X)$ then

$$|f^{\leftarrow}(f(x))| = 1.$$

Hence if $|X \backslash i(X)| \leqq 1$, it follows that $IP(X)$ contains only one element, namely id_X. Hence for any two such spaces X and Y, $IP(X)$ and $IP(Y)$ are trivially order-isomorphic. Let IN be the countably infinite discrete space, D the discrete space of cardinality $\aleph_1$, αIN (resp. αD) the one-point compactification of IN (resp. D), and LD the one-point Lindelöf extension of D (i.e., $LD = D \cup \{p\}$, and neighborhoods of p are $\{p\} \cup A$, where $|D \backslash A| \leqq \aleph_0$). Then $IP(\alpha IN)$, $IP(\alpha D)$, and $IP(LD)$ are order-isomorphic while αIN, αD, and LD are pairwise non-homeomorphic.

The above example makes it clear that X cannot have a lot of isolated points if the order structure of $IP(X)$ is to determine the topology of X. We now turn to some positive results.

Definition 3.2. A bijection f from a space X onto a space Y is called a *cn-bijection* if $\{f[A] : A$ is a compact nowhere dense subset of $X\} = \{B : B$ is a compact nowhere dense subset of $Y\}$.

Our proof of 1.1 can be split into three parts, as follows. First we show that if X and Y have no isolated points, and if $IP(X)$ and $IP(Y)$ are order isomorphic, then there is a *cn*-bijection from X onto Y (see 3.5). The proof of this is sketched only, as it is essentially identical to Magill's proof of 1.4. Second, we show that if X and Y are compact spaces without isolated points, then a cn-bijection from X to Y is a homeomorphism. Finally, we prove the same assertion for k-spaces (rather than compact spaces). The proofs of the latter two assertions involve a number of new techniques.

Definition 3.3. Let X be a space without isolated points and let $f \in IP(X)$.

(a) f is *primary* in $IP(X)$ if $\mathcal{P}(f)$ has at most one non-singleton member.

(b) f is *dual* in $IP(X)$ if f is primary and $\mathcal{P}(f)$ contains (precisely) one doubleton.

One can easily adapt the proofs of Lemmas 9 and 10 of **[10]** to prove:

PROPOSITION 3.4. *Let X be a space, let $f \in IP(X)$, and $f \neq \mathrm{id}_X$. Then:*

(a) *f is dual if and only if there is no $g \in IP(X)$ such that $f < g < \mathrm{id}_X$.*

(b) *f is primary if and only if whenever g and h are distinct dual members of $IP(X)$ for which $f \wedge g = f \wedge h \neq f$, then*

$$|\{k \in IP(X) : k \geqq g \wedge h\}| = 5.$$

Note that one consequence of 3.4 is that dual and primary members of $IP(X)$ can be characterized in purely order-theoretic terms.

Using 3.4 and exactly the same techniques used in the proof of theorem 12 of [**10**], we obtain the following result (whose proof we do not include).

THEOREM 3.5. *Let X and Y be spaces without isolated points, and let $\varphi : IP(X) \longrightarrow IP(Y)$ be an order isomorphism. Then there is a cn-bijection $F : X \longrightarrow Y$ such that if $f \in IP(X)$, then*

$$\mathcal{P}(\varphi(f)) = \{F[A] : A \in \mathcal{P}(f)\}.$$

It is clear from 3.5 that in order to prove 1.1 we need only show that the topology of a k-space without isolated points is completely determined by its family of compact nowhere dense subsets. To this end, we now show that cn-bijections between k-spaces without isolated points are homeomorphisms. (Recall that a space X is a *k-space* if whenever $A \subseteq X$ and $A \cap K$ is closed in X for each compact subspace K of X, then if follows that A is closed in X.) We begin with a technical lemma and then give a characterization of k-spaces without isolated points.

Originally this lemma was proved only for compact spaces, and by a different means. We are grateful to the referee for suggesting the proof below, which applies to countably compact T_3 spaces.

LEMMA 3.6. *Let X be a countably compact T_3 space without isolated points. Suppose A is a subset of X satisfying this condition.*

(*) *If $B \subseteq A$ and $\text{cl}_X B$ is nowhere dense in X then $\text{cl}_X B \subseteq A$.*

Then A is a closed subset of X.

Proof. Suppose $x \in \text{cl}_X A \backslash A$. Let $\mathcal{V}$ be a maximal family of pairwise disjoint non-empty open sets of A for which

$$x \notin \cup\{\text{cl}_X U : U \in \mathcal{V}\}.$$

Let V be an open set containing x, and let W be open such that $x \in W \subseteq \text{cl}_X W \subseteq V$. Let

$$\mathcal{W} = \{U \in \mathcal{V} : W \cap U \neq \emptyset\}.$$

If $\mathcal{W}$ is finite, then

$$x \in W \backslash \cup \{\text{cl}_X U : U \in \mathcal{W}\} = T.$$

Let $a \in T \cap A$; as X is Hausdorff there exists an open set S of X such that

$$x \in S \subseteq \text{cl}_X S \subseteq T \quad \text{and} \quad a \in T \backslash \text{cl}_X S.$$

Then $\mathcal{V} \subset \mathcal{V} \cup \{S \cap A\}$, contradicting the maximality of $\mathcal{V}$. Thus $\mathcal{W}$ is infinite. Choose $x_U \in W \cap U$ for each $U \in \mathcal{W}$, and let

$$D = \{x_U : U \in \mathcal{W}\}.$$

Then D is a discrete subset of A, and as X has no isolated points, it follows that $\mathrm{cl}_X D$ is nowhere dense. Thus by hypothesis $\mathrm{cl}_X D \subseteq A$. As X is countably compact, there exists $y \in \mathrm{cl}_X D \backslash D$. Evidently $y \in A \backslash \cup \mathcal{V}$, and since $\mathrm{cl}_X D \subseteq \mathrm{cl}_X W$, it follows that

$$V \cap (A \backslash \cup \mathcal{V}) \neq \emptyset.$$

Thus

$$x \in \mathrm{cl}_X (A \backslash \cup \mathcal{V}).$$

The maximality of $\mathcal{V}$ implies that $\mathrm{cl}_X (A \backslash \cup \mathcal{V})$ is nowhere dense, so by hypothesis

$$\mathrm{cl}_X (A \backslash \cup \mathcal{V}) \subseteq A.$$

Hence $x \in A$, which is a contradiction. Hence A is closed as claimed.

