Canad. Math. Bull. Vol. **27** (1), 1984

SPLIT GRAPHS WITH SPECIFIED DILWORTH NUMBERS

BY

CHIÊ NARA AND IWAO SATO

ABSTRACT. Let G be a split graph with the independent part I_G and the complete part K_G. Suppose that the Dilworth number of $(I_G, \leqslant)$ with respect to the vicinal preorder $\leqslant$ is two and that of $(K_G, \leqslant)$ is an integer k. We show that G has a specified graph H_k, defined in this paper, as an induced subgraph.

§1. **Introduction.** For a finite set S and a binary relation $\leqslant$ on S, we call the pair $(S, \leqslant)$ a *preordered set* if $\leqslant$ satisfies both the reflexive and the transitive laws, and call a subset S_0 of S *incomparable* if for any two elements x and y of S_0 it holds neither $x \leqslant y$ nor $y \leqslant x$. The Dilworth number of a preordered set $(S, \leqslant)$ is, by definition, the maximum cardinality of all incomparable subsets of S, which is equal to the minimum number of chains covering S (see [2]).

We denote by $V(G)$ the vertex set of a simple graph G and by $N(v)$ the neighborhood of a vertex v in G. Let S be a subset of $V(G)$. Then the *vicinal preorder* $\leqslant$ on S is defined by

$$u \leqslant v \quad \text{if and only if} \quad N(u) \subset N(v) \cup \{v\}$$

and Dilworth number of $(S, \leqslant)$ is written by $\nabla_G(S)$. We use the symbol $\nabla(G)$ instead of $\nabla_G(V(G))$ for the sake of simplicity and call it *Dilworth number of G*. If $V(G)$ is partitioned into two subsets, denoted by I_G and K_G, such that any two vertices of I_G are not adjacent to each other and the subgraph induced by K_G is complete, G is called a *split graph*. It is easy to see that we have $u \leqslant v$ for every $u \in I_G$ and every $v \in K_G$ of a split graph G and hence it holds $\nabla(G) = \max\{\nabla_G(I_G), \nabla_G(K_G)\}$. Split graphs are characterized by V. Chvátal and P. L. Hammer [1] as graphs which have no induced subgraph isomorphic to $2K_2$, C_4 or C_5. Furthermore, they showed that *threshold graphs* (see [1]) are graphs with Dilworth number one, i.e., graphs having no induced subgraph isomorphic to $2K_2$, C_4 or P_4, and hence threshold graphs are split graphs. A characterization of split graphs with Dilworth number two is given by S. Foldes and P. L. Hammer [3], and that of split graphs with Dilworth number three is given by the first author [4].

Received by the editors July 2, 1982 and, in final revised form, February 26, 1983.
AMS subject-classification. 05C75

In this paper we show that if a split graph G has $\nabla_G(I_G)=2$ and $\nabla_G(K_G)=k$, then G has a specified graph H_k (see §2) as an induced subgraph.

§2. **Definition and theorem**

DEFINITION. For an integer $k\geq 2$, we define a split graph H_k with the independent part I_{H_k} and the complete part K_{H_k} as follows,

$$I_{H_k}=\{x_1, x_2, \ldots, x_{k-1}, y_1, y_2, \ldots, y_{k-1}\},$$
$$K_{H_k}=\{v_1, v_2, \ldots, v_k\},$$

$N(x_i)=\{v_1, \ldots, v_i\}$ and $N(y_i)=\{v_k, \ldots, v_{k-i+1}\}$ for all i with $1\leq i\leq k-1$.

Then it is easy to see $x_1\leqslant x_2\leqslant\cdots\leqslant x_{k-1}$, $y_1\leqslant y_2\leqslant\cdots\leqslant y_{k-1}$ and K_{H_k} is incomparable. Hence H_k satisfies

$$\nabla_{H_k}(I_{H_k})=2 \quad \text{and} \quad \nabla_{H_k}(K_{H_k})=k.$$

It must be noticed that H_k has H_{k-1} as an induced subgraph because $H_k-\{x_1, y_{k-1}, v_1\}$ is isomorphic to H_{k-1}.

Now we state our theorem and prove it in the following section.

THEOREM. *Let G be a split graph. If G satisfies $\nabla_G(I_G)=2$ and $\nabla_G(K_G)=k$ for an integer k, then G has an induced subgraph isomorphic to H_k.*

§3. **Lemmas and proofs.** We shall identify two graphs which are isomorphic to each other if there is no cause of confusion. We set

$$\nabla(2, k)=\{\text{a split graph } G:\nabla_G(I_G)=2 \text{ and } \nabla_G(K_G)=k\}.$$

First, we show two lemmas.

