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SIGN-CHANGING SOLUTIONS OF (e_1, B)-LIMIT INCREASING OPERATOR EQUATION*

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Abstract. In this paper, by using the fixed point index method first we obtain some existence and multiplicity results for sign-changing solutions of an (e_1, B)-limit increasing operator equation. The main results can be applied to many non-linear boundary value problems to obtain the existence and multiplicity results for sign-changing solutions. We also give a clear description of locations of these sign-changing solutions through strict lower and upper solutions. As an example, in the last section we obtain some existence and multiplicity results for sign-changing solutions of some Sturm–Liouville differential boundary value problems.

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1. Introduction. As is well known, when studying non-linear problems, better results may be established if the non-linear terms are assumed to have some monotonicity properties. For instance, when studying non-linear differential boundary value problems, if the non-linear terms are increasing and there exists a pair of well-ordered upper and lower solutions, then the differential boundary value problems always have maximal and minimal solutions on the ordered interval defined by the well-ordered upper and lower solutions. For another instance, recently some authors obtained the existence results for critical points by combing the lower and upper solutions method with the descending flowing invariant set method; see [**6**, **27**] and the references therein. In order to establish these results, the authors of these papers always assume that the non-linear terms satisfy some kinds of monotonicity properties. However, in many cases the non-linear terms may not have any monotonicity properties. In order to overcome this difficulty, in [**13**] we introduced a new concept of (e_1, B)-limit increasing operator and studied the existence of solutions of (e_1, B)-limit increasing operator equations under the condition of pairs of paralleled upper and lower solutions.

The sign-changing solutions have attracted much attention in recent years; see [**1**–**12**, **21**–**30**] and the references therein. Generally speaking, there are three approaches to establish the sign-changing solution results. They are critical point theory, the topological degree method and the global bifurcation theory. In our papers [**8**–**11**], we established some existence results for sign-changing solutions of some differential boundary value problems via the topological degree method and the global bifurcation theories. Concerning the method of global bifurcation theories, we also refer readers to

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Rabinnowitz's classical results [**29**, **30**]. Since there are explicit geometrical significance, recently the method of critical point theory has been employed extensively to find sign-changing solutions. Bartsch and Wang [**27**] established an abstract critical point theory in the partially ordered Hilbert spaces by virtue of critical groups and studied sign-changing solutions of some elliptic boundary value problems. Concerning the method of critical point theory we also refer readers to Refs. [**1**–**7**, **21**–**26**, **28**]. Moreover, there were some interesting results in Zou's recent monographs [**28**] concerning sign-changing critical point theory. However, we should point out that in many cases this method is invalid because a suitable variational setting can not be found for some non-linear boundary value problems, e.g. impulsive differential boundary value problems, multi-point differential boundary value problems etc.; nevertheless we may establish existence results for sign-changing solutions for these differential boundary value problems by using the methods of the topological degree or global bifurcation theories.

The main purpose of this paper is to study the sign-changing solutions of (e_1, B)-limit increasing operator equations. Some existence and multiplicity results for the sign-changing solutions of an equation of (e_1, B)-limit increasing operator are established and the locations of these sign-changing solutions are clearly described in terms of the cone structure of the space. The abstract theorems are applied to the Sturm–Liouville equations to obtain the existence results for sign-changing solutions. Of course, the abstract theorems may also be applied to many other non-linear boundary value problems to obtain sign-changing solutions, such as impulsive differential boundary value problems and multi-point differential boundary value problems.

2. Main results. Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be the two real Banach spaces, and P and P_1 normal cones of X and Z, respectively. Denoted by $\leqslant$, the both partial orderings in X and Z are induced by the cones P and P_1, respectively. For more discussions about cone and partial ordering we refer the reader to [**20**]. Let $u_0, v_0 \in X$ and $[u_0, v_0] = \{x \in X | u_0 \leqslant x \leqslant v_0\}$ denote an ordered interval of X. In this paper we will consider the existence of solutions of the operator equation

$$Lx = Fx, \tag{2.1}$$

where $L : \mathrm{dom}L \subset X \mapsto Z$ is a linear operator and $F : X \mapsto Z$ is a continuous and bounded operator.

Let $B : X \mapsto Z$ be a linear bounded operator such that $B : P\backslash\{\theta\} \to P_1\backslash\{\theta\}$, $e_1 \in P_1\backslash\{\theta\}$, $\bar{e}_0 = L^{-1}e_1 \in P\backslash\{\theta\}$. For any $x, y \in X$, denote by $x \prec_0 y$ if there exists $\delta > 0$ such that $y - x \geqslant \delta\bar{e}_0$. For any $x, y \in Z$, denote by $x \prec_1 y$ if there exists $\delta > 0$ such that $y - x \geqslant \delta e_1$. Let $\mathbb{N}$ denote the set of all natural numbers, and $\mathbb{N}^+ = \{0\} \cup \mathbb{N}$.

First recall the concept of (e_1, B)-limit-increasing operator.

Definition 2.1. Let $D \subset X$ be a bounded and closed set, and $F : D \mapsto Z$ a bounded and continuous operator. Then F is called an (e_1, B)-limit-increasing operator on the set D if there exist a sequence of continuous operators $\{F_n\}_{n=1}^{\infty}$ and a sequence of positive numbers $\{M_n\}_{n=1}^{\infty}$ such that for all $n \in \mathbb{N}$,

$$-\frac{1}{n}e_1 \leqslant Fu - F_nu \leqslant \frac{1}{n}e_1, \quad \forall u \in D, \tag{2.2}$$

$$G_nv > G_nu, \quad \forall v,\ u \in D,\ v > u, \tag{2.3}$$

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where the operator $G_n : D \mapsto Z$ is defined by $G_n x = F_n x + M_n Bx$ for all $x \in D$ and $n \in \mathbb{N}$. An (e_1, B)-limit-increasing operator F on a bounded closed set D whose interior is non-empty is called locally increasing at an interior point x_0 of D if there exists $r_0 > 0$ such that $U(x_0, r_0) \subset D$ and

$$Fu = F_1 u = F_2 u = \cdots = F_n u = \cdots, \quad \forall u \in U(x_0, r_0). \tag{2.4}$$

REMARK 2.1. The concept of (e_1, B)-limit-increasing operator was put forward in [**13**, **18**, **19**]. However, there are some subtle differences between the concept of (e_1, B)-limit-increasing operator in this paper and those in [**13**, **18**, **19**]. In this paper, we will always assume that (2.2) and (2.3) hold if we assume that $F : D \mapsto Z$ is an (e_1, B)-limit-increasing operator.

DEFINITION 2.2. ([**15**]) An operator $T : \mathscr{D}(T) \subset Z \mapsto X$ is called e-continuous at $x_0 \in \mathscr{D}(T)$ if for every $\varepsilon > 0$, there is number $\delta > 0$, such that

$$-\varepsilon e \leqslant Tx - Tx_0 \leqslant \varepsilon e$$

for every $x \in \mathscr{D}(T) \cap U(x_0, \delta)$. An operator T is called e-continuous on $\mathscr{D}(T)$ if T is e-continuous at every $x \in \mathscr{D}(T)$.

DEFINITION 2.3. ([**13**]) Let α_0 and $\beta_0 \in X$. Then α_0 and β_0 are said to be strict lower and upper solutions of (2.1), respectively, if $L\alpha_0 \prec_1 F\alpha_0$ and $F\beta_0 \prec_1 L\beta_0$.

DEFINITION 2.4. ([**13**]) Let α_0 and β_0 be strict lower and upper solutions of (2.1), respectively. Then α_0 and β_0 is called a pair of well-ordered strict lower and upper solutions of (2.1) if $\alpha_0 \prec_0 \beta_0$.

DEFINITION 2.5. ([**16**]) Let $e \in P \backslash \{\theta\}$. An operator $T : \mathscr{D}(T) \subset Z \to X$ is said to be e-positive if for any $u \in P_1 \setminus \{\theta\}$, there exist $\alpha = \alpha(u)$, $\beta = \beta(u) > 0$ such that

$$\alpha e \leqslant Tu \leqslant \beta e.$$

DEFINITION 2.6. Let $\bar{x}$ be a non-zero solution of the operator equation (2.1). If $\bar{x} \in (-P)$ (or $\bar{x} \in P$, or $\bar{x} \in E \setminus (P \cup (-P))$), then $\bar{x}$ is called a negative (or positive, or sign-changing) solution of (2.1).

From [**14**, Lemma 5.2] we have the following result.

LEMMA 2.1. *Let E be an ordered Banach space with solid cone P. Let $K : E \mapsto E$ be a compact, e-positive, linear operator and let $F : E \mapsto E$ be a map such that for some $u_0 \in E$, $u_0 = KF(u_0)$. Suppose F is Gâteaux differentiable at u_0 with strictly positive derivative $F'(u_0)$. Denote by $r(T)$ the spectral radius of the operator $T = KF'(u_0)$ and by h_0 the positive eigenfunction of T corresponding to $r(T)$. Then there exists a $\tau_0 > 0$ such that for all $0 < \tau < \tau_0$,*

$$r(T) > 1 \quad \textit{implies} \quad \begin{cases} KF(u_0 + \tau h_0) > u_0 + \tau h_0, \\ KF(u_0 - \tau h_0) < u_0 - \tau h_0, \end{cases}$$

and

$$r(T) < 1 \quad \textit{implies} \quad \begin{cases} KF(u_0 + \tau h_0) < u_0 + \tau h_0, \\ KF(u_0 - \tau h_0) > u_0 - \tau h_0. \end{cases}$$

*From [**17**, Theorem 19.2] we have the following Lemma 2.2.*

LEMMA 2.2. *(Krein–Rutman) Let E be a Banach space, $P \subset E$ a total cone and $K \in L(E)$ a compact positive with $r(K) > 0$. Then $r(K)$ is an eigenvalue with a positive eigenvector.*

For convenience, let us introduce the following conditions to be used in the sequel.

