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DECOMPOSITIONS OF LOCALIZED FIBRES AND COFIBRES

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ABSTRACT. In this paper p-local versions of the Rational Fibre and Cofibre Decomposition Theorems are given. In particular, if there exists an element in the nth Gottlieb group of a space F such that its image under the Hurewicz map has infinite order, then $F \simeq_p \mathscr{F} \times S^n$ for almost all primes p. A dual result is proved for cofibrations.

Introduction. In topology it has become commonplace to study spaces in terms of the fibrations of which they are a part. The fact of a space's position in a given fibration often reveals a great deal about the structure of the space itself. This viewpoint is exemplified by Cartan's analysis [1] of the cohomological structure of homogeneous spaces via consideration of the fibration

$$G/K \to BK \to BG.$$

As a further example of this approach we mention the following result independently proved by several authors. (See the list in [10] and add to it I. Berstein, who, according to John Harper, knew the result in the early 1960's.)

THEOREM 1. *If $F \to E \xrightarrow{p} B$ is a fibration of simply connected rational spaces of finite type and $p_\#$ is injective, then F has the homotopy type of a product of $K(\mathbf{Q}, n)$'s.*

This theorem and the result of Cartan form a starting point for an investigation into the structure of spaces which occur as fibres. The rational situation was considered in [10]:

THEOREM 2. *If $F \to E \to B$ is a fibration of simply connected rational spaces of finite type, then there is a subproduct $K \subset \Omega B$ and a space $\mathscr{F}$ such that $F \simeq \mathscr{F} \times K$ and $H^*(K) \cong Im(\partial^*: H^*(F) \to H^*(\Omega B))$.*

(Recall that if B is rational, then ΩB is a product of $K(\mathbf{Q}, n)$'s.) If, for example, F is the rationalization of a finite complex, then $H^*(F)$ has finite dimension. This implies that no factor of K may have the form $K(\mathbf{Q}, 2n)$ since the cohomology algebra of such a space is a polynomial algebra of infinite

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dimension. Each factor of K then looks like a rationalized sphere $S_{\mathbf{Q}}^{2n+1} \simeq K(\mathbf{Q}, 2n + 1)$ and we have,

$$F \simeq \mathscr{F} \times \Pi S_{\mathbf{Q}}^{2n+1}.$$

It is natural to ask if there are non-rational splittings of this form. In [9] McGibbon and Wilkerson showed that the decomposition of a rational loop space into $K(\mathbf{Q}, n)$'s has a p-local analogue *for almost all* primes p. Namely,

THEOREM 3. *If X is a finite simply connected complex with* $\dim(\pi_*(X) \otimes \mathbf{Q}) < \infty$, *then for almost all p*,

$$\Omega X \simeq_p \Pi S^{2n-1} \times \Pi \Omega S^{2m-1}.$$

This result indicates that a p-local version of Theorem 2 might also exist. The purpose of this note is to verify this belief in an elementary fashion using the argument for the rational situation as a model and amplifying it via the introduction of an idea borrowed from the McGibbon-Wilkerson proof.

CONVENTIONS. Throughout this note we freely use the techniques and fundamental results of homotopy theoretic localization. Standard references are [8] and [13] for example. For a set of primes Q, we write $\mathbf{Z}[Q^{-1}] = \{(a/b)|a, b \in \mathbf{Z}$ and b is a product of primes in $Q\}$. Of course, $X[Q^{-1}]$ then denotes a corresponding homotopical localization of the space X.

1. *P*-local fibre decomposition.