Examples 3.7. (a) The referee has pointed out that 3.6 fails if "countably compact" is replaced by "pseudocompact". To see this, let T denote the set of remote points of $I\!R$. Then T is dense in $\beta I\!R \backslash I\!R$ (see 4.2 of [**1**]) and so $Y = I\!R \cup T$ is pseudocompact (see 3.1 of [**4**] and 17.1(d) of [**1**]). By definition of "remote point", no point of T is in the Y-closure of a closed nowhere dense subset of $I\!R$. Hence $I\!R$ satisfies the hypothesis on A in 3.6, yet $I\!R$ is not closed in Y.

(b) Lemma 3.6 also fails if "countably compact" is replaced by "H-closed and semiregular"; the space $(\beta Q)(D^2)$ discussed in example 8 of [**12**] provides a counterexample.

Now we establish that the compact nowhere dense subsets of a k-space without isolated points completely determine its topology.

THEOREM 3.8. *Let X be a space without isolated points. The following are equivalent*

(a) *X is a k-space.*

(b) *If $X \subseteq X$ and if $\mathrm{cl}_X B \subseteq A$ whenever $B \subseteq A$ and $\mathrm{cl}_X B$ is compact and nowhere dense, then A is closed in X.*

(c) *If A is a subset of X such that $A \cap K$ is closed in X for each compact nowhere dense subset K of X, then A is closed in X.*

Proof. (c) $\Rightarrow$ (c) This is trivial.

(b) $\Rightarrow$ (c) Suppose that $A \subseteq X$ and that $A \cap K$ is closed in K whenever K is a compact nowhere dense subset of X. Suppose that $B \subseteq A$ and that

$\mathrm{cl}_X B$ is compact and nowhere dense. By hypothesis $A \cap \mathrm{cl}_X B$ is closed in X; as $B \subseteq A \cap \mathrm{cl}_X B$, it follows that $\mathrm{cl}_X B \subseteq A$. It follows from (b) that A is closed in X. Hence (c) follows.

(a) $\Rightarrow$ (b) Suppose that X is a k-space without isolated points, $A \subseteq X$, and that $\mathrm{cl}_X B \subseteq A$ whenever $B \subseteq A$ and $\mathrm{cl}_X B$ is compact and nowhere dense. We must show that A is closed in X; by hypothesis, it suffices to show that if L is a compact subset of X then $A \cap L$ is closed in X. Let

$$K = \mathrm{cl}_L(L \backslash \mathrm{cl}_L i(L))$$

(see 3.1 for notation). Then K is a compact subset of L with no isolated points. Observe that

$$(1) \qquad A \cap L = [A \cap \mathrm{cl}_L i(L)] \cup [A \cap K].$$

We claim that $A \cap K$ is closed in X. As K is compact it suffices to show that $A \cap K$ is closed in K. To show this, by 3.6 it suffices to show that if $B \subseteq A \cap K$ and $\mathrm{cl}_K B$ is nowhere dense in K, then $\mathrm{cl}_K B \subseteq A \cap K$. If $B \subseteq A \cap K$ and $\mathrm{cl}_K B$ is nowhere dense in K, then $\mathrm{cl}_K B = \mathrm{cl}_X B$ (as K is compact) and so $\mathrm{cl}_X B$ is nowhere dense in X. By hypothesis on A, $\mathrm{cl}_X B \subseteq A$, i.e., $\mathrm{cl}_K B \subseteq A$. Obviously $\mathrm{cl}_K B \subseteq K$, so $\mathrm{cl}_K B \subseteq A \cap K$. Hence $A \cap K$ is closed in X as claimed above.

Let $M = \mathrm{cl}_L i(L)$. Arguing as in 3.6 we see that the compact set M is nowhere dense in X. Thus $\mathrm{cl}_X(A \cap M)$ is a compact nowhere dense subset of X. By hypothesis on A,

$$\mathrm{cl}_X(A \cap M) \subseteq A.$$

Thus

$$\mathrm{cl}_X(A \cap M) \subseteq A \cap M$$

and so $A \cap M$ is closed in X. By (1) $A \cap L$ is the union of two closed subsets of X and hence is closed in X. The theorem follows.

We now can prove the main result of this section.

Proof of 1.1. Obviously if X and Y are homeomorphic then $IP(X)$ and $IP(Y)$ are order-isomorphic. Conversely, suppose $IP(X)$ and $IP(Y)$ are order-isomorphic. As X and Y have no isolated points, by 3.5 there is a cn-bijection $f : X \longrightarrow Y$ (obviously $f^{\leftarrow} : Y \longrightarrow X$ is also a cn-bijection). We will prove that f is a closed map. By symmetry $f^{\leftarrow}$ will also be closed, and hence f will be a homeomorphism.

Let $M \subseteq X$ and $\mathrm{cl}_X M$ be compact and nowhere dense. First we show that

$$f[\mathrm{cl}_X M] = \mathrm{cl}_Y f[M].$$

Since $f[\mathrm{cl}_X M]$ is compact and nowhere dense, then

$$\mathrm{cl}_Y f[M] \subseteq f[\mathrm{cl}_X M]$$

and $\mathrm{cl}_Y f[M]$ is compact and nowhere dense. So, $f^{\leftarrow}[\mathrm{cl}_Y f[M]]$ is compact and nowhere dense. Since

$$M \subseteq f^{\leftarrow}[\mathrm{cl}_Y f[M]],$$

it follows that

$$\mathrm{cl}_X M \subseteq f^{\leftarrow}[\mathrm{cl}_Y f[M]]$$

and that

$$f[\mathrm{cl}_X M] \subseteq \mathrm{cl}_Y f[M].$$

Next, we show that f is closed. Let C be a closed subset of X. Suppose that $B \subseteq f[C]$ and that $\mathrm{cl}_Y B$ is compact and nowhere dense. By 3.8 (a) and (b), since Y is a k-space without isolated points, to show $f[C]$ is closed in Y it suffices to show that $\mathrm{cl}_Y B \subseteq f[C]$. By the assertion at the beginning of the paragraph,

$$f^{\leftarrow}[\mathrm{cl}_Y B] = \mathrm{cl}_X f^{\leftarrow}[B].$$

As $f^{\leftarrow}[B] \subseteq C$ and C is closed, it follows that

$$f^{\leftarrow}[\mathrm{cl}_Y B] \subseteq C.$$

Hence, $\mathrm{cl}_Y B \subseteq f[C]$. This completes the proof that f is closed and by symmetry, $f^{\leftarrow}$ is closed. Thus, f is a homeomorphism.