(H_1) *There exists a pair of well-ordered strict lower and upper solutions u_0 and v_0 of (2.1) such that $u_0, v_0 \in \mathring{P}$.*

(H_2) *F is (e_1, B)-limit increasing on any bounded set of X and is locally increasing at θ, $F(\theta) = \theta$, F is Fréchet differentiable at θ, $F'(\theta) = \beta_0 B$, and F is e_1-continuous on $[u_0, v_0]$, where $\beta_0 > 0$.*

(H_3) *For each $M \geqslant 0$, $K_M := (L + MB)^{-1} : Z \mapsto X$ exists and is completely continuous, K_M is $\bar{e}_M$-continuous and $\bar{e}_M$-positive on any ordered interval of Z for some $\bar{e}_M \in P\backslash\{\theta\}$; the algebraic multiplicity of each positive eigenvalue of $K_M B$ is 1.*

(H_4) *There exists a pair of well-ordered strict lower and upper solutions, u_1 and v_1, of (2.1) such that $u_1, v_1 \in X\backslash(P \cup (-P))$ and $u_0 \prec_0 u_1 \prec_0 v_1 \prec_0 v_0$.*

Assume in the sequel that the positive eigenvalues of KB decreasingly are $\{\lambda_n\}_{n=1}^{\infty}$, where $K := K_0$. Then we have the following main results.

THEOREM 2.1. *Suppose that (H_1)–(H_3) hold, $\beta_0 \in (\frac{1}{\lambda_{2n_0}}, \frac{1}{\lambda_{2n_0+1}})$, where n_0 is a natural number. Then (2.1) has at least one sign-changing solution. Moreover, (2.1) has at least one positive and one negative solution.*

Proof. The proof is achieved in the following six steps:

Step 1. Since F is an (e_1, B)-limit-increasing operator, which is locally increasing at θ, there exist a sequence of continuous operators $\{F_n\}$, a sequence of positive numbers $\{M_n\}$ and $r_0 > 0$ such that (2.2)–(2.4) hold, where $x_0 = \theta$ and $D = [u_0, v_0]$. For brevity, let us denote $F_0 = F$, $G_0 = F$, $M_0 = 0$ and $K_n = K_{M_n}$ for each $n \in \mathbb{N}$ in the sequel of this section. Now, since u_0 and v_0 are strict lower and upper solutions and F is e_1-continuous on $[u_0, v_0]$, by (2.2) and the Definitions 2.1 and 2.3, for sufficiently large enough $n \in \mathbb{N}$ (assume without loss of generality that for all $n \in \mathbb{N}$), we have

$$Lu_0 \prec_1 F_n u_0, \quad F_n v_0 \prec_1 Lv_0.$$

Since $F'(\theta) = \beta_0 B$, we have

$$\lim_{x\in X, \|x\|_X \to 0} \frac{\|Fx - F\theta - \beta_0 Bx\|_Z}{\|x\|_X} = 0.$$

By (2.4) we have for each $n \in \mathbb{N}^+$,

$$\lim_{x\in X, \|x\|_X \to 0} \frac{\|K_n G_n x - K_n G_n \theta - (\beta_0 + M_n) K_n Bx\|_Z}{\|x\|_X} \leqslant \lim_{x\in X, \|x\|_X \to 0} \frac{\|K_n\| \|Fx - F\theta - \beta_0 Bx\|_Z}{\|x\|_X} = 0.$$

This implies that $K_n G_n$ is Fréchet differentiable at θ for each $n \in \mathbb{N}^+$, and $(K_n G_n)'(\theta) = (\beta_0 + M_n) K_n B$. Let μ be a positive eigenvalue of the linear operator $(\beta_0 + M_n) K_n B$ and x_μ be the corresponding eigenfunction, that is

$$(\beta_0 + M_n) K_n B x_\mu = \mu x_\mu.$$

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Then, we have

$$(\beta_0 + M_n)Bx_\mu = \mu(L + M_nB)x_\mu,$$

and so

$$KBx_\mu = \frac{\mu}{\beta_0 + (1-\mu)M_n}x_\mu.$$

Since the positive eigenvalues of KB are $\{\lambda_i\}$, there exists i_0 such that

$$\lambda_{i_0} = \frac{\mu}{\beta_0 + (1-\mu)M_n},$$

and so

$$\mu = \frac{\lambda_{i_0}(\beta_0 + M_n)}{1 + \lambda_{i_0}M_n}. \tag{2.5}$$

On other hand, we can easily prove that every positive number μ satisfying (2.5) is an eigenvalue of the linear operator $(\beta_0 + M_n)K_nB$. Thus, the sequence of positive eigenvalues of the linear operator $(\beta_0 + M_n)K_nB$ is

$$\left\{\frac{\lambda_i(\beta_0 + M_n)}{1 + \lambda_iM_n}\right\}_{i=1}^{\infty}.$$

Step 2. For each $n \in \mathbb{N}^+$, let

$$S_n^+ = \{x \in [u_0, v_0] | x > \theta, x = K_nG_nx\},$$
$$S_n^- = \{x \in [u_0, v_0] | x < \theta, x = K_nG_nx\}.$$

By (H_3), there exists $\overline{e}_n \in P\backslash\{\theta\}$ such that, for each $n \in \mathbb{N}^+$, K_nB is $\overline{e}_n$-positive and $\overline{e}_n$-continuous. Now we shall show that for each $n \in \mathbb{N}^+$, there exists $\zeta_n > 0$ such that

$$S_n^+ \geqslant \zeta_n\overline{e}_n, \quad S_n^- \leqslant -\zeta_n\overline{e}_n. \tag{2.6}$$

Let $x_0 \in S_n^+$. Now, since $G_nx_0 > G_n\theta = \theta$ and K_n is $\overline{e}_n$-positive, there exists $\beta_{x_0}^{(n)} > 0$ such that $x_0 = K_nG_nx_0 \geqslant \beta_{x_0}^{(n)}\overline{e}_n$. On the other hand, since $G_n : X \mapsto Z$ is continuous and $K_n : Z \mapsto X$ is $\overline{e}_n$-continuous, for $\frac{\beta_{x_0}^{(n)}}{2} > 0$, there exits $r_{x_0} > 0$ such that for any $x \in U(x_0, r_{x_0}) \cap S_n^+$,

$$-\frac{1}{2}\beta_{x_0}^{(n)}\overline{e}_n \leqslant K_nG_nx - K_nG_nx_0 \leqslant \frac{1}{2}\beta_{x_0}^{(n)}\overline{e}_n,$$

and thus, for any $x \in U(x_0, r_{x_0}) \cap S_n^+$, we have

$$x \geqslant x_0 - \frac{1}{2}\beta_{x_0}^{(n)}\overline{e}_n \geqslant \frac{1}{2}\beta_{x_0}^{(n)}\overline{e}_n.$$

Obviously, $\{U(x_0, r_{x_0}) | x_0 \in S_n^+\}$ is an open cover of S_n^+. Since $P \subset X$ and $P_1 \subset Z$ are normal, (2.3) implies that G_n is bounded on $[u_0, v_0]$ for each $n \in \mathbb{N}$. Thus, $S_n^+ = K_nG_nS_n^+ \subset X$ is a relative compact set for each $n \in \mathbb{N}^+$ (note $G_0 = F$ also is bounded). Therefore, there exist finite subsets of $\{U(x_0, r_{x_0}) | x_0 \in S_n^+\}$, say, $U(x_1, r_{x_1}), \ldots, U(x_{k_n}, r_{k_n})$, such that

$$S_n^+ \subset \bigcup_{i=1}^{k_n} U(x_i, r_{x_i}).$$

Let $\beta_+^{(n)} = \min\{\frac{1}{2}\beta_{x_1}^{(n)}, \frac{1}{2}\beta_{x_2}^{(n)}, \ldots, \frac{1}{2}\beta_{x_{k_n}}^{(n)}\} > 0$ for each $n \in \mathbb{N}^+$. Then, we have $S_n^+ \geqslant \beta_+^{(n)}\bar{e}_n$. Similarly, there exists $\beta_-^{(n)} > 0$ such that $S_n^- \leqslant -\beta_-^{(n)}\bar{e}_n$. Let $\beta^{(n)} = \frac{1}{2}\min\{\beta_+^{(n)}, \beta_-^{(n)}\}$, then (2.6) holds.