In order to motivate the proof of p-local fibre decomposition given below, we briefly review the rational situation. Let $F \to E \to B$ be a fibration of simply connected rational spaces with finite Betti numbers. Denote the Hurewicz map of F by h and the connecting map of the fibration by ∂. Because $\Omega B \simeq \Pi K(\mathbf{Q}, i)$, there is a subproduct K such that the restriction $h\partial_\#:\pi_*(K) \to H_*(F)$ is an isomorphism onto $h\partial_\#(\pi_*(\Omega B))$. We may dualize via the bijections

$$\mathrm{Hom}(H_nF, \mathbf{Q}) \cong H^n(F; \mathbf{Q}) \cong [F, K(\mathbf{Q}, n)]$$

to obtain a map $\theta:F \to K$ with $\theta\partial \simeq \mathrm{id}_K$. If $\mathscr{F}$ denotes the homotopy fibre of θ, then we have a fibration with homotopy section,

$$\mathscr{F} \xrightarrow{i} F \underset{\partial}{\overset{\theta}{\rightleftarrows}} K$$

which induces $\pi_*(F) \cong \pi_*(\mathscr{F}) \oplus \pi_*(K)$. Now, the operation of ΩB on F derived from the original fibration allows the realization of this isomorphism by the composition,

$$K \times \mathscr{F} \to \Omega B \times F \to F.$$

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Hence, $F \simeq \mathscr{F} \times K$.

These same ingredients, together with one very important addition, go into p-local decomposition as well. Note, however, although the vector space structure inherent in working over $\mathbf{Q}$ allowed us to split K in one step, we cannot expect this in general. We therefore split off one sphere at a time.

THEOREM 4. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of simply connected spaces of finite type and suppose F is a finite complex. If there exists $\alpha \in \operatorname{Ker}(i_{\#}:\pi_n F \to \pi_n E)$ such that the Hurewicz image $h(\alpha)$ has infinite order in $H_n(F; \mathbf{Z})$, then for almost all primes p,*

$$F \simeq_p \mathscr{F} \times S^n.$$

REMARKS. (1) Note that n must be *odd* because $\operatorname{Ker} i_{\#} = \operatorname{Im} \partial_{\#}$ and, for finite complexes, $\operatorname{Im} \partial_{2i}$ consists of torsion elements ([2]). Hence $h(\alpha)$ is torsion for all $\alpha \in \pi_{2i}(F)$.

(2) By our previous remarks, this decomposition is compatible with the rational decomposition of Theorem 2.

(3) Theorem 2 includes the information that $H^*(K) \cong \operatorname{Im}(\partial^*:H^*(F) \to H^*(\Omega B))$. This fact has many applications (see [11]). A p-local analogue is unknown.

PROOF OF THEOREM 4. Let $H_n(F) = A \oplus T$ where A and T denote the free and torsion parts of $H_n(F)$ respectively. Now $h(\alpha)$ has the form,

$$h(\alpha) = \lambda_1 a_1 + \ldots + \lambda_k a_k + t$$

where $\{a_i\}$ is a basis for A and $t \in T$. Because $h(\alpha)$ has infinite order, some $\lambda_i \neq 0$. Without loss of generality, assume $\lambda_1 = \lambda \neq 0$ and let Q_1 denote the set of primes occurring in the prime factorization of λ.

We can now define a map $\phi:F \to K(\mathbf{Z}[Q_1^{-1}], n)$ by observing that the following inclusion and bijection,

$$\operatorname{Hom}(H_n F, \mathbf{Z}[Q_1^{-1}]) \to H^n(F; \mathbf{Z}[Q_1^{-1}]) \cong [F, K(\mathbf{Z}[Q_1^{-1}], n)]$$

allow the construction of ϕ by specifying $\Phi \in \operatorname{Hom}(H_n F, \mathbf{Z}[Q_1^{-1}])$. Define Φ by:

$$\Phi(a_1) = 1/\lambda,\ \Phi(a_i) = 0 \text{ for } i \geqq 2 \text{ and } \Phi(T) = 0.$$

Let $i:S^n \to K(\mathbf{Z}[Q_1^{-1}], n)$ represent $1 \in \mathbf{Z}[Q_1^{-1}]$ and denote by Q_2 the set of primes p which divide the order of some $\pi_i(S^n)$ with $n < i \leqq \dim F$. Then, for $Q = Q_1 \cup Q_2$,

$$S^n[Q^{-1}] \xrightarrow{\bar{i}} K(\mathbf{Z}[Q_1^{-1}], n)[Q_2^{-1}] = K(\mathbf{Z}[Q^{-1}], n)$$

is a ($\dim F$)-equivalence. This implies that the groups $H^{s+1}(F; \pi_s \text{ (fibre of } \bar{i}))$