The above proof essentially consisted of showing that a cn-bijection between two k-spaces without isolated points is of necessity a homeomorphism. To show that this result can fail if the spaces involved are not k-spaces, we present the following example.

Example 3.9. Recall (see [**1**]) that r is a remote point of Q (the space of rational numbers) if

$$r \in \beta Q \backslash \cup \{\mathrm{cl}_{\beta Q} A : A \text{ is a closed, nowhere dense subset of } Q\}.$$

It is known [**1**] that

$$|\{r \in \beta Q \backslash Q : r \text{ is a remote point of } Q\}| = \exp(\exp \aleph_0).$$

Since

$$|\{h : Q \longrightarrow Q : h \text{ is a homeomorphism}\}| \leqq \exp(\aleph_0),$$

there are remote points p and q of Q such that if $h : Q \longrightarrow Q$ is a homeomorphism and $h^\beta : \beta Q \longrightarrow \beta Q$ is the continuous extension of h, then $h^\beta(p) \neq q$. Define

$$f : Q \cup \{p\} \longrightarrow Q \cup \{q\}$$

by $f(x) = x$ for $x \in Q$ and $f(p) = q$. Let C be a closed nowhere dense in $Q \cup \{p\}$. If $p \notin C$, then

$$f[C] = C \subseteq Q \quad \text{and} \quad q \notin \mathrm{cl}_{Q\cup\{q\}} C.$$

So, C is closed and nowhere dense in $Q \cup \{q\}$. If $p \in C$, then $f[C] = (C \cap Q) \cup \{q\}$ is closed and nowhere dense in $Q \cup \{q\}$. If C is also compact, then so is $f[C]$. A similar argument shows that $f^{\leftarrow}$ preserves compact, nowhere dense sets. However, f is not a homeomorphism; in fact, $Q \cup \{p\}$ and $Q \cup \{q\}$ are not homeomorphic. Thus cn-bijections between spaces without isolated points need not be homeomorphisms.

The similar form of 1.1 and 1.4 suggests that they might have a common generalization. In fact they do, but its statement is rather cumbersome. We briefly sketch this generalization.

Suppose that X is a space, U is an open set of X (possibly empty), and $i(X) \subseteq U$. As noted in 2.7, $IP(X, U)$ is a complete upper semilattice. The same technique of proof that we indicated for 3.5 can be used to prove the following:

THEOREM 3.10. *Let X_i be a space, U_i be open in X_i, and $i(X_i) \subseteq U_i (i = 1, 2)$. Suppose*

$$\varphi : IP(X_1, U_1) \longrightarrow IP(X_2, U_2)$$

is an order isomorphism. Then there is a bijection

$$F : X_1 \backslash U_1 \longrightarrow X_2 \backslash U_2$$

such that $\{F[A] : A$ is a compact nowhere dense subset of X_1 and $A \subseteq X_1 \backslash U_1\} = \{B : B$ is a compact nowhere dense subset of X_2 and $B \subseteq X_2 \backslash U_2\}$; and if $f \in IP(X_1, U_1)$ then

$$\mathcal{P}(\varphi(f)) = (\{x\} : x \in U_2\} \cup \{F(A) : A \in \mathcal{P}(F) \text{ and } A \subseteq X_1 \backslash U_1\}.$$

Note that 3.5 is the special case of 3.10 in which $U_1 = U_2 = \emptyset$.

LEMMA 3.11. *Let X be a locally compact space. The function*

$$\psi : IP(\beta X, X) \longrightarrow \mathcal{E}_{\mathcal{K}}(X)$$

defined by

$$\psi(f) = \beta X / \mathcal{P}(f)$$

is an order isomorphism (here $\beta X/\mathcal{P}(f)$ is the obvious quotient space of βX). (See 1.3 for notation.)

Proof. Define $\varphi : IP(\beta X, X) \mathcal{S}(X)$ (the set of covering partitions of βX) as follows: $\varphi(f) = \mathcal{P}(f)$. It follows quickly from 2.5(b) that φ is an order isomorphism onto its image. By Lemma 1 of [**10**], there is an order isomorphism

$$\lambda : \varphi[IP(\beta X, X)] \longrightarrow \mathcal{E}_{\mathcal{K}}(X).$$

Then $\lambda \circ \varphi$ is the required ψ.

We can now deduce the non-trivial half of 1.4 as follows. If X and Y are locally compact and if $\mathcal{E}_{\mathcal{K}}(X)$ and $\mathcal{E}_{\mathcal{K}}(Y)$ are order-isomorphic, then $IP(\beta X, X)$ and $IP(\beta Y, Y)$ are order-isomorphic by 3.11. By 3.10 there is a bijection

$$F : \beta X \backslash X \longrightarrow \beta Y \backslash Y$$

such that A is a compact nowhere dense subset of βX contained in $\beta X \backslash X$ if and only if $F[A]$ is a compact nowhere dense subset of βY contained in $\beta Y \backslash Y$. As all closed subsets of $\beta X \backslash X$ are nowhere dense in $\beta X, F$ is a closed map. By symmetry so is $F^{\leftarrow}$, and so F is a homeomorphism.

Hence 1.1 and 1.4 can both be viewed as consequences of 3.10.

4. Other uses of $IP(X)$**.** By 3.5, we know that if X and Y are spaces without isolated points and if $IP(X)$ and $IP(Y)$ are order-isomorphic, then there is a *cn*-bijection between X and Y. In this section, we show that the converse is false. We are thankful to the referee for many suggestions, some of which led to the results in this section.

Let X be an infinite space without isolated points and let $IP_d(X)$ denote $\{f \in IP(X) : \mathcal{P}_2(f)$ contains only one member, and this member is a doubleton$\}$. An *ideal point of* $IP_d(X)$ is a subset $\mathcal{A}$ of $IP_d(X)$ satisfying (1) $f, g \in \mathcal{A}$ implies $f \wedge g$ exists and is primary and (2) $\mathcal{A}$ is maximal with respect to (1). The set of ideal points, denoted by $S(X)$, is determined by the poset $IP(X)$.