Step 3. Let $n \in \mathbb{N}^+$ be fixed at present. The spectrum radius of $(K_nG_n)'(\theta) = (\beta_0 + M_n)K_nB$ is

$$r((\beta_0 + M_n)K_nB) = \frac{\lambda_1(\beta_0 + M_n)}{1 + \lambda_1 M_n} > 1.$$

By Lemma 2.2, there exists $h_n \in P\backslash\{\theta\}$ such that

$$(\beta_0 + M_n)K_nBh_n = r((\beta_0 + M_n)K_nB)h_n. \tag{2.7}$$

Since K_n is $\bar{e}_n$-positive and $B : P\backslash\{\theta\} \mapsto P_1\backslash\{\theta\}$, by (2.7), there exist $\alpha(h_n) > 0$ and $\beta(h_n) > 0$ such that

$$\alpha(h_n)\bar{e}_n \leqslant h_n \leqslant \beta(h_n)\bar{e}_n. \tag{2.8}$$

By Lemma 2.1, there exists $\tau_n > 0$ such that for any $\tau \in (0, \tau_n]$,

$$-\tau h_n > K_nG_n(-\tau h_n), \quad K_nG_n(\tau h_n) > \tau h_n. \tag{2.9}$$

By (2.6), (2.8) and (2.9), we may take $\tau_n > 0$ small enough such that

$$u_0 < v_{2,n} < \theta < u_{2,n} < v_0, \tag{2.10}$$

$$S_n^+ \geqslant u_{2,n}, S_n^- \leqslant v_{2,n} \tag{2.11}$$

and

$$u_{2,n} < K_nG_nu_{2,n}, K_nG_nv_{2,n} < v_{2,n}, \tag{2.12}$$

where $u_{2,n} = \tau_nh_n$ and $v_{2,n} = -\tau_nh_n$. Since 1 is not an eigenvalue of the linear operator $(K_nG_n)'(\theta)$, there exists $\bar{r}_n > 0$ such that K_nG_n has the unique fixed point θ in $U(\theta, \bar{r}_n)$, and

$$\deg(I - K_nG_n, U(\theta, \bar{r}_n), \theta) = (-1)^{2n_0} = 1. \tag{2.13}$$

Assume that $\bar{r}_n > 0$ small enough such that $v_{2,n}, u_{2,n} \notin U(\theta, \bar{r}_n)$, $U(\theta, \bar{r}_n) \subset [u_0, v_0]$ and $\bar{r}_n < \bar{r}_0 < r_0$.

Step 4. For each $n \in \mathbb{N}$, let

$$\Omega_{1,n} = \{x \in [u_0, v_0] | \text{there exists } \tau > 0 \text{ such that } K_nG_nx \leqslant K_nG_nv_{2,n} - \tau\bar{e}_n\},$$
$$\Omega_{2,n} = \{x \in [u_0, v_0] | \text{there exists } \tau > 0 \text{ such that } K_nG_nx \geqslant K_nG_nu_{2,n} + \tau\bar{e}_n\}.$$

It is easy to see that $u_0 \in \Omega_{1,n}$ and $v_0 \in \Omega_{2,n}$. By the $\bar{e}_n$-continuity of K_nG_n, it is easy to see that $\Omega_{1,n}$ and $\Omega_{2,n}$ are open subsets of $[u_0, v_0]$. Now assume that n is a fixed natural number. Now we shall show that for any $x \in \partial_{[u_0,v_0]}\Omega_{1,n}$ and $t \in [0, 1]$,

$$x \neq tK_nG_nx + (1-t)u_0. \tag{2.14}$$

If (2.14) is not true, then there exists $x_0 \in \partial_{[u_0,v_0]}\Omega_{1,n}$ and $t_0 \in [0, 1]$ such that

$$x_0 = t_0K_nG_nx_0 + (1-t_0)u_0. \tag{2.15}$$

Notice that $x_0 \in \partial_{[u_0,v_0]}\Omega_{1,n}$, so we have

$$K_nG_nx_0 \leqslant K_nG_nv_{2,n} < v_{2,n}.$$

It follows from (2.10) and (2.15) that $x_0 < v_{2,n}$. Thus, by the $\bar{e}_n$-positivity of K_n we have

$$K_nG_nv_{2,n} - K_nG_nx_0 = K_n(G_nv_{2,n} - G_nx_0) \geqslant \alpha(G_nv_{2,n} - G_nx_0)\bar{e}_n,$$

where $\alpha(G_nv_{2,n} - G_nx_0) > 0$. This implies that $x_0 \in \Omega_{1,n}$, which contradicts $x_0 \in \partial_{[u_0,v_0]}\Omega_{1,n}$. Thus, (2.14) holds, and so

$$i(K_nG_n, \Omega_{1,n}, [u_0, v_0]) = 1. \tag{2.16}$$

Similarly, we have

$$i(K_nG_n, \Omega_{2,n}, [u_0, v_0]) = 1. \tag{2.17}$$

Obviously, we have

$$i(K_nG_n, [u_0, v_0], [u_0, v_0]) = 1. \tag{2.18}$$

From (2.3.5) of [**20**], we have

$$i(K_nG_n, U(\theta, \bar{r}_n), [u_0, v_0]) = \deg(I - K_nG_n \cdot r, U(\theta, \bar{R}_n) \cap r^{-1}(U(\theta, \bar{r}_n)), \theta), \tag{2.19}$$

where $r : X \mapsto [u_0, v_0]$ is a retraction and $\bar{R}_n > \bar{r}_0$. It is easy to see that each fixed point of $K_nG_n \cdot r$ in $U(\theta, \bar{R}_n) \cap r^{-1}(U(\theta, \bar{r}_n))$ must belong to $U(\theta, \bar{r}_n)$. Thus, by the properties of the Leray–Schauder degree,

$$\deg(I - K_nG_n, U(\theta, \bar{r}_n), \theta) = \deg(I - K_nG_n \cdot r, U(\theta, \bar{R}_n) \cap r^{-1}(U(\theta, \bar{r}_n)), \theta). \tag{2.20}$$

It follows from (2.13), (2.19) and (2.20) that

$$i(K_nG_n, U(\theta, \bar{r}_n), [u_0, v_0]) = 1. \tag{2.21}$$

By (2.16)–(2.18) and (2.21), we have

$$i\big(K_nG_n, [u_0, v_0]\backslash\big(Cl_{[u_0,v_0]}\Omega_{1,n} \cup Cl_{[u_0,v_0]}\Omega_{2,n} \cup \bar{U}(\theta, \bar{r}_n)\big), [u_0, v_0]\big) = -2. \tag{2.22}$$

By (2.16), (2.17) and (2.22), K_nG_n has three fixed points $x_{1,n} \in \Omega_{1,n}$, $x_{2,n} \in \Omega_{2,n}$ and $x_{3,n} \in [u_0, v_0]\backslash\big(Cl_{[u_0,v_0]}\Omega_{1,n} \cup Cl_{[u_0,v_0]}\Omega_{2,n} \cup \bar{U}(\theta, \bar{r}_n)\big)$, respectively. Let us show that

$$\|x_{i,n}\|_X \geqslant \bar{r}_0, \quad i = 1, 2, 3. \tag{2.23}$$

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We only show that $\|x_{3,n}\|_X \geqslant \bar{r}_0$. By contradiction, assume that $\|x_{3,n}\|_X < \bar{r}_0 < r_0$. Since

$$x_{3,n} = K_n G_n x_{3,n} = (L + M_n)^{-1}(F_n x_{3,n} + M_n B x_{3,n}),$$

we have $x_{3,n} = KFx_{3,n}$. Now, since θ is the unique solution of KF in $U(\theta, \bar{r}_0)$, we see that $x_{3,n} = \theta$, which contradicts $x_{3,n} \notin \bar{B}(\theta, \bar{r}_n)$, and so (2.23) holds.

Step 5. Since $[u_0, v_0]$ is bounded and $F : [u_0, v_0] \mapsto Z$ is bounded, there exists $R' > 0$ such that $\|Fx_{1,n}\|_Z < R'$. It follows from (2.2) that

$$\theta \leqslant F_n x_{1,n} - F x_{1,n} + \frac{1}{n} e_1 \leqslant \frac{2}{n} e_1 \leqslant 2e_1,$$

and so

$$\begin{aligned}\|F_n x_{1,n}\|_Z &\leqslant \|F_n x_{1,n} - F x_{1,n} + \tfrac{1}{n} e_1\|_Z + \|F x_{1,n} - \tfrac{1}{n} e_1\|_Z \\ &\leqslant 2\gamma_0 \|e_1\|_Z + \|F x_{1,n} - \tfrac{1}{n} e_1\|_Z \\ &\leqslant 2\gamma_0 \|e_1\|_Z + \|F x_{1,n}\|_Z + \|e_1\|_Z \\ &\leqslant (2\gamma_0 + 1)\|e_1\|_Z + R',\end{aligned}$$

where $\gamma_0 > 0$ is the normal constant of the cone P_1. Since $x_{1,n} = KF_n x_{1,n}$ and K is a linear completely continuous operator, $\{x_{1,n}\}_{n=1}^{\infty}$ is a relatively compact set. Thus, there exist a sub-sequence of $\{x_{1,n}\}_{n=1}^{\infty}$ (assume without loss of generality that the sub-sequence is $\{x_{1,n}\}_{n=1}^{\infty}$ itself) and x_1^* such that $x_{1,n} \to x_1^* (n \to \infty)$. Since

$$-\frac{1}{n} e_1 \leqslant F_n x_{1,n} - F x_{1,n} \leqslant \frac{1}{n} e_1,$$

we have $F_n x_{1,n} - F x_{1,n} \to \theta (n \to \infty)$ and so $K(F_n x_{1,n} - F x_{1,n}) \to \theta (n \to \infty)$. Consequently, we have

$$\begin{aligned}KFx_{1,n} &= K(Fx_{1,n} - F_n x_{1,n}) + KF_n x_{1,n} \\ &= K(Fx_{1,n} - F_n x_{1,n}) + x_{1,n} \to x_1^* (n \to \infty).\end{aligned}$$

On the other hand, $KFx_{1,n} \to KFx_1^* (n \to \infty)$. Therefore, $x_1^* = KFx_1^*$, and so $Lx_1^* = Fx_1^*$, that is, x_1^* is a solution of (2.1). Similarly, there are two sub-sequences of $\{x_{2,n}\}_{n=1}^{\infty}$ and $\{x_{3,n}\}_{n=1}^{\infty}$ (assume that the two sub-sequences are themselves) and x_2^*, x_3^*, such that $x_{2,n} \to x_2^*$ and $x_{3,n} \to x_3^* (n \to \infty)$. Then, x_2^* and x_3^* are two solutions of (2.1).