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vanish for all s. Hence, by obstruction theory, we obtain a lifting θ in the following diagram,

$$\begin{array}{ccccc} & & & & S^n[Q^{-1}] \\ & & & \overset{\theta}{\nearrow} & \downarrow \bar{i} \\ S^n & \xrightarrow{\alpha} & F & \xrightarrow{\bar{\phi}} & K(\mathbf{Z}[Q^{-1}], n). \end{array}$$

($\bar{\phi}$ denotes the composition of ϕ with the localization of spaces corresponding to the inclusion $\mathbf{Z}[Q_1^{-1}] \to \mathbf{Z}[Q^{-1}]$ determined by $1 \mapsto 1$.)

We intend to use the localized maps $\bar{\theta}$ and $\bar{\alpha}$ as our fibre map and splitting respectively. To show that $\bar{\theta}\bar{\alpha} \simeq \mathrm{id}_{S^n[Q^{-1}]}$ we first compute:

$$\begin{aligned} \bar{i}_*\theta_*\alpha_*(1) &= \bar{\phi}_*\alpha_*(1) \\ &= \bar{\phi}_* h\alpha_\#(1) \\ &= \bar{\phi}_* h(\alpha) \\ &= \Phi(\lambda a_1 + \ldots + \lambda_k a_k + t) \\ &= \lambda(1/\lambda) \\ &= 1\,. \end{aligned}$$

Since $\bar{i}_* = \mathrm{id}$ in dimension n, we see $\theta_*\alpha_*(1) = 1$. Hence $(\theta\alpha)_*$ is the localization homomorphism $\mathbf{Z} \to \mathbf{Z}[Q^{-1}]$ given by $x \mapsto x \otimes 1$. This inclusion extends naturally to the identity $\mathbf{Z}[Q^{-1}] \to \mathbf{Z}[Q^{-1}]$. Now, the universal coefficient theorem and the Hopf-Whitney classification provide bijections,

$$\begin{aligned} [S^n, S^n[Q^{-1}]] &\cong H^n(S^n; \mathbf{Z}[Q^{-1}]) \cong \mathrm{Hom}(H_n S^n, \mathbf{Z}[Q^{-1}]) \\ &\cong \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}[Q^{-1}]), \end{aligned}$$

which reveal $\theta\alpha$ to be the homotopical localization $S^n \to S^n[Q^{-1}]$ realizing $(\theta\alpha)_*$. Consequently, with respect to this localization, the induced map $\theta\bar{\alpha}: S^n[Q^{-1}] \to S^n[Q^{-1}]$ realizes $\mathrm{id}: \mathbf{Z}[Q^{-1}] \to \mathbf{Z}[Q^{-1}]$ and is therefore homotopic to the identity on $S^n[Q^{-1}]$.

Now, let $\mathscr{F}$ denote the homotopy fibre of $\bar{\theta}$. We then have a fibration with homotopy section,

$$\mathscr{F} \xrightarrow{j} F[Q^{-1}] \underset{\bar{\alpha}}{\overset{\bar{\theta}}{\rightleftarrows}} S^n[Q^{-1}]$$

which induces $\pi_*(F[Q^{-1}]) \cong j_\#(\pi_*\mathscr{F}) \oplus \bar{\alpha}_\#(\pi_* S^n[Q^{-1}])$.

We now invoke the hypothesis $\alpha \in \mathrm{Ker}\, i_\#$ for the fibration $F \xrightarrow{i} E \xrightarrow{p} B$. Because $\mathrm{Ker}\, i_\# = \mathrm{Im}\, \partial_\#$, there exists $\beta \in \pi_n(\Omega B)$ with $\partial_\#(\beta) = \alpha$. It follows that $\partial_\#(\bar{\beta}) = \bar{\alpha}$ for the localized fibration $F[Q^{-1}] \to E[Q^{-1}] \to B[Q^{-1}]$ as well.

