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PRESENTATIONS OF AFFINE KAC–MOODY GROUPS

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Abstract

How many generators and relations does $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$ need? In this paper we exhibit its explicit presentation with 9 generators and 44 relations. We investigate presentations of affine Kac–Moody groups over finite fields. Our goal is to derive finite presentations, independent of the field and with as few generators and relations as we can achieve. It turns out that any simply connected affine Kac–Moody group over a finite field has a presentation with at most 11 generators and 70 relations. We describe these presentations explicitly type by type. As a consequence, we derive explicit presentations of Chevalley groups $G(\mathbb{F}_q[t, t^{-1}])$ and explicit profinite presentations of profinite Chevalley groups $G(\mathbb{F}_q[[t]])$.

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1. Introduction

Recently there has been a number of papers showing that various infinite families of groups have presentations with bounded number of generators and relations. In particular, the results of the following type were proved:

Let $\mathcal{A}$ be a certain family of groups. There exists $C > 0$ such that for any group $G \in \mathcal{A}$, G admits a presentation $\sigma(G) = \langle D_{\sigma(G)} \mid R_{\sigma(G)} \rangle$ such that

$$|D_{\sigma(G)}| + |R_{\sigma(G)}| < C.$$

This result is known if the family $\mathcal{A}$ is a family of finite simple groups [11–13], a family of affine Kac–Moody groups defined over finite fields [3], and more recently a family of Chevalley groups over various rings [5]. In fact, Guralnick *et al.* [11, 13] provide a numerical bound on C in the case when $\mathcal{A}$ is a family of finite simple groups (with the possible exception of ${}^2G_2(3^a)$, $a \geqslant 1$):

If G is a nonsporadic quasisimple finite group (with the possible exception of ${}^2G_2(3^a)$, $a \geqslant 1$), then G has a presentation with at most 2 generators and 51 relations.

The goal of the current paper is to provide a quantitative statement in the case when $\mathcal{A}$ is a family of affine Kac–Moody groups over finite fields and use this to derive numerical bounds for some arithmetic groups defined over fields of positive characteristic.

THEOREM 1.1. *Let G be a simply connected affine Kac–Moody group of rank $n \geqslant 3$ defined over a finite field $\mathbb{F}_q$. If $q \geqslant 4$, G has a presentation with 2 generators and at most 72 relations.*

The result also holds if $q \in \{2, 3\}$ provided that the Dynkin diagram of G is not of type $\tilde{A}_2$ and does not contain a subdiagram of type B_2 or G_2 for $q = 2$, and of type G_2 for $q = 3$. If $G = \widetilde{A}_2(2)$ or $\widetilde{A}_2(3)$, G has a presentation with at most 3 generators and 29 relations.

Our results are not restricted to simply connected groups. Using our techniques we derive quantitative bounds on the presentations of arbitrary affine Kac–Moody groups over finite fields. These results can be found in Section 6.

The upper bound of 72 in Theorem 1.1 comes from the groups of type $\widetilde{C}_n^t$. In other types the bounds are better, as stated in the next theorem.

THEOREM 1.2. *Let G be a simply connected affine Kac–Moody group of rank $n \geqslant 3$ defined over a finite field $\mathbb{F}_q$. If $q \geqslant 4$, G has a presentation $\sigma_G = \langle D_\sigma \mid R_\sigma \rangle$ where $|D_\sigma|$ and $|R_\sigma|$ are given in Table 1. The result also holds if $q \in \{2, 3\}$ provided that the Dynkin diagram of G does not contain a subdiagram of type B_2 or G_2 for $q = 2$, and of type G_2 for $q = 3$.*

Many mathematicians encounter affine Kac–Moody groups defined over $\mathbb{F}_q$ as Chevalley groups defined over $\mathbb{F}_q[t, t^{-1}]$: $\mathbf{G}(\mathbb{F}_q[t, t^{-1}]) \cong \widetilde{\mathbf{G}}(\mathbb{F}_q)/Z$ where $Z \cong \mathbb{F}_q^\times$ is a central subgroup of $\widetilde{\mathbf{G}}(\mathbb{F}_q)$ [16, Section 2]. For example, $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$ is the quotient of the simply connected affine Kac–Moody group $\widetilde{A}_{n-1}(q)$ by its central subgroup $Z \cong \mathbb{F}_q^\times$. Therefore we can obtain a presentation of $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$ from a presentation of $\widetilde{\mathbf{G}}(\mathbb{F}_q)$ (as in Table 1) by adding one extra relation to kill a generator of Z.

The groups $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$ can be generated by two elements (cf. Theorem 2.7). Therefore we can change our presentation to a presentation of $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$ in

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Table 1. Generators and relations of $\widetilde{X}(q)$.