Step 6. Note that $x_{1,n} \in \Omega_{1,n}$, so we have

$$x_{1,n} = K_n G_n x_{1,n} \leqslant K_n G_n v_{2,n} - \tau \bar{e}_n < v_{2,n} < \theta$$

for some $\tau > 0$, and so $x_1^* \leqslant \theta$. It follows from (2.23) that $\|x_1^*\|_X \geqslant \bar{r}_0$. Thus, x_1^* is a negative solution of (2.1). Similarly, x_2^* is a positive solution of (2.1). Finally we shall show that x_3^* is a sign-changing solution of (2.1). By contradiction, assume that x_3^* is not a sign-changing solution of (2.1). It follows from (2.23) that x_3^* is not a zero solution of (2.1). Assume that x_3^* is a positive solution of (2.1), then by (2.6) we have

$$x_3^* \geqslant \zeta_0 \bar{e}_0. \tag{2.24}$$

It is easy to see that

$$\begin{aligned} x_{3,n} - x_3^* &= KF_n x_{3,n} - KFx_3^* \\ &= K(F_n x_{3,n} - Fx_{3,n}) + K(Fx_{3,n} - Fx_3^*) \\ &\geqslant -\tfrac{1}{n}Ke_1 + K(Fx_{3,n} - Fx_3^*) \\ &= -\tfrac{1}{n}\bar{e}_0 + K(Fx_{3,n} - Fx_3^*). \end{aligned} \tag{2.25}$$

Since $\frac{1}{n} \to 0$, $x_{3,n} \to x_3^*(n \to \infty)$ and K is $\bar{e}_0$-continuous, for $\dfrac{\zeta_0}{2} > 0$, there exists $n_1 > 0$ large enough such that

$$-\frac{1}{n}\bar{e}_0 + K(Fx_{3,n_1} - Fx_3) \geqslant -\frac{1}{2}\zeta_0\bar{e}_0. \tag{2.26}$$

It follows from (2.24)–(2.26) that

$$x_{3,n_1} \geqslant x_3^* - \frac{1}{2}\zeta_0\bar{e}_0 \geqslant \frac{1}{2}\zeta_0\bar{e}_0,$$

that is, $x_{3,n_1} \in S_{n_1}^+$. From (2.11), we have for any $x \in S_n^+$ and $n \in \mathbb{N}$,

$$x = K_nG_nx \geqslant K_nG_nu_{2,n} > u_{2,n},$$

and so by (2.2) and the $\bar{e}_n$-positive property of K_n, we have

$$K_nG_nx - K_nG_nu_{2,n} \geqslant \alpha(G_nx - G_nu_{2,n})\bar{e}_n,$$

where $\alpha(G_nx - G_nu_{2,n}) > 0$. This implies that $x \in \Omega_{2,n}$, and so $S_n^+ \subset \Omega_{2,n}$ for each $n \in \mathbb{N}$. Similarly, we have $S_n^- \subset \Omega_{1,n}$ for each $n \in \mathbb{N}$. Hence,

$$x_{3,n_1} \in [u_0, v_0]\backslash\big(Cl_{[u_0,v_0]}\Omega_{1,n_1} \cup Cl_{[u_0,v_0]}\Omega_{2,n_1} \cup \bar{U}(\theta, \bar{r}_{n_1})\big) \subset [u_0, v_0]\backslash(S_{n_1}^+ \cup S_{n_1}^-),$$

which is a contradiction. Therefore, x_3^* is a sign-changing solution of (2.1). The proof is complete.

□

THEOREM 2.2. *Suppose that* (H_1)–(H_4) *hold,* $\beta_0 > \frac{1}{\lambda_1}$, $\beta_0 \neq \frac{1}{\lambda_n}$ *for all* $n \geqslant 2$. *Then* (2.1) *has at least four sign-changing solutions. Moreover,* (2.1) *has at least one positive and one negative solution.*

Proof. In a similar way as that of Theorem 2.1, take a sequence of continuous operators $\{F_n\}$, a sequence of positive numbers $\{M_n\}$ and $r_0 > 0$ such that (2.2)–(2.4) hold. Let S_n^+ and S_n^- be defined as in Theorem 2.1. Then there exists $\zeta_n > 0$ such that (2.6) holds. A similar argument as in Theorem 2.1 shows that K_nG_n is Fréchet differentiable at θ, and $r((K_nG_n)'(\theta)) > 1$. Thus, there exist $u_{2,n}$, $v_{2,n}$ such that (2.10)–(2.12) hold. For each $n \in \mathbb{N}^+$, take $0 < \bar{r}_n < \bar{r}_0 < r_0$ small enough such that $u_{2,n}, v_{2,n} \notin U(\theta, \bar{r}_n)$, $U(\theta, \bar{r}_n) \subset [u_0, v_0]$, K_nG_n has the unique fixed point θ in $U(\theta, \bar{r}_n)$, and

$$i(K_nG_n, U(\theta, \bar{r}_n), [u_0, v_0]) = \deg(I - K_nG_n, U(\theta, \bar{r}_n), \theta) = (-1)^k = \pm 1, \tag{2.27}$$

where k is the sum of all algebraic multiplicities of eigenvalues of $(K_nG_n)'(\theta)$ larger than 1. Since F is e_1-continuous, we may take $\delta_0 > 0$ small enough such that $\widetilde{u}_1 \prec_0 \widetilde{v}_1$, and

$$L\widetilde{u}_1 \prec_1 F\widetilde{u}_1, \quad F\widetilde{v}_1 \prec_1 L\widetilde{v}_1.$$

where $\widetilde{u}_1 = u_1 + \delta_0\bar{e}_0$ and $\widetilde{v}_1 = v_1 - \delta_0\bar{e}_0$. This means that $\widetilde{u}_1$ and $\widetilde{v}_1$ is a pair of well-ordered strict lower and upper solutions of (2.1). Assume without loss of generality that for all $n \in \mathbb{N}$,

$$L\widetilde{u}_1 \prec_1 F_n\widetilde{u}_1, \quad F_n\widetilde{v}_1 \prec_1 L\widetilde{v}_1, \tag{2.28}$$

$$Lu_1 \prec_1 F_nu_1, \quad F_nv_1 \prec_1 Lv_1, \tag{2.29}$$

$$Lu_0 \prec_1 F_nu_0, \quad F_nv_0 \prec_1 Lv_0. \tag{2.30}$$

For each $n \in \mathbb{N}$, let

$$\begin{aligned}
O_{1,n} &= \{x \in [u_0, v_0] \mid \text{there exists } \tau > 0 \text{ such that } K_nG_nx \geqslant K_nG_n\widetilde{u}_1 + \tau\bar{e}_n\},\\
\Omega_{2,n} &= \{x \in [u_0, v_0] \mid \text{there exists } \tau > 0 \text{ such that } K_nG_nx \geqslant K_nG_nu_{2,n} + \tau\bar{e}_n\},\\
\Omega_{3,n} &= \{x \in [u_0, v_0] \mid \text{there exists } \tau > 0 \text{ such that } K_nG_nx \leqslant K_nG_nv_{2,n} - \tau\bar{e}_n\},\\
O_{4,n} &= \{x \in [u_0, v_0] \mid \text{there exists } \tau > 0 \text{ such that } K_nG_nx \leqslant K_nG_n\widetilde{v}_1 - \tau\bar{e}_n\},\\
\Omega_{1,n} &= \{x \in [u_0, v_0] \mid \text{there exists } \tau > 0 \text{ such that } K_nG_nx \geqslant K_nG_nu_1 + \tau\bar{e}_n\},\\
\Omega_{4,n} &= \{x \in [u_0, v_0] \mid \text{there exists } \tau > 0 \text{ such that } K_nG_nx \leqslant K_nG_nv_1 - \tau\bar{e}_n\}.
\end{aligned}$$

Similar to the proof of Theorem 2.1, we have

$$i(K_nG_n, \Omega_{i,n}, [u_0, v_0]) = 1, \quad i = 1, 2, 3, 4, n \in \mathbb{N}, \tag{2.31}$$

$$i(K_nG_n, O_{i,n}, [u_0, v_0]) = 1, \quad i = 1, 4, n \in \mathbb{N}. \tag{2.32}$$