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The operation of $\Omega B[Q^{-1}]$ on $F[Q^{-1}]$ provides a composition,

$$S^n[Q^{-1}] \times \mathscr{F} \xrightarrow{\bar{\beta} \times j} \Omega B[Q^{-1}] \times F[Q^{-1}] \xrightarrow{c} F[Q^{-1}]$$

where $c_\#$ is given by $c_\#(x, y) = \partial_\#(x) + y$ (see [7]). We then have

$$c_\#(\bar{\beta}_\# \times j_\#)(x, y) = \partial_\# \bar{\beta}_\#(x) + j_\#(y) = \bar{\alpha}_\#(x) + j_\#(y).$$

Since this is the form of the direct sum decomposition of $\pi_*(F[Q^{-1}])$ derived above, we see that $c_\#(\bar{\beta}_\# \times j_\#)$ is an isomorphism. Hence, $c(\bar{\beta} \times j)$ is a homotopy equivalence.

Finally, observe that Q is a *finite* set of primes and, for any $p \notin Q$, we have $F \simeq_p \mathscr{F} \times S^n$.

EXAMPLES. (1) The Hopf fibration $S^3 \xrightarrow{i} S^7 \to S^4$ is the first example which comes to mind. Here, $\mathrm{Ker}(i_\#)_3 = \pi_3 S^3 \cong \mathbf{Z}$ (and, of course, h_3 is an isomorphism), so $\mathscr{F} = pt$.

(2) In contrast with the Hopf fibration, it is usually too much to ask that α as in Theorem 4 generates a $\mathbf{Z}$-summand. In fact, except for $n = 1, 3, 7$, the Stiefel fibration with $n = odd$,

$$S^n \to V_{n+2,2} \xrightarrow{p} S^{n+1}$$

has $\mathrm{Ker}(i_\#)_n = 2\mathbf{Z} \subset \mathbf{Z} \cong \pi_n S^n$. (Here, $V_{n,k}$ denotes the Stiefel manifold of k-planes in $\mathbf{R}^n$ and we have identified $S^{n+1} = V_{n+2,1}$.) Also, if X is any finite 1-connected complex, then the fibration

$$X \times S^n \to X \times V_{n+2,2} \xrightarrow{* \times p} S^{n+1}$$

gives $X \simeq_p \mathscr{F}$ for almost all p. In both of these examples, the splitting of the theorem results from inverting 2 (since $h(2\iota) = 2\iota$, where ι generates $\pi_n S^n \cong H_n S^n$), rather than from the original product structure.

(3) Note that the theorem is not true if F is not finite. For example, consider the fibration $\mathbf{C}P(\infty) \xrightarrow{i} E \to S^3$ obtained by pulling back the path fibration of $K(\mathbf{Z}, 3)$ via the degree 1 map $S^3 \to K(\mathbf{Z}, 3)$. The pullback gives $\mathrm{Ker}\, i_\# = \pi_2 \mathbf{C}P(\infty) \cong H_2 \mathbf{C}P(\infty) \cong \mathbf{Z}$, but clearly $\mathbf{C}P(\infty) \not\simeq_p \mathscr{F} \times S^2$ for any p. To see this, compare homotopy groups or note that the cohomology algebra of $\mathbf{C}P(\infty)$ is a polynomial algebra on a degree 2 generator for any coefficients, while the degree 2 generator of the cohomology of $\mathscr{F} \times S^2$ is the "spherical" generator and so has square zero.

(4) Theorem 4 may be extended to the non-simply connected situation (see [12]). Let X be a space with $H_1(X; \mathbf{Z})$ finitely generated and let $X \xrightarrow{i} E \to B$ be a fibration. If there exists $\alpha \in \mathrm{Ker}\, i_\#$ with $h(\alpha)$ of infinite order, then there is a finite cyclic cover $\tilde{X}$ of X with $\tilde{X} \simeq Y \times S^1$.

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2. Some consequences.