For each $x \in X$, let

$$e(x) = \{f \in IP_d(X) : x \in \cup \mathcal{P}_2(f)\}.$$

Another path to establishing 3.5 is to start with the next result which is easy to verify.

PROPOSITION 4.1. *Let X be an infinite space without isolated points. Then e is a bijection from X onto $S(X)$ such that if $x, y \in X$ are distinct, then $e(x) \cap e(y)$ is a singleton and if $f \in e(x) \cap e(y)$, then $\cup \mathcal{P}_2(f) = \{x, y\}$.*

For $B \subseteq S(X)$, let $B^* = \{f \in IP_d(X)$: there are $\mathcal{A}, \mathcal{A}' \in B$ such that $\mathcal{A} \neq \mathcal{A}'$ and $f \in \mathcal{A} \cap \mathcal{A}'\}$. Now, we show that the order structure of $IP(X)$ determines the compact nowhere dense subsets and the convergent sequences of X.

PROPOSITION 4.2. *Let X be an infinite space without isolated points and let $A \subseteq X$ have at least two points.*

(a) *A is nowhere dense and has compact closure if and only if $(e[A])^*$ has a primary lower bound.*

(b) *A is compact and nowhere dense if $(e[A])^*$ has a primary lower bound and if $g \in IP_d(X)$ and $g \geqq \wedge(e[A])^*$, then $g \in (e[A])^*$.*

(c) *Let $A = \{x_n : n \in \omega\} \cup \{y\}$ where $y \neq x_n$ for $n \in \omega$. Then $(x_n) \longrightarrow y$ if and only if A and $A \backslash \{x_n\}$ are compact and nowhere dense sets for $n \in \omega$ and $A \backslash \{y\}$ is not compact and nowhere dense.*

Proof. The proof is straightforward and left to the reader.

Example 4.3. Consider the infinite space $X = \mathbf{Q} \cup \{p\}$ where $p \in (\mathrm{cl}_{\beta Q}\mathbf{N}) \backslash \mathbf{Q}$ and the infinite space $Y = \mathbf{Q} \cup \{q\}$ where q is a remote point. The function which takes p to q and is the identity on $\mathbf{Q}$ is a cn-bijection from X onto Y (use the same argument as in 3.9).

Assume there exists an order-isomorphism $\varphi : IP(X) \longrightarrow IP(Y)$. By 3.5, φ induces a cn-bijection $F : X \longrightarrow Y$ such that

$$\mathcal{P}(\varphi(f)) = \{F[A] : A \in \mathcal{P}(f) \text{ for } f \in IP(X)\}.$$

By 4.2, we have that $(r_n) \longrightarrow r$ in X if and only if $(F(r_n)) \longrightarrow F(r)$ in Y. Since no sequence converges to p or q and sequences converge to every point of $\mathbf{Q}$, it follows that $F[\mathbf{Q}] = \mathbf{Q}$ and $F(p) = q$. Also, $F|\mathbf{Q} : \mathbf{Q} \longrightarrow \mathbf{Q}$ is a cn-bijection; by the proof of 1.1, $F|\mathbf{Q}$ is a homeomorphism. Thus, $F[\mathbf{N}]$ is a closed and discrete subspace of $\mathbf{Q}$ and hence of Y since q is a remote point. For each $n \in \omega$, let $f_n \in IP_d(X)$ such that $\mathcal{P}_2(f_n) = \{\{2n, 2n+1\}\}$. Assume that $\wedge\{f_n : n \in \omega\}$ exists and is denoted by f. Since the elements of $\mathcal{P}_2(f)$ are compact and only finite subsets of $\mathbf{N}$ have compact closure in X, it follows that an element of $\mathcal{P}_2(f)$ meets $\mathbf{N}$ in a finite set. Let $D \in \mathcal{P}(f)$ be the element such that $p \in D$: so, $D \cap \mathbf{N}$ is finite. Note that $D \backslash \{p\}$ is compact, for if not then $D \backslash \{p\}$ is a closed subset of $\mathbf{Q}$ such that D is its one-point compactification. As $\mathbf{Q}$ is normal, $D \backslash \{p\}$ is C^*-embedded in $\mathbf{Q}$ and hence in $\beta\mathbf{Q}$; thus $D \backslash \{p\}$ would be C^*-embedded in D. No countable space has its one-point compactification as its Stone-Cech compactification (see 6J of **[6]**, for example), so this is a contradiction. Hence $D \backslash \{p\}$ is compact as claimed. Similarly, $\mathbf{N}$ is C^*-embedded in $\mathbf{Q}$; hence,

$$A = \{2n : n \in \omega\} \backslash D \quad \text{and} \quad B = \mathbf{N} \backslash (A \cup D)$$

have disjoint closures in $\beta\mathbf{Q}$. Thus, there are disjoint open sets U and V in X such that

$$\text{cl}_X A \subseteq U, \quad \text{cl}_X B \subseteq V, \quad \text{and } D\backslash\{p\} \subseteq W$$

where

$$W = X\backslash\text{cl}_X(U \cup V).$$

Now, $p \in \text{cl}_X A \cup \text{cl}_X B$. If $p \in \text{cl}_X A$, then $D \subseteq W \cup U$. As $\mathcal{P}(f)$ is an upper semicontinuous decomposition of X, there is an open set R such that

$$D \subseteq R \subseteq W \cup U \quad \text{and} \quad R = \cup\{E \in \mathcal{P}(f) : E \cap R \neq \phi\}.$$

So, $p \in R \cap \text{cl}_X A$ and for some $n \in \omega$, $2n \in R$ such that $2n + 1 \in B$ (the latter assertion is true since $D \cap \mathbf{N}$ is finite). There is some $E \in \mathcal{P}_2(f)$ such that $\{2n, 2n + 1\} \subseteq E$; so, $2n + 1 \in R$. This is a contradiction as $2n + 1 \in V$ and $V \cap R = \phi$. Similarly, the assumption that $p \in \text{cl}_X B$ leads to a contradiction. This completes the proof that $\{f_n : n \in \omega\}$ has no lower bound in $IP(X)$.