Obviously, $v_0 \in O_{1,n} \cap \Omega_{2,n}$. Next we shall show that for any $x \in \partial_{[u_0,v_0]}(O_{1,n} \cap \Omega_{2,n})$ and $t \in [0, 1]$,

$$x \neq tK_nG_nx + (1 - t)v_0, \tag{2.33}$$

Suppose this is not the case. Then there exists $x_0 \in \partial_{[u_0,v_0]}(O_{1,n} \cap \Omega_{2,n})$ and $t_0 \in [0, 1]$ such that $x_0 = t_0K_nG_nx + (1 - t_0)v_0$. Now we have three cases: (i) $x_0 \in (\partial_{[u_0,v_0]}O_{1,n}) \cap \Omega_{2,n}$; (ii) $x_0 \in O_{1,n} \cap \partial_{[u_0,v_0]}\Omega_{2,n}$; (iii) $x_0 \in \partial_{[u_0,v_0]}O_{1,n} \cap \partial_{[u_0,v_0]}\Omega_{2,n}$. Similar to the proof of (2.14) in Theorem 2.1, we can get contradictions for the above three cases, and so (2.33) holds. Thus, we have

$$i(K_nG_n, O_{1,n} \cap \Omega_{2,n}, [u_0, v_0]) = i(v_0, O_{1,n} \cap \Omega_{2,n}, [u_0, v_0]) = 1. \tag{2.34}$$

Similarly, we have

$$i(K_nG_n, O_{1,n} \cap O_{4,n}, [u_0, v_0]) = i\left(\frac{\widetilde{u}_1 + \widetilde{v}_1}{2}, O_{1,n} \cap O_{4,n}, [u_0, v_0]\right) = 1, \tag{2.35}$$

$$i(K_nG_n, O_{4,n} \cap \Omega_{3,n}, [u_0, v_0]) = i(u_0, O_{4,n} \cap \Omega_{3,n}, [u_0, v_0]) = 1, \tag{2.36}$$

$$i(K_nG_n, O_{1,n} \cap \Omega_{4,n}, [u_0, v_0]) = i\left(\frac{\widetilde{u}_1 + v_1}{2}, O_{1,n} \cap \Omega_{4,n}, [u_0, v_0]\right) = 1, \tag{2.37}$$

$$i(K_nG_n, O_{4,n} \cap \Omega_{1,n}, [u_0, v_0]) = i\left(\frac{u_1 + \widetilde{v}_1}{2}, O_{4,n} \cap \Omega_{1,n}, [u_0, v_0]\right) = 1. \tag{2.38}$$

Let

$$D_{1,n} = O_{1,n} \backslash \big(Cl_{[u_0,v_0]}(O_{1,n} \cap \Omega_{4,n}) \cup Cl_{[u_0,v_0]}(O_{1,n} \cap \Omega_{2,n})\big),$$
$$D_{2,n} = O_{4,n} \backslash \big(Cl_{[u_0,v_0]}(O_{4,n} \cap \Omega_{1,n}) \cup Cl_{[u_0,v_0]}(O_{4,n} \cap \Omega_{3,n})\big).$$

It follows from (2.32)–(2.34) and (2.36)–(2.38) that

$$i(K_nG_n, D_{1,n}, [u_0, v_0]) = -1, \tag{2.39}$$

$$i(K_nG_n, D_{2,n}, [u_0, v_0]) = -1. \tag{2.40}$$

By (2.34)–(2.40), K_nG_n has fixed points $x_{1,n} \in O_{1,n} \cap \Omega_{2,n}$, $x_{2,n} \in O_{4,n} \cap \Omega_{3,n}$, $x_{3,n} \in O_{1,n} \cap O_{4,n}$, $x_{4,n} \in D_{1,n}$, $x_{5,n} \in D_{2,n}$. Similar to the proof of (2.23) in Theorem 2.1, we have

$$\|x_{i,n}\|_X \geqslant \bar{r}_0, \, i = 1, 2, 3, 4, 5, \, n \in \mathbb{N}.$$

A similar argument as in Step 5 of Theorem 2.1 shows that for each $i = 1, 2, 3, 4, 5$ there are sub-sequences of $\{x_{i,n}\}_{n=1}^{\infty}$ (assume without loss of generality that the four sub-sequences are $\{x_{i,n}\}_{n=1}^{\infty}$ themselves) and x_i^* such that $x_{i,n} \to x_i^*(n \to \infty)$. Then, x_i^* is a solution of (2.1) for each $i = 1, 2, 3, 4, 5$. Moreover, by a similar argument as in Step 6 of Theorem 2.1, we see that x_1^* is a positive solution of (2.1), x_2^* is a negative solution of (2.1) and x_3^*, x_4^* and x_5^* are three sign-changing solutions of (2.1). Since $x_{3,n} \in O_{1,n} \cap O_{4,n}$, there exists $\tau > 0$ such that

$$x_{3,n} = K_nG_nx_{3,n} \leqslant K_nG_n\widetilde{v}_1 - \tau\bar{e}_n < K_nG_n\widetilde{v}_1 < \widetilde{v}_1$$

and so $x_3^* \leqslant \widetilde{v}_1$. Next we shall show that $x_3^* \neq x_4^*$. Suppose this is not the case. Then $x_4^* \leqslant \widetilde{v}_1 = v_1 - \delta_0\bar{e}_0$. On the other hand, a similar argument as in Step 6 of Theorem 2.1 shows that

$$\begin{aligned} x_{4,n} - x_4^* &= KF_nx_{4,n} - KFx_4^* \\ &= K(F_nx_{4,n} - Fx_{4,n}) + K(Fx_{4,n} - Fx_4^*) \\ &\leqslant \tfrac{1}{n}Ke_1 + K(Fx_{4,n} - Fx_4^*) \\ &= \tfrac{1}{n}\bar{e}_0 + K(Fx_{4,n} - Fx_4^*). \end{aligned} \tag{2.41}$$

Since F is e_1-continuous and $\frac{1}{n} \to 0(n \to +\infty)$, there exists $n_1 > 0$ large enough such that

$$\frac{1}{n}\bar{e}_0 + K(Fx_{4,n_1} - Fx_4^*) \leqslant \frac{1}{2}\delta_0\bar{e}_0.$$

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Then, by (2.41) we have

$$x_{4,n_1} \leqslant x_4^* + \frac{1}{2}\delta_0\bar{e}_0 \leqslant v_1 - \frac{1}{2}\delta_0\bar{e}_0 < v_1. \tag{2.42}$$

Since K_{n_1} is $\bar{e}_{n_1}$-positive and G_{n_1} is strictly increasing, we have

$$K_{n_1}G_{n_1}x_{4,n_1} \leqslant K_{n_1}G_{n_1}v_1 - \alpha(G_{n_1}v_1 - G_{n_1}x_{4,n_1})\bar{e}_{n_1}, \tag{2.43}$$

where $\alpha(G_{n_1}v_1 - G_{n_1}x_{4,n_1}) > 0$. This implies that $x_{4,n_1} \in \Omega_{4,n_1}$, which contradicts $x_{4,n_1} \in D_{1,n_1}$, and so $x_3^* \neq x_4^*$. Similarly, $x_3^* \neq x_5^*$. From $x_{4,n} \geqslant \widetilde{u}_1$, we have $x_4^* \geqslant \widetilde{u}_1 = u_1 + \delta_0\bar{e}_0$. Then, we can show that $x_5^* \neq x_4^*$. Thus, x_3^*, x_4^* and x_5^* are three distinct sign-changing solutions of (2.1).

Now we show the existence of the fourth sign-changing solution. For each $n \in \mathbb{N}$, let us define $\Omega_{i,n}(i = 1, 2, 3, 4)$ as above. For each $n \in \mathbb{N}$, we have

$$i(K_nG_n, \Omega_{1,n} \cap \Omega_{2,n}, [u_0, v_0]) = 1, \tag{2.44}$$

$$i(K_nG_n, \Omega_{3,n} \cap \Omega_{4,n}, [u_0, v_0]) = 1, \tag{2.45}$$

$$i(K_nG_n, \Omega_{4,n} \cap \Omega_{1,n}, [u_0, v_0]) = 1, \tag{2.46}$$

$$i(K_nG_n, [u_0, v_0], [u_0, v_0]) = 1. \tag{2.47}$$

Let

$$\widetilde{D}_{1,n} = \Omega_{1,n}\backslash\big(Cl_{[u_0,v_0]}(\Omega_{1,n} \cap \Omega_{4,n}) \cup (Cl_{[u_0,v_0]}(\Omega_{1,n} \cap \Omega_{2,n})\big),$$
$$\widetilde{D}_{2,n} = \Omega_{4,n}\backslash\big(Cl_{[u_0,v_0]}(\Omega_{1,n} \cap \Omega_{4,n}) \cup (Cl_{[u_0,v_0]}(\Omega_{4,n} \cap \Omega_{3,n})\big).$$

It follows from (2.31) and (2.44)–(2.46) that

$$i(K_nG_n, \widetilde{D}_{1,n}, [u_0, v_0]) = -1, \tag{2.48}$$

$$i(K_nG_n, \widetilde{D}_{2,n}, [u_0, v_0]) = -1. \tag{2.49}$$

By (2.27), (2.31) and (2.46)–(2.49), we have

$$i\big(K_nG_n, [u_0, v_0]\backslash\big(Cl_{[u_0,v_0]}\widetilde{D}_{1,n} \cup Cl_{[u_0,v_0]}\widetilde{D}_{2,n} \cup Cl_{[u_0,v_0]}(\Omega_{1,n} \cap \Omega_{4,n}) \cup Cl_{[u_0,v_0]}\Omega_{2,n}$$
$$\cup\, Cl_{[u_0,v_0]}\Omega_{3,n} \cup \bar{U}(\theta, \bar{r}_n)\big), [u_0, v_0]\big) = 1 - (-1) - (-1) - 1 - 1 - 1 - (\pm 1) = \mp 1.$$