LEMMA 1. *For a split graph G the following three statements are equivalent*;
(i) $\nabla_G(I_G)\geq 2$,
(ii) $\nabla(G)\geq 2$,
(iii) *G has an induced subgraph isomorphic to P_4.*

Proof. We shall show that (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i). (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(i) are obvious. To see (ii)$\Rightarrow$(iii), suppose $\nabla(G)\geq 2$. Then G has an induced subgraph isomorphic to $2K_2$, C_4 or P_4 (see [2]). Since G is a split graph, this subgraph is isomorphic to P_4.

LEMMA 2. *A split graph G belonging to $\nabla(2, k)$ and having minimal order satisfies $|K_G|=k$.*

Proof. Since $\nabla(G)=\max\{\nabla_G(I_G), \nabla_G(K_G)\}$, we have $\nabla(G)\geq 2$ and G has P_4 as an induced subgraph by Lemma 1. Hence we have $k=\nabla_G(K_G)\geq\nabla_{P_4}(K_{P_4})=2$.

By $\nabla_G(K_G)=k$, there is an incomparable subset S of K_G with $|S|=k$. Let

$$T=\{v\in I_G : v\in N(u) \text{ for some } u\in S\}$$

and let G_0 be the subgraph of G induced by $S\cup T$. Then we have $\nabla_{G_0}(K_{G_0})=k$. Using $k\geq 2$, we see $\nabla(G_0)\geq 2$ and hence $2\leq\nabla_{G_0}(I_{G_0})\leq\nabla_G(I_G)=2$ by Lemma 1. This implies $G=G_0$, because of the minimality of $|V(G)|$. therefore we have $|K_G|=k$.

We call a graph with minimal order belonging to $\nabla(2,k)$ a *critical graph.*

Proof of Theorem. Let G be a split graph belonging to $\nabla(2,k)$. We prove Theorem by the mathematical induction with respect to an integer k. If $k=2$, Theorem is true since we have $\nabla(G)=2$ and $G\succ P_4\cong H_2$ by Lemma 1.

Assuming the truth of the theorem for $k-1$ $(k\geq 3)$, we shall show that it is true for k. It is obvious that there is an induced subgraph G_0 of G which is a critical graph belonging to $\nabla(2,k)$. Then it suffices to show that G_0 is isomorphic to H_k. From now on, we shall use the symbol G instead of G_0 and so G is a critical graph belonging to $\nabla(2,k)$ and satisfies $|K_G|=k$ by Lemma 2. We will prove $G\cong H_k$.

CASE 1. $|N(u)|=1$ for some vertex $u\in I_G$.

Let $N(u)=\{v\}$. By $|K_G|=k$ and $\nabla_G(K_G)=k$, the set K_G is incomparable. By $k\geq 3$, it holds $\nabla(G-v)\geq k-1\geq 2$ and hence $\nabla_{G-v}(I_{G-v})=2$ by Lemma 1. Therefore $G-v$ belongs to the set $\nabla(2,k-1)$ and has the graph H_{k-1} as an induced subgraph by the induction's hypothesis. Since $\nabla_G(I_G)=2$, at least one of the pairs contained in the set $\{u, x_1, y_1\}$ is comparable. We have either $u\preccurlyeq x_1$ or $u\preccurlyeq y_1$ because the pair $\{x_1, y_1\}$ is incomparable, $x_1\not\preccurlyeq u$ and $y_1\not\preccurlyeq u$. Without loss of generality, we can assume $u\preccurlyeq x_1$. Then we have $\{v\}=N(u)\subset N(x_1)$ and $x_1\in N(v)$.

First, we show

$$N(v)\cap I_G=\{x_1,\ldots,x_{k-2},u\}.$$

If we have $x_i\notin N(v)$ for some i $(1<i\leq k-2)$, then the set $\{x_1, x_i, y_1\}$ is incomparable in G, which contradicts $\nabla_G(I_G)=2$. Hence we have $\{x_1,\ldots,x_{k-2},u\}\subset N(v)$. Suppose $y_i\in N(v)$ for some i $(1\leq i\leq k-2)$. Then $N(v)-\{u\}$ is not contained in $N(v_i)$ for any i $(1\leq i\leq k-1)$ and $N(v_i)\cap V(G-u)=N(v_i)$ for every i $(1\leq i\leq k-1)$ by $N(u)=\{v\}$. This implies that the set $\{v, v_1,\ldots,v_{k-1}\}$ is incomparable in $G-u$ and $G-u$ belongs to $\nabla(2,k)$, which contradicts being critical of G. Thus we have $N(v)\cap I_G=\{x_1,\ldots,x_{k-2},u\}$.