For $n \in \omega$, let $g_n = \varphi(f_n)$; so,

$$\mathcal{P}_2(g_n) = \{\{F(2n), F(2n + 1)\}\}.$$

Let

$$\mathcal{P} = \{\{F(2n), F(2n + 1)\} : n \in \omega\} \cup \{\{y\} : y \in Y\backslash F[\mathbf{N}]\}.$$

Since $q \notin \text{cl}_Y F[\mathbf{N}]$, it is straightforward to verify that $\mathcal{P}$ is a covering partition of Y. There is some h in $IP(Y)$ with $\mathcal{P} = \mathcal{P}(h)$. Now $g_n \geqq h$ for each $n \in \omega$ implying that

$$f_n = \varphi^{\leftarrow}(g_n) \geqq \varphi^{\leftarrow}(h) \quad \text{for every } n \in \omega.$$

So, $\{f_n : n \in \omega\}$ has a lower bound in $IP(X)$. This is a contradiction to the assumption $IP(X)$ and $IP(Y)$ are order-isomorphic.

5. When is $IP(X)$ a lattice? Unfortunately we do not have a complete answer to the above question. However, non-trivial partial results of interest appear in 5.1, 5.2, and 5.9. Although not stated in this form, 5.1 appears as theorem 2 of [**5**]. We includee a proof for completeness. We thank Professor A. W. Hager for calling this paper to our attention.

THEOREM 5.1. *The following are equivalent for a space X*
(a) $(IP(X), \leqq)$ *is a complete lattice.*
(b) *The set* $X\backslash i(X)$ *of non-isolated points of* X *is compact and nowhere dense.*

Proof. $(a) \Rightarrow (b)$. Suppose $X \backslash i(X)$ is either non-compact or has non-empty interior. In either case, if $f \in IP(X)$ then $f[X \backslash i[X]]$ contains distinct points p and q. Let H be the quotient space of $f(X)$ obtained by identifying p and q and let

$$g : f(X) \longrightarrow H$$

be the corresponding quotient map. Then $g \circ f \in IP(X)$ and $g \circ f < f$. Hence $(IP(X) \leqq)$ has no smallest member and hence is not a complete lattice.

$(b) \Rightarrow (a)$. If $X \backslash i(X)$ is compact and nowhere dense, identify it to a point; let H be the resulting quotient space and $f : X \longrightarrow H$ be the corresponding quotient map. Then $f \in IP(X)$. If $g \in IP(X)$ and $x \in i(X)$, then

$$|g^{\leftarrow}[g(x)]| = 1;$$

from this it follows that f is the smallest member of $IP(X)$. But a complete upper semilattice with a smallest member is a complete lattice (see 2.1(e) of [**13**], for example), so we are done.

THEOREM 5.2. *Let X be a space. If $X \backslash i(X)$ is not countably compact, then $IP(X)$ is not a lattice.*

Proof. If $X \backslash i(X)$ is not a countably compact space, it contains a countably infinite closed discrete subset $D = \{x_n : n \in IN\}$. Let

$$\mathcal{A} = \{\{x_{2n-1}, x_{2n}\} : n \in IN\} \cup \{\{y\} : y \in X \backslash D\} \quad \text{and}$$
$$\mathcal{B} = \{\{x_{2n}, x_{2n+1}\} : n \in IN\} \cup \{\{y\} : y \in (X \backslash D) \cup \{x_1\}\}.$$

It is easy to verify that $\mathcal{A}$ and $\mathcal{B}$ are covering partitions of X, so $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}} \in IP(X)$ (see 2.5(b)). If $g \in IP(X)$, $g \leqq \varphi_{\mathcal{A}}$, and $g \leqq \varphi_{\mathcal{B}}$ then by 2.4(b) $\mathcal{A}$ and $\mathcal{B}$ both refine $\mathcal{P}(g)$. It follows that D is a subset of some member of $\mathcal{P}(g)$, and hence is compact. This is a contradiction, and so $\varphi_{\mathcal{A}} \wedge \varphi_{\mathcal{B}}$ cannot exist. Thus $IP(X)$ is not a lattice.

Theorems 5.1 and 5.2 in some sense deal with the two "extreme cases". Next we show that $IP([0, 1])$ is not a lattice. This is done by exhibiting two specific covering partitions $\mathcal{A}$ and $\mathcal{B}$ of [0, 1] for which the corresponding covering maps $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}}$ (which will belong to $IP([0, 1])$) have no common lower bound in $IP([0, 1])$. We first record some preliminary facts.

PROPOSITION 5.3. *Let K be a compact metric space.*

(a) *Let $f : K \longrightarrow L$ be a covering map. Then there is a dense G_δ-set G of K such that*

$$|f^{\leftarrow}[f(x)]| = 1 \quad \textit{for each } x \in G.$$

(b) *If $\mathcal{P}$ is a covering partition of K then $\{z \in K : \{x\} \in \mathcal{P}\}$ is a dense G_δ-set of K.*

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Proof. Part (a) follows from 2.1 of [[**9**] and part (b) is an immediate consequence of part (a).

PROPOSITION 5.4. *Let (X, d) be a metric space and let $\mathcal{P}$ be an upper semi-continuous decomposition of X into closed sets. If $\delta > 0$ let*

$$\mathcal{P}(\delta) = \{P \in \mathcal{P} : \text{diam } (P) \geqq \delta\}.$$

Then $\cup\mathcal{P}(\delta)$ is closed in X.

Proof. Suppose $r \in \text{cl}_X[\cup\mathcal{P}(\delta)]\backslash\cup\mathcal{P}(\delta)$. There exists $Q \in \mathcal{P}$ such that $r \in Q$. Since $Q \notin \mathcal{P}(\delta)$, it follows that diam $(Q) = \sigma < \delta$. Let $\epsilon = (\delta - \sigma)/4$; then $\epsilon > 0$. Let

$$V = \cup\{S(\epsilon, q) : q \in Q\},$$

where $S(\lambda, y)$ is the open sphere of radius λ about y. If $a, b \in V$ there exist $q(a), q(b) \in Q$ such that

$$d(a, q(a)) < \epsilon \quad \text{and} \quad d(b, q(b)) < \epsilon.$$

Thus

$$\begin{aligned} d(a, b) &< 2\epsilon + d(q(a), q(b)) \\ &< 2\epsilon + \sigma; \end{aligned}$$

therefore $d(a, b) < \delta - 2\epsilon$. It follows that diam $(V) < \delta - \epsilon$.