Therefore, for each $n \in \mathbb{N}$, K_nG_n has a fixed point

$$x_{6,n} \in [u_0, v_0]\backslash\big(Cl_{[u_0,v_0]}\widetilde{D}_{1,n} \cup Cl_{[u_0,v_0]}\widetilde{D}_{2,n} \cup Cl_{[u_0,v_0]}(\Omega_{1,n} \cap \Omega_{4,n}) \cup Cl_{[u_0,v_0]}\Omega_{2,n}$$
$$\cup\, Cl_{[u_0,v_0]}\Omega_{3,n} \cup \bar{U}(\theta, \bar{r}_n)\big).$$

Then, by the method of Theorem 2.1, we see that $\|x_{6,n}\|_X \geqslant \bar{r}_0$ for each $n \in \mathbb{N}$. Similar to the proof of Steps 5 and 6 in Theorem 2.1, we see that there exist a sub-sequence of $\{x_{6,n}\}_{n=1}^{\infty}$ (assume without loss of generality that the sub-sequence is $\{x_{6,n}\}_{n=1}^{\infty}$ itself) and x_6^* such that $x_{6,n}^* \to x_6^*$ as $n \to \infty$ and x_6^* is a sign-changing solution of (2.1). A similar way to that of showing (2.41)–(2.43) yields that x_6^* is different from x_3^*, x_4^*, x_5^*. Thus, $x_3^*, \dots, x_6^*$ are four sign-changing solutions of (2.1). The proof is complete. □

REMARK 2.2. In [**10**] we obtained some existence results for sign-changing solutions of a three-point boundary value problem. The method to show the main results in [**10**] are different from that of this paper. We obtained the main results in [**10**] by using the modification functions technique and the Leray–Schauder degree method. But in this paper we obtained the main results by using the fixed point index method and the (e_1, B)-limit-increasing operator method.

THEOREM 2.3. *Suppose that* (H_1)*–*(H_4) *hold,* $\beta_0 < \frac{1}{\lambda_1}$. *Then* (2.1) *has at least four sign-changing solutions. Moreover,* (2.1) *has at least two positive and two negative solutions.*

Proof. The proof is similar to that of Theorem 2.2. For completeness, we will sketch the proof. Take $\{F_n\}$, $\{M_n\}$ and r_0 such that (2.2)–(2.4) hold. Then K_nG_n is Fréchet differentiable at θ, 1 is not an eigenvalue of $(K_nG_n)'(\theta) = (\beta_0 + M_n)K_nB$ and $r((K_nG_n)'(\theta)) < 1$ for each $n \in \mathbb{N}$. By Lemma 2.1, there exists $\tau_n > 0$ such that for any $\tau \in (0, \tau_n]$,

$$-\tau h_n < K_nG_n(-\tau h_n),\, K_nG_n(\tau h_n) < \tau h_n,$$

where h_n is the eigenfunction of $(K_nG_n)'(\theta)$ corresponding to the eigenvalue $r((K_nG_n)'(\theta))$.

For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, take $\varepsilon_i \in (0, \tau_n]$ small enough such that

$$u_{2,n}^{(i)} \not\leqslant v_1,\, u_1 \not\leqslant v_{2,n}^{(i)},$$

$$u_0 < u_{2,n}^{(i)} < \theta < v_{2,n}^{(i)} < v_0,$$

and $\varepsilon_i \to 0$ as $i \to \infty$, where $u_{2,n}^{(i)} = -\varepsilon_i h_n$, $v_{2,n}^{(i)} = \varepsilon_i h_n$. Similar to the proof of Theorem 2.2, we may take $\delta_0 > 0$ small enough such that $L\widetilde{u}_1 \prec_1 F\widetilde{u}_1, F\widetilde{v}_1 \prec_1 L\widetilde{v}_1$ and $\widetilde{u}_1 \prec_0 \widetilde{v}_1$, where $\widetilde{u}_1 = u_1 + \delta_0\bar{e}_0$ and $\widetilde{v}_1 = v_1 - \delta_0\bar{e}_0$. Assume without loss of generality that (2.28)–(2.30) hold for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, let

$$\Omega_{1,n} = \{x \in [u_0, v_0] | \text{ there exists } \tau > 0 \text{ such that } K_nG_nx \geqslant K_nG_nu_1 + \tau\bar{e}_n\},$$
$$\Omega_{2,n}^{(i)} = \{x \in [u_0, v_0] | \text{ there exists } \tau > 0 \text{ such that } K_nG_nx \geqslant K_nG_nu_{2,n}^{(i)} + \tau\bar{e}_n\},$$
$$\Omega_{3,n}^{(i)} = \{x \in [u_0, v_0] | \text{ there exists } \tau > 0 \text{ such that } K_nG_nx \leqslant K_nG_nv_{2,n}^{(i)} - \tau\bar{e}_n\},$$
$$\Omega_{4,n} = \{x \in [u_0, v_0] | \text{ there exists } \tau > 0 \text{ such that } K_nG_nx \leqslant K_nG_nv_1 - \tau\bar{e}_n\},$$
$$O_{1,n} = \{x \in [u_0, v_0] | \text{ there exists } \tau > 0 \text{ such that } K_nG_nx \geqslant K_nG_n\widetilde{u}_1 + \tau\bar{e}_n\},$$
$$O_{4,n} = \{x \in [u_0, v_0] | \text{ there exists } \tau > 0 \text{ such that } K_nG_nx \leqslant K_nG_n\widetilde{v}_1 - \tau\bar{e}_n\},$$
$$D_{1,n}^{(i)} = O_{1,n} \backslash \big(Cl_{[u_0,v_0]}(O_{1,n} \cap \Omega_{4,n}) \cup Cl_{[u_0,v_0]}(O_{1,n} \cap \Omega_{2,n}^{(i)})\big),$$
$$D_{2,n}^{(i)} = O_{4,n} \backslash \big(Cl_{[u_0,v_0]}(O_{4,n} \cap \Omega_{1,n}) \cup Cl_{[u_0,v_0]}(O_{4,n} \cap \Omega_{3,n}^{(i)})\big),$$

$$D_{3,n}^{(i)} = \Omega_{3,n}^{(i)} \backslash \big(Cl_{[u_0,v_0]}(\Omega_{4,n} \cap \Omega_{3,n}^{(i)}) \cup Cl_{[u_0,v_0]}(\Omega_{3,n} \cap \Omega_{2,n}^{(i)})\big),$$
$$D_{4,n}^{(i)} = \Omega_{1,n}^{(i)} \backslash \big(Cl_{[u_0,v_0]}(\Omega_{1,n} \cap \Omega_{4,n}) \cup Cl_{[u_0,v_0]}(\Omega_{1,n} \cap \Omega_{2,n}^{(i)})\big).$$

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Similar to the proof of Theorem 2.1, we can show that for each $n \in \mathbb{N}$ and some $i_0 \in \mathbb{N}$,

$$i(K_nG_n, O_{1,n} \cap O_{4,n}, [u_0, v_0]) = 1, \tag{2.50}$$

$$i\big(K_nG_n, D^{(i_0)}_{j,n}, [u_0, v_0]\big) = -1,\ j = 1, 2, 3, 4. \tag{2.51}$$

By (2.50) and (2.51), K_nG_n has fixed points $x_{1,n} \in O_{1,n} \cap O_{4,n}$, $x_{2,n} \in D^{(i_0)}_{1,n}$ and $x_{3,n} \in D^{(i_0)}_{2,n}$. Moreover,

$$i(K_nG_n, [u_0, v_0], [u_0, v_0]) = 1, \tag{2.52}$$

$$i(K_nG_n, \Omega^{(i_0)}_{2,n}, [u_0, v_0]) = 1, \tag{2.53}$$

$$i(K_nG_n, \Omega_{4,n}, [u_0, v_0]) = 1. \tag{2.54}$$

It follows from (2.51)–(2.54) that

$$\begin{aligned} i\big(K_nG_n, [u_0, v_0]\backslash\big(&Cl_{[u_0,v_0]}\Omega^{(i_0)}_{2,n} \cup Cl_{[u_0,v_0]}\Omega_{4,n} \cup Cl_{[u_0,v_0]}D^{(i_0)}_{3,n} \\ &\cup Cl_{[u_0,v_0]}D^{(i_0)}_{4,n}\big), [u_0, v_0]\big) = 1 - 1 - 1 - (-1) - (-1) = 1. \end{aligned}$$

Therefore, K_nG_n has a fixed point

$$x_{4,n} \in [u_0, v_0]\backslash\big(Cl_{[u_0,v_0]}\Omega^{(i_0)}_{2,n} \cup Cl_{[u_0,v_0]}\Omega_{4,n} \cup Cl_{[u_0,v_0]}D^{(i_0)}_{3,n} \cup Cl_{[u_0,v_0]}D^{(i_0)}_{4,n}\big).$$

Since $[\theta, v_0] \subset \Omega^{(i_0)}_{2,n}$ and $[u_0, \theta] \subset \Omega^{(i_0)}_{3,n}$, we easily see that $x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n} \in X\backslash(P \cup (-P))$. By a similar way as that of Theorem 2.2 we can show that for each $n \in \mathbb{N}^+$ there exists $\bar{r}_n > 0$ such that $\bar{r}_n < \bar{r}_0 < r_0$ and K_nG_n has the unique solution θ in $U(\theta, \bar{r}_n)$. Then as the proof of Theorem 2.1, we can show that $\|x_{i,n}\|_X \geqslant \bar{r}_0$ for $i = 1, 2, 3, 4$ and $n \in \mathbb{N}$, and there exist sub-sequences of $\{x_{i,n}\}_{n=1}^{\infty}(i = 1, 2, 3, 4)$ (assume without loss of generality that the sub-sequences are $\{x_{i,n}\}_{n=1}^{\infty}(i = 1, 2, 3, 4)$ themselves) and $x_i^*(i = 1, 2, 3, 4)$ such that $x_{i,n} \to x_i^*$ as $n \to \infty$ for each $i = 1, 2, 3, 4$. It is easy to see that x_1^*, x_2^*, x_3^* and x_4^* are four distinct sign-changing solutions of (2.1).