An $\alpha \in \operatorname{Ker} i_{\#}$ (as in Theorem 4) belongs to the nth *Gottlieb group* of F, denoted $G_n(F)$. (See [4] for the definition.) Gottlieb was the first to study this subgroup of $\pi_n F$ and use it to investigate spatial structure. In particular, he proved (see [5] or [6]) that if $\alpha \in G_n(F)$ and $h_p(\alpha) \neq 0$, then $H^*(F; \mathbf{Z}/p) \cong M \otimes H^*(S^n; \mathbf{Z}/p)$ as $\mathbf{Z}/p$-modules. (Here, h_p denotes the composition $\pi_n F \xrightarrow{h} H_n F \to H_n(F; \mathbf{Z}/p)$.) The hypothesis of Theorem 4 is stronger than Gottlieb's condition, but the equivalence $F \simeq_p \mathscr{F} \times S^n$ produces the stronger conclusion:

COROLLARY 5. *For $p \notin Q$, there is a $\mathbf{Z}/p$-algebra isomorphism*

$$H^*(F; \mathbf{Z}/p) \cong H^*(\mathscr{F}; \mathbf{Z}/p) \otimes H^*(S^n; \mathbf{Z}/p).$$

QUESTION. For $p \in Q$, can Gottlieb's $\mathbf{Z}/p$-module decomposition of $H^*(F; \mathbf{Z}/p)$ be strengthened to a $\mathbf{Z}/p$-algebra decomposition?

Of course, Corollary 5 has the following immediate consequence obtained from the product formula for Euler characteristic and the easily proven equality $\chi(F) = \chi(H_*(F; \mathbf{Z}/p))$.

COROLLARY 6. *The Euler characteristic of F vanishes.*

Finally, we present a corollary very much in the spirit of Theorem 5-2, Corollary 5-3 and Corollary 5-5 of [4].

COROLLARY 7. *If* $\dim F = n$, *then* (i) $F[Q_1^{-1}] \simeq S^n[Q_1^{-1}]$, (ii) $F[Q_1^{-1}]$ *is an H-space,* (iii) $n = 1, 3$ *or* 7 *when* $2 \notin Q_1$ *and* (iv) $F \simeq S^1, S^3$ *or* S^7 *when $h(\alpha)$ is a generator.*

PROOF. (i) First, note that we may take Q_2 to be empty since $S^n[Q_1^{-1}]$ and $K(\mathbf{Z}[Q_1^{-1}], n)$ are already $n = \dim F$ equivalent. From Theorem 4 we obtain, $F[Q_1^{-1}] \simeq \mathscr{F} \times S^n[Q_1^{-1}]$. However, this can only occur when $\mathscr{F}$ is trivial, for the "product" of a non-trivial class in $H_*(\mathscr{F})$ with a class in $H_n(S^n[Q_1^{-1}])$ would produce homology in $H_*(F[Q_1^{-1}])$ of degree greater than n.

(ii) From the proof of Theorem 4, $\bar{\theta}\bar{\alpha} \simeq \mathrm{id}$ while $\bar{\alpha} = \partial\bar{\beta}$ is a homotopy equivalence by (i). Hence, $\partial\bar{\beta}\bar{\theta} \simeq \bar{\alpha}\bar{\theta} \simeq \mathrm{id}$ as well, so $F[Q_1^{-1}]$ is a weak retract of $\Omega B[Q_1^{-1}]$ and therefore is an H-space.

(iii) If $2 \notin Q_1$, then it is well known that $S^n[Q_1^{-1}]$ is an H-space only when $n = 1, 3$ or 7.

(iv) If $h(\alpha)$ is a generator of infinite order, then we may take $\lambda = 1$. Hence, Q_1 is empty and, by (i) and (iii), $F \simeq S^1, S^3$ or S^7.