Since $Q \subseteq V$ and $Q \in \mathcal{P}$, there exists a $\mathcal{P}$-saturated open set W such that $Q \subseteq W \subseteq V$. So

$$r \in W \cap \text{cl}_X[\cup\mathcal{P}(\delta)],$$

and there exists $P \in \mathcal{P}(\delta)$ for which $W \cap P \neq \emptyset$. As W is saturated, $P \subseteq W$. But then

$$\delta \leqq \text{diam}(P) \leqq \text{diam}(V) \leqq \delta - \epsilon.$$

This contradiction implies no such r exists, and it follows that $\cup\mathcal{P}(\delta)$ is closed.

Definition 5.5. For $n \in IN$, let

$$D_n = \{m/2^n : m \text{ is an odd integer and } 1 \leqq m \leqq 2^n - 1\}$$

Let

$$D = \cup\{D_n : n \in IN\}.$$

The set D is just the dyadic rationals in (0, 1); it is obviously dense in [0, 1]. We will construct two covering partitions $\mathcal{A}$ and $\mathcal{B}$ of [0, 1], each of which will consist of singletons and doubletons. Each doubleton will be a subset of D_n for some n, and the corresponding covering maps $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}}$ will have no lower bound in $IP([0,1])$. To prove that $\mathcal{A}$ and $\mathcal{B}$ are upper semicontinuous, we will need the following elementary number-theoretic fact.

LEMMA 5.6. *Suppose that* $n, s, j, m \in IN$ *and that* m *is odd. Suppose* $s \geqq n+2$ *and* $1 \leqq j \leqq 2^{s-2}$. *Then:*

(a) $m/2^n < 4j - 1/2^s$ *implies* $m/2^n < 4j - 3/2^s$, *and*

(b) $m/2^n > 4j - 3/2^s$ *implies* $m/2^n > 4j - 1/2^s$.

Proof. We sketch the proof of (a); (b) is proved similarly. The hypothesis of (a) implies that

$$0 < 4j - 1/2^s - m/2^n = 4j - 1 - (4m)(2^{s-n-2})/2^s.$$

Thus $4(j - m \cdot 2^{s-n-2}) - 1$ is a positive integer and hence is at least as big as 3. Thus

$$4(j - m \cdot 2^{s-n-2}) - 3 > 0.$$

The conclusion of (a) quickly follows.

Definition 5.7. For each $n \in IN$, let

$$\mathcal{A}_n = \{\{4m - 3/2^n, 4m - 1/2^n\} : m \in IN \text{ and } 1 \leqq m \leqq 2^{n-2}\},$$

$$\mathcal{B}_n = \{\{4m - 1/2^n, 4m + 1/2^n\} : m \in IN \text{ and } 1 \leqq m \leqq 2^{n-2} - 1\},$$

$$\mathcal{A} = \cup\{\mathcal{A}_n : n \in IN\} \cup \{\{x\} : x \in [0,1] \text{ and } x \notin \cup\{A : A \in \mathcal{A}_n \text{ for some } n\},$$

and

$$\mathcal{B} = \cup\{\mathcal{B}_n : n \in IN\} \cup \{\{x\} : x \in [0,1] \text{ and } x \notin \cup\{B : B \in \mathcal{B}_n \text{ for some } n\},$$

LEMMA 5.8. $\mathcal{A}$ *and* $\mathcal{B}$ *are covering partitions of* [0, 1].

Proof. Evidently $\mathcal{A}$ and $\mathcal{B}$ are partitions of [0, 1] into compact sets. Only countably many points of [0, 1] belong to some non-singleton member of $\mathcal{A}$ (resp. $\mathcal{B}$), so if $\mathcal{A}$ (resp. $\mathcal{B}$) is upper semicontinuous, then it will be a covering partition (see 2.4). It remains to prove upper semicontinuity. We do this for $\mathcal{A}$; the proof for $\mathcal{B}$ is similar.

Let $A \in \mathcal{A}$ and let $\mathcal{A} = \{r, s\}$ (where r and s can be distinct or equal). Suppose V is open and $A \subseteq V$. As D is dense in [0, 1] there exist $n \in IN$, and $m, k \in \{1, 3, \ldots, 2^n - 1\}$, such that

$$A \subseteq (m/2^n, m + 2/2^n) \cup (k/2^n, k + 2/2^n) \subseteq V$$

(these intervals are chosen to be disjoint or equal according to whether $|A| = 2$ or $|A| = 1$). Let

$$W = (m/2^n, m + 2/2^n) \cup (k/2^n, k + 2/2^n),$$

let

$$\mathcal{S} = \{C \in \mathcal{A} : C \cap W \neq \emptyset \neq C \backslash W\},$$

and put $S = \cup \mathcal{S}$. It follows from 5.6 that

$$\mathcal{S} \subseteq \cup\{\mathcal{A}_j : 1 \leqq j \leqq n + 1\},$$

and so $\mathcal{S}$ is a finite collection of closed sets. Hence S is closed and so $W \backslash S$ is open. Evidently $A \subseteq W \backslash S \subseteq V$, and $W \backslash S$ is an $\mathcal{A}$-saturated open set. It follows that $\mathcal{A}$ is upper semicontinuous. (This argument needs obvious and easy modifications if $A = \{0\}$ or $\{1\}$.) Thus $\mathcal{A}$ is a covering partition of [0, 1]; a similar argument shows that $\mathcal{B}$ is too.

THEOREM 5.9. *$IP([0, 1])$ is not a lattice.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be as in 5.7. Suppose $g \in IP([0, 1])$ with $g \leqq \varphi_{\mathcal{A}}$ and $g \leqq \varphi_{\mathcal{B}}$ (see 2.4(c) and 2.5(b)). As $\mathcal{P}(g)$ is refined by both $\mathcal{A}$ and $\mathcal{B}$, it is evident that for each $n \in IN$ there exists $P_n \in \mathcal{P}(g)$ for which $D_n \subseteq \mathcal{P}_n$. But $\mathrm{diam}(D_n) = 1 - 1/2^{n-1}$ for each $n \in IN$; hence $\mathrm{diam}(P_n) \geqq 1/2$ whenever $n \geqq 2$. Thus

$$\cup\{R \in \mathcal{P}(f) : \mathrm{diam}(R) \geqq 1/2\} \supseteq \cup\{D_n : n \geqq 2\} = D \backslash \{1/2\},$$

which is dense in [0, 1]. It follows from 5.4 that

$$\cup\{R \in \mathcal{P}(f) : \mathrm{diam}(R) \geqq 1/2\} = [0, 1].$$

Hence $\mathcal{P}(g)$ has no singleton sets, which contradicts 5.3(b). Thus $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}}$ have no common lower bound, and $IP([0, 1])$ is not a lattice.