Next we shall show that (2.1) has at least two positive solutions. It is easy to see that for each $i, n \in \mathbb{N}$,

$$i(K_nG_n, O_{1,n} \cap \Omega^{(i)}_{2,n}, [u_0, v_0]) = 1. \tag{2.55}$$

Then K_nG_n has fixed point $x^{(i)}_{5,n} \in O_{1,n} \cap \Omega^{(i)}_{2,n}$. Similar to the above argument, we see that $\|x^{(i)}_{5,n}\|_X \geqslant \bar{r}_0$ for all $i, n \in \mathbb{N}$. Since $x^{(i)}_{5,n} \in O_{1,n} \cap \Omega^{(i)}_{2,n}$, we have

$$x^{(i)}_{5,n} = K_nG_nx^{(i)}_{5,n} \geqslant K_nG_nu^{(i)}_{2,n} + \tau\bar{e}_n \geqslant K_nG_nu^{(i)}_{2,n} > u^{(i)}_{2,n}. \tag{2.56}$$

A similar argument as given in Theorem 2.1 shows that there exist a sub-sequence of $\{x^{(i)}_{5,n}\}$ (assume without loss of generality that the sub-sequence is $\{x^{(i)}_{5,n}\}$ itself) and $x^{(i)}_5$ such that $x^{(i)}_{5,n} \to x^{(i)}_5$ as $n \to \infty$. Obviously, $x^{(i)}_5$ is a solution of (2.1), and so $x^{(i)}_5 = KFx^{(i)}_5$. From $\{x^{(i)}_5 | i = 1, 2, \ldots\} \subset [u_0, v_0]$, we see that $\{x^{(i)}_5 | i = 1, 2, \ldots\}$ is bounded. Thus, $\{x^{(i)}_5 | i = 1, 2, \ldots\}$ is a relatively compact set because K is a completely continuous

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operator. Assume that $x_5^{(i)} \to x_5^*$ as $i \to \infty$, then $x_5^* = KFx_5^*$, that is $Lx_5^* = Fx_5^*$. This means that x_5^* is a solution of (2.1). It follows from (2.56) that $x_5^* \geqslant \theta$. Since $\|x_{5,n}^{(i)}\|_X \geqslant \bar{r}_0$, x_5^* is not a zero solution of (2.1). Thus, x_5^* is a positive solution of (2.1).

Also, we can show that for each $i \in \mathbb{N}$ and $n \in \mathbb{N}$ and some $j_0 \in N$,

$$i\big(K_nG_n, \Omega_{2,n}^{(i)} \backslash \big(Cl_{[u_0,v_0]}(\Omega_{2,n}^{(i)} \cap \Omega_{3,n}^{(j_0)}) \cup Cl_{[u_0,v_0]}(\Omega_{2,n}^{(i)} \cap \Omega_{1,n})\big), [u_0, v_0]\big) = -1.$$

Thus, K_nG_n has a fixed point $x_{6,n}^{(i)} \in \Omega_{2,n}^{(i)} \backslash \big(Cl_{[u_0,v_0]}(\Omega_{2,n}^{(i)} \cap \Omega_{3,n}^{(j_0)}) \cup Cl_{[u_0,v_0]}(\Omega_{2,n}^{(i)} \cap \Omega_{1,n})\big)$. Similar to the above argument, we may assume that $x_{6,n}^{(i)} \to x_6^{(i)}$ as $n \to \infty$ and $x_6^{(i)} \to x_6^*$ as $i \to \infty$. Then x_6^* is a positive solution of (2.1). It is easy to see that $x_6^* \neq x_5^*$. Therefore, x_5^* and x_6^* are two positive solutions of (2.1). Similarly, we can show that (2.1) has at least two negative solutions, x_7^* and x_8^*. The proof is complete. □

REMARK 2.3. In Theorems 2.2 and 2.3 we not only obtained multiplicity results for sign-changing solutions but also made a clear description of positions of these solutions of the non-linear operator equation (2.1). In order to show the main results, we have constructed some strict upper or lower solutions of (2.1). Some pairs of these strict upper and lower solutions are well ordered and others are not well ordered. Especially, some pairs of strict upper and lower solutions are parallel to each other. For other discussions concerning the parallel pairs of upper and lower solutions, the reader is refereed to [**10**, **13**].

3. Applications of the abstract results in differential boundary value problems. In this section we will apply the main results of Section 2 to study the Sturm–Liouville differential boundary value problem

$$\begin{cases} -(p(t)u')' - q(t)u = f(t, u(t)), & t \in I, \\ R_0(u) := au(0) - bu'(0) = 0, \\ R_1(u) := cu(1) + du'(1) = 0, \end{cases} \tag{3.1}$$

where $I = [0, 1]$, $p(t) \in C^1(I)$, $q(t) \leqslant 0$, $p(t) > 0 (\forall t \in I)$, $a, b, c, d \geqslant 0$, $a^2 + b^2 \neq 0$, $c^2 + d^2 \neq 0$.

Let $X = C(I)$ denote the Banach space of all continuous functions on I with the maximum norm $\|\cdot\|_X$, and $Z = X \times R^2$. For each $\widetilde{x} = (x(t), l, m) \in Z$, let

$$\|\widetilde{x}\|_Z = \|x(t)\|_X + |l| + |m|.$$

Then $(Z, \|\cdot\|_Z)$ is the real Banach space. Let $P = \{x = x(t) \in X | x(t) \geqslant 0, t \in I\}$ and $P_1 = \{\widetilde{x} = (x(t), l, m) \in Z | x(t) \in P, l \geqslant 0, m \geqslant 0\}$. Then P and P_1 are normal cones of X and Z, respectively.

Now let us introduce the following conditions to be used.

(A_1) $f \in C(I \times R^1, R^1)$, $f(t, 0) = 0$, and $f(t, x)$ is locally continuous differentiable with x at $x = 0$.

(A_2) There exists $\beta_0 > 0$ such that $\lim\limits_{x \to 0} \dfrac{f(t, x)}{x} = \beta_0$ uniformly with $t \in I$.

(A_3) There exist $u_0(t), v_0(t) \in X$, $u_0(t) < 0 < v_0(t)$ for all $t \in I$, u_0, v_0 are strict lower and upper solutions of (3.1).

(A_4) There exist $u_1, v_1 \in X$ such that u_1 and v_1 are sign-changing on I and $u_0(t) < u_1(t) < v_1(t) < v_0(t)$ for all $t \in I$, u_1, v_1 are strict lower and upper solutions of (3.1).

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Let $\{\lambda_n\}$ denote the decreasing sequence of the positive eigenvalues of the linear problem:

$$\begin{cases} -(p(t)u')' - q(t)u = \dfrac{1}{\lambda_n}u, \ t \in (0,1), \\ R_0(u) = R_1(u) = 0. \end{cases}$$

The following Lemmas 3.1–3.3 can be found in Section 3 of [**13**].

LEMMA 3.1. *For each $M \geqslant 0$, let $\varphi_M(t)$ and $\psi_M(t)$ satisfy*

$$\begin{cases} (p(t)\varphi_M'(t))' + (q(t) - M)\varphi_M(t) = 0, \quad t \in I, \\ \varphi_M(1) = d, \varphi_M'(1) = -c, \end{cases} \tag{3.2}$$

$$\begin{cases} (p(t)\psi_M'(t))' + (q(t) - M)\psi_M(t) = 0, \quad t \in I, \\ \psi_M(0) = b, \psi_M'(0) = a, \end{cases} \tag{3.3}$$

respectively. Then we have

(i) $\varphi_M(t)$ is non-increasing in $[0, 1)$ with $\varphi_M(t) > 0 (t \in [0, 1))$.
(ii) $\psi_M(t)$ is non-decreasing in $(0, 1]$ with $\psi_M(t) > 0 (t \in (0, 1])$.
(iii) φ_M and ψ_M are linearly independent.
(iv) $p(t)(\varphi_M(t)\psi_M'(t) - \varphi_M'(t)\psi_M(t)) = \omega$, where ω is a positive constant.
(v) $R_0(\varphi_M) \neq 0, R_1(\varphi_M) = 0$.
(vi) $R_0(\psi_M) = 0, \ R_1(\psi_M) \neq 0$.

