A NOTE ON THE CARADUS CLASS $\mathcal{G}$ OF BOUNDED LINEAR OPERATORS ON A COMPLEX BANACH SPACE

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1. In a recent paper (1) on meromorphic operators, Caradus introduced the class $\mathcal{G}$ of bounded linear operators on a complex Banach space $X$. A bounded linear operator $T$ is put in the class $\mathcal{G}$ if and only if its spectrum consists of a finite number of poles of the resolvent of $T$. Equivalently, $T$ is in $\mathcal{G}$ if and only if it has a rational resolvent (8, p. 314).

Some ten years ago (in May, 1957), I discovered a property of the class $\mathcal{G}$ which may be of interest in connection with Caradus' work, and is the subject of the present note.

2. Theorem. Let $X$ be a complex Banach space. If $T$ belongs to the class $\mathcal{G}$, and the linear operator $S$ commutes with every bounded linear operator which commutes with $T$, then there is a polynomial $p$ such that $S = p(T)$.

Suppose that $T$ and $S$ satisfy the hypothesis of the theorem. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the points of the spectrum of $T$, which by hypothesis are poles of the resolvent of $T$, and let $\nu_1, \nu_2, \ldots, \nu_n$ be the orders of those poles, respectively. Let $M_r$ be the kernel (or “null manifold”) of $(T - \lambda_r I)^r$ ($r = 1, 2, \ldots, n$). Then $X = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ (8, p. 317, Theorem 5.9-E). For typographical convenience we write $T_r$ for $T - \lambda_r I$ ($r = 1, 2, \ldots, n$).

Now let $x$ be any member of $M_r$ (where $r$ is any integer with $1 \leq r \leq n$). Choose a bounded linear functional $f$ on $X$ such that $(T_{r-1} f) T_r x = 0$ but $(T_r^* f) T_r x \neq 0$; such an $f$ exists since $\lambda_r$ is also a pole of order $\nu_r$ of the resolvent of the adjoint $T^*$ of $T$ (3, p. 568, Theorem VII.3.7). We now consider the bounded linear operator

$$V = \sum_{s=1}^{r} T_r^{-s} (x \otimes f) T_r^{r-s},$$

where $x \otimes f$ denotes the operator $y \rightarrow f(y)x$ on $X$ into itself; cf. (7, p. 110). In view of our choice of $x$ and $f$, we have:

$$T_r V = \sum_{s=1}^{r-1} T_r^{-s} (x \otimes f) T_r^{r-s} = VT_r,$$

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so that \( V \) commutes with \( T_r \), and thus with \( T \). Hence (by hypothesis), \( V \) commutes with \( S \).

Now

\[
f, \ T_r \cdot f, \ (T_r \cdot f)_2, \ldots, \ (T_r \cdot f)_{r-1}\f
\]

are clearly linearly independent (if \( \sum_{s=1}^{r} \alpha_s (T_r \cdot f)^{s-1}\f = 0 \), then

\[
\sum_{s=1}^{r} \alpha_s (T_r \cdot f)^{s-2}\f = 0,
\]

and hence \( \alpha_1 = 0, \ \sum_{s=1}^{r} \alpha_s (T_r \cdot f)^{r+s-2}\f = 0 \), and therefore \( \alpha_2 = 0 \), and so on), and thus a point \( y \) of \( X \) can be found such that

\[
[(T_r \cdot f)^{r-1}\f](y) = 1, \quad [(T_r \cdot f)^{s-1}\f](y) = 0 \quad (s = 1, 2, \ldots, r - 1),
\]

that is,

\[
f(T_r \cdot f)^{r-1}\f = 1, \quad f(T_r \cdot f)^{s-1}\f = 0 \quad (s = 2, 3, \ldots, r)
\]

(cf. 2, p. 6, Theorem I.2.2, Corollary 2). Then \( SVy = VSy \), and therefore

\[
\sum_{s=1}^{r} \ T_r^{s-1} (x \otimes f) T_r^{r-s} y = \sum_{s=1}^{r} \ T_r^{s-1} (x \otimes f) T_r^{r-s} Sy,
\]

that is,

\[
Sx = \sum_{s=1}^{r} f(T_r \cdot f)^{s-1}\f T_r^{s-1} x
\]

\[
= \sum_{s=1}^{r} f(T_r \cdot f)^{s-1}\f (T - \lambda I)^{s-1} x.
\]

However, the choice of \( f \) and \( y \) was quite independent of the choice of \( x \in M_r \). Hence,

\[
Sx = p_r(T)x
\]

for every \( x \in M_r \), where \( p_r \) is the polynomial given by

\[
p_r(\lambda) = \sum_{s=1}^{r} f(T_r \cdot f)^{s-1}\f (\lambda - \lambda_r)^{s-1}.
\]

Having chosen a polynomial \( p_r \) as above for each \( r = 1, 2, \ldots, n \), we now choose a polynomial \( p \) such that

\[
p^{(s)}(\lambda_r) = p_r^{(s)}(\lambda_r) \quad (s = 0, 1, 2, \ldots, r - 1; r = 1, 2, \ldots, n).
\]

This can certainly be done; for example we can take

\[
p = p_1 \cdot \phi_1 + p_2 \cdot \phi_2 + \ldots + p_n \cdot \phi_n,
\]

where \( \phi_r \) is given by

\[
\phi_r(\lambda) = \left[ \prod_{s=1}^{r} \ (\lambda - \lambda_s)^{r-s} \right] \Phi_r(\lambda),
\]
A. F. RUSTON

\[ \Phi_r(\lambda) \text{ being the sum of the first } v_r \text{ terms in the expansion of } \left[ \prod_{s=1}^{\infty} (\lambda - \lambda_s)^{v_r} \right]^{-1} \]

as a power series in \( \lambda - \lambda_r \) (this generalizes, in effect, the Lagrange interpolation formula, which corresponds to the case \( v_1 = v_2 = \ldots = v_n = 1 \); that such a generalization is possible is, of course, well known; cf. \( (6; 5; 4) \); the last two refer specifically to the Hermite interpolation formula, which corresponds to the case \( v_1 = v_2 = \ldots = v_n = 2 \). Then

\[ p(T)x = p_r(T)x = Sx \]

for every \( x \in M_r \) (3, p. 571, Theorem VII.3.16; 8, p. 307, Theorem 5.8-B). Hence,

\[ p(T)x = Sx \]

for every \( x \in M_1 \oplus M_2 \oplus \ldots \oplus M_n = X \), and therefore \( S = p(T) \), as required. Incidentally, \( \phi_r(T) \) is the spectral projection of \( X \) onto \( M_r \); cf. (8, § 5.9, p. 319, Problem 3).

**Note.** Since \( V \) is of finite rank, and thus a member of \( \mathfrak{F} \), we have in fact proved the following, slightly stronger, result.

**If** \( T \in \mathfrak{F} \), **and the linear operator** \( S \) **commutes with every member** \( \mathfrak{F} \) **which commutes with** \( T \), **then there is a polynomial** \( p \) **such that** \( S = p(T) \).

**References**


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