PROPOSITION 5.10. *Let X be a space, let $f \in IP(X)$, and suppose $IP(Rf)$ is not a lattice. Then $IP(X)$ is not a lattice.*

Proof. Suppose $g, k \in IP(Rf)$ have no common lower bound. Observe that $g \circ f, k \circ f \in IP(X)$. If h were a common lower bound of $g \circ f$ and $k \circ f$ in $IP(X)$, there would be covering maps $i : Rg \longrightarrow Rh$ and $j : Rk \longrightarrow Rh$ such that $i \circ g \circ f = j \circ k \circ f$. As f is surjective it follows that $i \circ g = j \circ k$, and so $i \circ g \leqq g, i \circ g \leqq k$, and $i \circ g \in Ir(Rf)$. This contradicts our hypothesis; hence $IP(X)$ is not a lattice.

COROLLARY 5.11. *If X is a space and $f : X \longrightarrow [0, 1]$ is a covering map, then $IP(X)$ is not a lattice; in particular $IP(C)$ is not a lattice if X is either the Cantor set or the absolute of* [0, 1].

Proof. The first assertion follows from 5.9 and 5.10. The absolute of a T_3 space Y is mapped onto Y by a covering map (see 2.1 of [**17**] or 6.6(e) of [**13**], and the Cantor set can be mapped onto [0, 1] by a covering map (seee 6I of [**13**], for example); this verifies the second assertion.

We conclude this paper by generalizing 5.9 and proving that if K is a compact metric space without a dense set of isolated points, then $IP(K)$ is not a lattice. This will not render 5.9 redundant, as we will use 5.9 to prove its generalization.

The following result is well-known and easily verified.

LEMMA 5.12. *Let X be a space, let $f \in IP(X)$, and let U be open in X. Define $f^{\#}[U]$ to be $Rf \backslash f[X \backslash U]$. Then $f^{\#}[U]$ is open in Rf and*

$$f^{\leftarrow}[f^{\#}[U]] = \cup\{P \in \mathcal{P}(f) : P \subseteq U\}.$$

LEMMA 5.13. *Let K be a compact space without isolated points and let $f, g \in IP(K)$. Suppose that*

$$\cup[\mathcal{P}_2(f) \cap \mathcal{P}_2(g)] = [\cup\mathcal{P}_2(f)] \cap [\cup\mathcal{P}_2(g)].$$

Then $f \wedge g \in IP(K)$ (see 2.4(b) for notation).

Proof. First observe that

$$[\cup\mathcal{P}_2(f)] \cap [\cup\mathcal{P}_2(g)] = \cup[\mathcal{P}_2(f) \cap \mathcal{P}_2(g)]$$

if and only if whenever $A \in \mathcal{P}_2(f)$, $B \in \mathcal{P}_2(g)$, and $A \cap B \neq \emptyset$, then $A = B$. Now let

$$\mathcal{P} = \mathcal{P}_2(f) \cup \mathcal{P}_2(g) \cup \{\{x\} : x \in K \backslash [(\cup\mathcal{P}_2(f)) \cup (\cup\mathcal{P}_2(g))]\}.$$

By hypothesis and the preceding sentence, $\mathcal{P}$ is a partition of K into compact sets. To show that $\mathcal{P}$ is upper semicontinuous, suppose that $P \in \mathcal{P}$, U is open in K, and $P \subseteq U$. Let

$$V = f^{\leftarrow}[f^{\#}[g^{\leftarrow}[g^{\#}[U]]]].$$

By 5.12 V is open in K and

$$V \subseteq g^{\leftarrow}[g^{\#}[U]] \subseteq U.$$

If P is a singleton set, obviously $\mathcal{P} \notin \mathcal{P}_2(g)$ so

$$P \subseteq g^{\leftarrow}[g^{\#}[U]]$$

by 5.12. Hence $P \subseteq V$ (also by 5.12). If $P \in \mathcal{P}_2(g)$, then

$$P \subseteq g^{\leftarrow}[g^{\#}[U]]$$

by 5.12. By hypothesis (see the first sentence of this proof) either $P \in \mathcal{P}_2(f)$, in which case $P \subseteq V$ by 5.12, or else $f^{\leftarrow}[f(x)] = \{x\}$ for each $x \in P$, in which case $P \subseteq V$. If $P \in \mathcal{P}_2(f)$, a similar argument yields that $P \subseteq V$. Thus $P \subseteq V \subseteq U$. An argument essentially identical to the above also shows that V is $\mathcal{P}$-saturated. Hence $\mathcal{P}$ is upper semicontinuous.

If W is a non-empty open subset of K, there exists $P \in \mathcal{P}_2(f)$ such that $P \subseteq W$ (since f is a covering map). As $\mathcal{P}_2(f) \subseteq \mathcal{P}$, it follows that W contains a member of $\mathcal{P}$, and so $\mathcal{P}$ is a covering partition. By 2.5(b) $\varphi_{\mathcal{P}} \leqq f$ and $\varphi_{\mathcal{P}} \leqq g$ as $\mathcal{P}(f)$ and $\mathcal{P}(g)$ refine $\mathcal{P}$ (see 2.4(c) for notation). Obviously $\varphi_{\mathcal{P}} = f \wedge g$ from the construction of $\mathcal{P}$.

LEMMA 5.14. *Let K be a compact metric space without isolated points, and let $f, g, h \in IP(K)$. Suppose that $g \wedge h$ does not exist and that*

$$[(\cup\mathcal{P}_2(g)) \cup (\cup\mathcal{P}_2(h))] \cap [\cup\mathcal{P}_2(f)] = \emptyset.$$

Then $IP(Rf)$ is not a lattice.