3. Cofibrations.

If $X \to Y \to C$ is a cofibration of simply connected rational spaces, then there is a decomposition of the cofibre C which is Eckmann-Hilton dual to the

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fibre decomposition of Theorem 2. Briefly, the decomposition is obtained as follows (see [10]). Extend the cofibration to the next term in the Puppe sequence, $X \to Y \to C \xrightarrow{\partial} \Sigma X$ and note that $\Sigma X \simeq VS^n_{\mathbf{Q}}$ (i.e., the dual to $\Omega B \simeq \Pi K(\mathbf{Q}, n)$). Now consider $\mathrm{Im}(\partial_* h: \pi_* C \to H_* C \to H_*(\Sigma X))$ and observe that, since $H_*(\Sigma X) \cong H_*(VS^n_Q) \cong \oplus \mathbf{Q}$, Im $\partial_* h$ is realized by a subwedge $S \subset \Sigma X$. We may choose a subspace V in $\pi_* C$ represented by $VS^{n_j} \to C$ such that the restriction $\partial_* h: V \to H_*(S)$ is an isomorphism. An inverse of this isomorphism is realized by $S \to VS^{n_j}$ and the composition gives $S \to C$. Now, there is a projection $\Sigma X \to S$ and the composition

$$S \to C \to \Sigma X \to S$$

is then homotopic to the identity on S. If $\mathscr{C}$ denotes the cofibre of $S \to C$, we have obtained a cofibration with homotopy retraction

$$S \rightleftarrows C \to \mathscr{C}$$

which gives $H_*(C) \cong H_*(S) \oplus H_*(\mathscr{C})$. The cooperation of ΣX on C (see [7]), $C \to CV\Sigma X$ then gives,

$$C \to CV\Sigma X \to \mathscr{C}VS,$$

which realizes the isomorphism on homology. Hence $C \simeq \mathscr{C}VS$.

Using this approach as a model, we can now prove an Eckmann-Hilton dual to Theorem 4. Because the proof is dual (with a minor adaptation) to that of Theorem 4, we omit some details.

THEOREM 8. *Let $X \to Y \to C$ be a cofibration of simply connected, finite CW complexes. If $\alpha \in \pi_n C$ and $\partial_* h(\alpha)$ is of infinite order in $H_n(\Sigma X; \mathbf{Z})$, then for almost all primes p,*

$$C \simeq_p \mathscr{C}VS^n.$$

PROOF. Let $H_n(\Sigma X) = A \oplus T$ and suppose $\partial_* h(\alpha) = \lambda a_1 + \ldots + \lambda_k a_k + t$, $\lambda \neq 0$. Let Q_1 denote the primes in λ and obtain $\phi: \Sigma X \to K(\mathbf{Z}[Q_1^{-1}], n)$ by defining $\Phi \in \mathrm{Hom}(H_n \Sigma X, \mathbf{Z}[Q_1^{-1}])$:

$$\Phi(a_1) = 1/\lambda,\ \Phi(a_i) = 0,\ i \geqq 2 \text{ and } \Phi(T) = 0.$$

Let Q_2 denote the primes required to make

$$S^n[Q^{-1}] \xrightarrow{i} K(\mathbf{Z}[Q^{-1}], n)$$

a dim(ΣX)-equivalence, where $Q = Q_1 \cup Q_2$. As before, obstruction theory provides a map $\theta: \Sigma X \to S^n[Q^{-1}]$ with $i\theta \simeq \phi$. The same computation as before shows that the composition

$$S^n \xrightarrow{\alpha} C \xrightarrow{\partial} \Sigma X \xrightarrow{\theta} S^n[Q^{-1}]$$

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induces the localization $x \mapsto x \otimes 1$ in homology. Hence, the localized map $\overline{\theta}\overline{\partial}\overline{i}$ is homotopic to the identity.

The cofibration

$$S^n[Q^{-1}] \rightleftarrows C[Q^{-1}] \xrightarrow{j} \mathscr{C}$$

then splits $H_*(C[Q^{-1}])$ as $H_*(\mathscr{C}) \oplus H_*(S^n[Q^{-1}])$. The cooperation map of the original cofibration provides (after localization),

$$C[Q^{-1}] \to C[Q^{-1}] V \Sigma X[Q^{-1}] \to \mathscr{C} V S^n[Q^{-1}]$$

which induces an isomorphism on homology and so is a homotopy equivalence. Hence, for $p \notin Q$, $C \simeq_p \mathscr{C} V S^n$.

REMARK. The results of Felix-Lemaire [3] indicate that the techniques presented here have implications for decompositions in the tame homotopy category as well.

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