Next, we shall show that there is a vertex $w\in N(v_1)-N(v)$ such that

$$N(w)=\{v_1,\ldots,v_{k-1}\}.$$

Since G is critical, the pair $\{v, v_1\}$ is incomparable and so there is a vertex $w \in N(v_1) - N(v)$, which is different from all x_i and y_i $(1 \le i \le k-2)$. If $v_i \notin N(w)$ for some i $(1 < i \le k-1)$, then the set $\{u, w, y_{k-2}\}$ is incomparable in G, which contradicts $\nabla_G(I_G) = 2$. Hence we have $N(w) = \{v_1, \ldots, v_{k-1}\}$.

From the above, by reordering the vertices of I_G and putting $x'_1 = u$, $x'_2 = x_1, \ldots, x'_{k-1} = x_{k-2}$, $y'_1 = y_1, \ldots, y'_{k-2} = y_{k-2}$ and $y'_{k-1} = w$, we see $G \cong H_k$.

CASE 2. $|N(u)| \ge 2$ for all $u \in I_G$.

We shall show that this leads to a contradiction. Let v be a vertex of K_G. Then by being critical of G, the subgraph $G - v$ belongs to the set $\nabla(2, k-1)$ and $G - v$ has the graph H_{k-1} as an induced subgraph by the induction hypothesis. Since $|N(x_1)| \ge 2$ and $|N(y_1)| \ge 2$, the set $\{x_1, y_1\}$ is contained in $N(v)$. Furthermore, we shall show that all x_i and all y_i $(1 \le i \le k-2)$ belong to $N(v)$. If $x_i \notin N(v)$ for some i $(1 < i \le k-2)$, then the set $\{x_1, x_i, y_1\}$ is incomparable in G, which contradicts $\nabla_G(I_G) = 2$. Thus $x_i \in N(v)$, and we can see $y_i \in N(v)$ $(1 \le i \le k-2)$ similarly.

On the other hand, since the pair $\{v, v_1\}$ is incomparable, there is a vertex w not belonging to $N(v)$ but to $N(v_1)$, which is different from all x_i and y_i $(1 \le i \le k-2)$. By $|N(w)| \ge 2$, we have $v_i \in N(w)$ for some i $(2 \le i \le k-1)$. Now the set $\{x_1, w, y_1\}$ is incomparable in G, which contradicts $\nabla_G(I_G) = 2$. Thus the proof is completed.

The following corollaries are easily seen.

COROLLARY 1. *Let G be a split graph with $\nabla_G(I_G) = 2$. Then $\nabla(G)$ is the largest k such that G has an induced subgraph isomorphic to H_k.*

COROLLARY 2 (S. Foldes and P. L. Hammer [3]). *Let G be a split graph with $\nabla_G(I_G) = 2$. Then we have $\nabla(G) = 2$ if and only if G has no induced subgraph isomorphic to H_3.*

A graph is an interval graph if there is a mapping i which associates to every $v \in V(G)$ a non-empty interval of the naturally ordered set $\mathbb{R}$ of all real numbers such that u is adjacent to v if and only if $u \ne v$ and $i(u) \cap i(v) \ne \emptyset$. It is known that a split graph is an interval graph if and only if $\nabla_G(I_G) \le 2$ (see [3]). Thus the following is an immediate corollary.

COROLLARY 3. *Let G be an interval, split graph with Dilworth number at least two. Then $\nabla(G)$ is the largest number k such that G has an induced subgraph isomorphic to H_k.*

REFERENCES

1. V. Chvátal and P. L. Hammer, *Set-packing problem and threshold graph*, Univ. of Waterloo, CORR 73-21, August 1973.

2. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. **51** (1950), 161–166.

3. S. Foldes and P. L. Hammer, *Split graphs having Dilworth number two*, Can. J. of Math. Vol. **29** (1977), 666–672.

4. C. Nara, *Split graphs with Dilworth number three*, Nat. Sci. Rep. Ochanomizu Univ., Vol. **33** (1982), 37–44.

DEPARTMENT OF MATHEMATICS,
MUSASHI INSTITUTE OF TECHNOLOGY,
TAMAZUTSUMI, SETAGAYA-KU,
TOKYO 158, JAPAN.
TSURUOKA TECHNICAL COLLEGE,
104 SAWATA, AZA, IOKA, ÓAZA,
TSURUOKA CITY 997, JAPAN.

https://doi.org/10.4153/CMB-1984-006-6 Published online by Cambridge University Press