Proof. Our hypotheses imply that

$$(\cup\mathcal{P}_2(h)) \cap (\cup\mathcal{P}_2(f)) = \cup(\mathcal{P}_2(h) \cap \mathcal{P}_2(f)),$$

and similarly for g and f. Hence by 5.13 $f \wedge h \in IP(K)$ and so there exists $m \in IP(Rf)$ such that $m \circ f = f \wedge h$. Similarly $f \wedge g \in IP(K)$ and there exists $n \in IP(RF)$ such that $n \circ f = f \wedge g$. If $m \wedge n$ existed in $IP(Rf)$, then

$$m \circ f \geqq (m \wedge n) \circ f.$$

Thus

$$h \geqq f \wedge h = m \circ f \geqq (m \wedge f) \circ f$$

and similarly

$$g \geqq f \wedge g = n \circ f \geqq (m \wedge n) \circ f.$$

Thus g and h have a common lower bound and so $g \wedge h$ exists by 2.3, in contradiction to hypothesis. Hence $m \wedge n$ cannot exist in $IP(Rf)$, and the lemma follows.

THEOREM 5.15 *Let K be a compact metric space without isolated points. Then $IP(K)$ is not a lattice.*

Proof. As noted in the proof of 5.11, there is a covering map $t : C \longrightarrow K$. (Here C denotes the Cantor set.) By 5.3(a) the set

$$\{x \in C : |t^{\leftarrow}(t(x))| = 1\}$$

contains a dense G_δ-set of C, say G. There is a covering map $f : C \longrightarrow [0, 1]$ such that $\cup\mathcal{P}_2(f)$ is a countable dense subset of C (see 3.2B of [[**3**]). By 5.9 and its proof there exist $g, h \in IP([0, 1])$ for which $g \wedge h$ does not exist and $[\cup\mathcal{P}_2(g)]\cup[\cup\mathcal{P}_2(h)]$ is a countable dense subset of [0, 1]. It follows that $[\cup\mathcal{P}_2(h\circ f)]\cup[\cup\mathcal{P}_2(g \circ f)]$ is a countable dense subset of C. Let D be a countable dense subset of G. By 4.3H of [[**3**] there is a homeomorphism $k : C \longrightarrow C$ for which

$$k[D] = [\cup\mathcal{P}_2(h \circ f)] \cup [\cup\mathcal{P}_2(g \circ f)].$$

By our choice of G it follows that

$$[(\cup\mathcal{P}_2(h \circ f \circ k)) \cup (\cup\mathcal{P}_2(g \circ f \circ h))] \cap (\cup\mathcal{P}_2(t)) = \emptyset.$$

By 5.11 $(g \circ f \circ k) \wedge (h \circ f \circ k)$ does not exist in $IP(C)$. Hence by 5.10 $IP(Rt)$ is not a lattice, i.e., $IP(K)$ is not a lattice.

COROLLARY 5.16. *Let K be a compact metric space. The following are equivalent:*

(a) *K has a dense set of isolated points.*

(b) *$IP(K)$ is a complete lattice.*

(c) *$IP(K)$ is a lattice.*

Proof. (a) ⇒(b). This follows immediately from 5.1.

(b) ⇒(c) is obvious.

(c) ⇒(a). Suppose (a) fails; let

$$L = \mathrm{cl}_K(K\backslash \mathrm{cl}_K i(K))$$

where $i(K)$ is the set of isolated points of K. It is immediate that L is a compact metric space without isolated points, so by 5.15 there exist $f, g \in IP(L)$ for which $f \wedge g$ does not exist in $IP(L)$. Let

$$\mathcal{A} = \mathcal{P}(f) \cup \{\{x\} : x \in K\backslash L\} \quad \text{and } \mathcal{B} = \mathcal{P}(g) \cup \{\{x\} : x \in K\backslash L\}.$$

A straightforward calculation verifies that $\mathcal{A}$ and $\mathcal{B}$ are covering partitions of K. If (c) were to hold, there would be a space Y and covering maps

$$s : R(\varphi_{\mathcal{A}}) \longrightarrow Y, \quad t : R(\varphi_{\mathcal{B}}) \longrightarrow Y$$

for which $s \circ \varphi_{\mathcal{A}} = t \circ \varphi_{\mathcal{B}}$. Then (up to equivalence)

$$s \circ (\varphi_{\mathcal{A}}|L) \leqq f \quad \text{and} \quad s \circ (\varphi_{\mathcal{A}}|L) \leqq g$$

(in $IP(L)$), which is a contradiction. Hence (c) must fail, and so (c) implies (a).

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References

1. E. K. van Douwen, *Remote points*, Dissertationes Math. *188* (1981).
2. F. Dashiell, A. Hager and M. Henriksen, *Order-Cauchy completions of ring and vector lattices of continuous functions*, Can. J. Math. *32* (1980), 657–685.
3. R. Engelking, *General topology* (Polish Scientific Publishers, Warsaw, 1977).
4. N. Fine and L. Gillman, *Remote points in βIR*, Proc. Amer. Math. Soc. *13* (1962), 29–36.
5. H. Gonshor, *Injective hulls of C^*-algebras. II*, Proc. Amer. Math. Soc. *24* (1970), 486–491.
6. L. Gillman and M. Jerison, *Rings of continuous functions* (Princeton, Von Nostrand, 1960).
7. A. Hager, *Isomorphisms of some completions of $C(X)$*, Top. Proc. *4* (1979), 407–436.
8. H. Herrlich and J. van der Slot, *Properties which are closely related to compactness*, Indag. Math. *29* (1967), 524–529.
9. J. van Mill and R. G. Woods, *Perfect images of zero-dimensional separable metric spaces*, Canad. Math. Bull. *25* (1982), 41–47.
10. K. D. Magill, *The lattice of compactifications of a locally compact space*, Proc. Lond. Math. Soc. *28* (1968), 231–244.
11. J. Mack, M. Rayburn and R. G. Woods, *Lattice of topological extensions*, Trans. Amer. Math. Soc. *189* (1974), 163–174.
12. J. R. Porter and R. G. Woods, *Accumulation points of nowhere dense sets in H-closed spaces*, Proc. Amer. Math. Soc. *93* (1985), 539–542.
13. ——— *Extensions and absolutes of Hausdorff spaces* (Springer-Verlag, New York, 1987).
14. M. C. Rayburn, *On Hausdorff compactifications*, Pacific J. Math. *44* (1973), 707–714.
15. J. Vermeer, *The smallest basically disconnected preimage of a space*, Top. Applic. *17* (1984), 217–232.
16. R. G. Woods, *Zero-dimensional compactifications of locally compact spaces*, Can. J. Math. *26* (1974), 920–930.
17. ——— *A survey of absolutes of topological spaces*, Topological Structures II, Mathematische Centrum Tract *116* (1979), 323–362 (Amsterdam).

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