LEMMA 3.2. *Let $g \in C(I)$, $m, l \in R^1$. Then $u \in C^2(I)$ is a solution to the following boundary value problem*

$$\begin{cases} -(p(t)u')' - (q(t) - M)u = g(t), \quad t \in I, \\ R_0(u) = l, R_1(u) = m, \end{cases}$$

if and only if

$$u(t) = \frac{\varphi_M(t)}{R_0(\varphi_M)}l + \frac{\psi_M(t)}{R_1(\psi_M)}m + \int_0^1 G(t,s)g(s)\,ds, \quad t \in I,$$

where

$$G_M(t,s) = \frac{1}{\omega}\begin{cases} \varphi_M(t)\psi_M(s), & s \leqslant t, \\ \varphi_M(s)\psi_M(t), & s > t. \end{cases}$$

LEMMA 3.3. *For any $M \geqslant 0$, $x \in P$, $l, m \geqslant 0$, let*

$$y(t) = \frac{\varphi_M(t)}{R_0(\varphi_M)}l + \frac{\psi_M(t)}{R_1(\psi_M)}m + \int_0^1 G_M(t,s)x(s)\,ds, \quad t \in I.$$

Then

$$y(t) \geqslant \|y\|_X \bar{e}_M(t), \quad t \in I,$$

where $\bar{e}_M(t) = \frac{1}{\gamma_M^2}\varphi_M(t)\psi_M(t)$, $\gamma_M = \max\{\|\varphi_M\|_X, \|\psi_M\|_X\}$.

THEOREM 3.1. *Suppose* (A_1)–(A_3) *hold, and* $\beta_0 \in (\frac{1}{\lambda_{2n_0}}, \frac{1}{\lambda_{2n_0+1}})$, *where* n_0 *is a natural number. Then (3.1) has at least one sign-changing solution. Moreover, (3.1) has at least one positive and one negative solution.*

Proof. Let us define the linear operators $L : \text{dom L} = C^2(I) \subset X \mapsto Z, B : X \mapsto Z$ and the non-linear operator $F : X \mapsto Z$ by

$$Lx = (-(p(t)x'(t))' - q(t)x(t), ax(0) - bx'(0), cx(1) + dx'(1)),$$
$$Bx = (x(t), 0, 0), \ \ Fx = (f(t, x(t)), 0, 0),$$

respectively. Then, we need to consider the operator equation

$$Lu = Fu, \quad u \in \text{dom}L.$$

For each $M \geqslant 0$, by Lemma 3.1–3.3 and a method in [**13**] we can easily show that $(L + MB)^{-1}$ exists, $(L + MB)^{-1}$ is $\bar{e}_M$-positive and $\bar{e}_M$-continuous (the details of the proof, one can find in Section 3 of [**13**]). By the well-known Sturm–Liouville theory of linear boundary value problems, we easily see that the algebraic multiplicity of each positive eigenvalue of $K_M B$ is 1. Thus, (H_3) holds.

Since $f(t, x)$ is locally continuous differentiable with x at $x = 0$, there exist $r_0 > 0$ and $\tau_0 > 0$ such that

$$f(t, x_2) - f(t, x_1) \geqslant -\tau_0(x_2 - x_1), \quad \forall x_2, x_1 \in [-r_0, r_0], x_2 > x_1, \quad t \in I.$$

Let $R_0 > r_0$. Now since $f : I \times [r_0, R_0] \mapsto R^1$ is continuous, for any $n \in \mathbb{N}$, there exists $\widetilde{g}_1^{(n)} : I \times [r_0, R_0] \mapsto R^1$ infinitely differentiable such that

$$|f(t, x) - g_1^{(n)}(t, x)| < \frac{1}{2n}, \forall (t, x) \in I \times [r_0, R_0].$$

Let

$$g_{1,n}(t, x) = \widetilde{g}_1^{(n)}(t, x) - (\widetilde{g}_1^{(n)}(t, r_0) - f(t, r_0)), \quad \forall (t, x) \in I \times [r_0, R_0].$$

Then, $g_{1,n}$ is infinitely differentiable with x, and $g_{1,n}(t, r_0) = f(t, r_0)(\forall t \in I)$. For any $n \in \mathbb{N}$, we have

$$|f(t, x) - g_{1,n}(t, x)| \leqslant |f(t, x) - \widetilde{g}_1^{(n)}(t, x)| + |\widetilde{g}_1^{(n)}(t, r_0) - f(t, r_0)| < \frac{1}{n}, \quad \forall (t, x) \in I \times [r_0, R_0].$$

Since $g_{1,n}$ is infinitely differentiable with x, there exists $\tau_{1,n} > 0$ such that

$$g_{1,n}(t, x_2) - g_{1,n}(t, x_1) > -\tau_{1,n}(x_2 - x_1), \quad \forall (t, x_1), \ (t, x_2) \in I \times [r_0, R_0], \ x_2 > x_1.$$

Similarly, there exists $g_{2,n}(t, x)$, which is continuous on $I \times [-R_0, -r_0]$ and infinitely differentiable such that $g_{2,n}(t, -r_0) = f(t, -r_0)(\forall t \in I)$, and

$$|f(t, x) - g_{2,n}(t, x)| < \frac{1}{n}, \quad \forall (t, x) \in I \times [-R_0, -r_0].$$

Also, there exists $\tau_{2,n} > 0$ such that

$$g_{2,n}(t, x_2) - g_{2,n}(t, x_1) > -\tau_{2,n}(x_2 - x_1), \quad \forall (t, x_2), (t, x_1) \in I \times [-R_0, -r_0], x_2 > x_1.$$

For each $n \in \mathbb{N}$, let $f_n : I \times [-R_0, R_0] \mapsto R^1$ be defined by

$$f_n(t,x) = \begin{cases} g_{1,n}(t,x), & (t,x) \in I \times [r_0, R_0]; \\ f(t,x), & (t,x) \in I \times [-r_0, r_0]; \\ g_{2,n}(t,x), & (t,x) \in I \times [-R_0, -r_0]. \end{cases}$$

Then, $f_n \in C(I \times [-R_0, R_0])$, and for each $n \in \mathbb{N}$,

$$|f(t,x) - f_n(t,x)| < \frac{1}{n}, \forall (t,x) \in I \times [-R_0, R_0].$$

Let $M_n = \tau_{1,n} + \tau_{2,n} + \tau_0$. Then we have

$$f_n(t,x_2) - f_n(t,x_1) > -M_n(x_2 - x_1), \quad \forall (t,x_2), (t,x_1) \in I \times [-R_0, R_0], x_2 > x_1.$$

Define $F_n : X \mapsto Z$ by

$$F_n x = (f_n(t, x(t)), 0, 0).$$

Then we have

$$-\frac{1}{n} e_1 < F_n x - F x < \frac{1}{n} e_1,$$

where $\bar{x}_0(t) \equiv 1 (\forall t \in I)$ and $e_1 = (\bar{x}_0(t), 0, 0)$. For each $x \in U(\theta, r_0)$ we have

$$F_n x = F x, n = 1, 2, \ldots .$$

Therefore, F is (e_1, B)-limit increasing on any bounded set of X and is locally increasing at θ. It follows from (A_2) that F is Fréchet differentiable at θ and $F'(\theta) = \beta_0 B$. We see from the continuity of f that F is e_1-continuous. This means that (H_2) holds. From (A_3) we see that u_0 and v_0 are strict lower and upper solutions of (3.1), respectively. This means that (H_1) holds. Thus, all conditions of Theorem 2.1 are satisfied. Now the conclusion of Theorem 3.1 follows from Theorem 2.1. The proof is complete. □

By Theorems 2.2 and 2.3 we can show the following Theorems 3.2. and 3.3.

THEOREM 3.2. *Suppose that* (A_1)–(A_4) *hold*, $\beta_0 > \frac{1}{\lambda_1}$, $\beta_0 \neq \frac{1}{\lambda_n}$ *for all* $n \geqslant 2$. *Then the boundary value problem* (3.1) *has at least four sign-changing solutions. Moreover, the boundary value problem* (3.1) *has at least one positive and one negative solution.*

THEOREM 3.3. *Suppose that* (A_1)–(A_4) *hold*, $\beta_0 < \frac{1}{\lambda_1}$. *Then the boundary value problem* (3.1) *has at least four sign-changing solutions. Moreover, the boundary value problem* (3.1) *has at least two positive and two negative solutions.*

REMARK 3.1. In Theorems 2.2 and 2.3 we have employed a condition of a pair of well-ordered strict lower and upper solutions that are sign-changing. As pointed out in [**10**], it is very difficult to construct a pair of well-ordered strict lower and upper solutions. However, a concrete numeral example shows that such a pair of well-ordered strict lower and upper solutions does exist, see [**10**].

REMARK 3.2. In Theorems 2.1–2.3, by combining the fixed point index method and the concept of (e_1, B)-limit-increasing operator we have obtained some multiplicity results for sign-changing solutions. Obviously, if we combine the critical point theory

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method with the concept of (e_1, B)-limit-increasing operator, we may obtain some interesting results for sign-changing critical point of non-linear boundary value problems.

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