Group	$\lvert D_\sigma\rvert$	$\lvert R_\sigma\rvert$ q odd	$\lvert R_\sigma\rvert$ q even	Group	$\lvert D_\sigma\rvert$ q odd	$\lvert R_\sigma\rvert$ q odd	$\lvert D_\sigma\rvert$ q even	$\lvert R_\sigma\rvert$ q even
$\widetilde{A}_2(q)$	5	26	22	$\widetilde{B}_3(q)$, $\widetilde{B}^t_3(q)$	7	42	8	35
$\widetilde{A}_3(q)$	7	34	30	$\widetilde{B}_4(q)$, $\widetilde{B}^t_4(q)$	8	51	9	44
$\widetilde{A}_n(q)$	7	35	31	$\widetilde{B}_n(q)$, $\widetilde{B}^t_n(q)$	8	52	9	45
$(4 \leqslant n \leqslant 7)$				$(5 \leqslant n \leqslant 8)$				
$\widetilde{A}_n(q)$	9	43	39	$\widetilde{B}_n(q)$, $\widetilde{B}^t_n(q)$	9	56	10	49
$(n \geqslant 8)$				$(n \geqslant 9)$				
$\widetilde{D}_4(q)$	7	38	34	$\widetilde{C}_2(q)$, $\widetilde{C}'_2(q)$	7	49	9	39
$\widetilde{D}_5(q)$	7	39	35	$\widetilde{C}^t_2(q)$	8	50	9	39
$\widetilde{D}_n(q)$	7	38	34	$\widetilde{C}_3(q)$, $\widetilde{C}'_3(q)$	8	58	10	48
$(6 \leqslant n \leqslant 8)$				$\widetilde{C}^t_3(q)$	9	59	10	48
$\widetilde{D}_n(q)$	8	42	38	$\widetilde{C}_4(q)$, $\widetilde{C}'_4(q)$	9	64	11	54
$(n \geqslant 9)$				$\widetilde{C}^t_4(q)$	10	65	11	54
$\widetilde{E}_6(q)$	7	36	32	$\widetilde{C}_n(q)$, $\widetilde{C}'_n(q)$,	9	65	11	55
$\widetilde{E}_7(q)$	6	30	26	$\widetilde{C}^t_n(q)$	10	66	11	55
$\widetilde{E}_8(q)$	7	34	30	$(5 \leqslant n \leqslant 8)$				
$\widetilde{G}_2(q)$	7	40	32	$\widetilde{C}_n(q)$, $\widetilde{C}'_n(q)$,	10	69	12	59
$\widetilde{G}^t_2(q)$	7	40	32	$\widetilde{C}^t_n(q)$	11	70	12	59
				$(n \geqslant 9)$				
				$\widetilde{F}_4(q)$, $\widetilde{F}^t_4(q)$	8	50	9	43

these two generators. This change of generators costs two extra relations (cf. Lemma 2.1). The next theorem summarizes this.

THEOREM 1.3. *Let* $\mathbf{G}$ *be a simple simply connected Chevalley group scheme of rank* $n \geqslant 2$. *Take* $q = p^a$, $a \geqslant 1$ *with* p *a prime and set* $G = \mathbf{G}(\mathbb{F}_q[t, t^{-1}])$. *Then* G *has a presentation with* 2 *generators and at most* 72 *relations with the possible exceptions of* $A_2(\mathbb{F}_2[t, t^{-1}])$, $B_n(\mathbb{F}_2[t, t^{-1}])$, $C_n(\mathbb{F}_2[t, t^{-1}])$, $G_2(\mathbb{F}_2[t, t^{-1}])$, $F_4(\mathbb{F}_2[t, t^{-1}])$, $A_2(\mathbb{F}_3[t, t^{-1}])$ *and* $G_2(\mathbb{F}_3[t, t^{-1}])$. *If* $G = A_2(\mathbb{F}_2[t, t^{-1}])$ *or* $A_2(\mathbb{F}_3[t, t^{-1}])$, G *has a presentation with at most* 3 *generators and* 30 *relations.*

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Table 2. Relations of $\mathbf{G}(\mathbb{F}_q[t,t^{-1}])$ with 2 generators.

Type	G	$\lvert R_\sigma\rvert$ q odd	$\lvert R_\sigma\rvert$ q even	Type	G	$\lvert R_\sigma\rvert$ q odd	$\lvert R_\sigma\rvert$ q even
	SL_3	29	25		Spin_7	45	38
	SL_4	37	33		Spin_9	54	47
A_{n-1}	SL_n $(4 \leqslant n \leqslant 8)$	38	34	B_n	Spin_{2n+1} $(5 \leqslant n \leqslant 8)$	55	48
	SL_n $(n \geqslant 9)$	46	42		Spin_{2n+1} $(n \geqslant 9)$	59	52
	Spin_8	41	37		Sp_4	52	42
	Spin_{10}	42	38		Sp_6	61	51
D_n	Spin_{2n} $(6 \leqslant n \leqslant 8)$	41	37	C_n	Sp_8	67	57
					Sp_{2n} $(5 \leqslant n \leqslant 8)$	68	58
	Spin_{2n} $(n \geqslant 9)$	45	41		Sp_{2n} $(n \geqslant 9)$	72	62
	E_6	39	35				
E_n	E_7	33	29	F_4	F_4	53	46
	E_8	37	33	G_2	G_2	43	35

The precise number of generators and relations in a presentation of $\mathbf{G}(\mathbb{F}_q[t,t^{-1}])$ can be deduced from Table 1 by adding 1 relation, and for a presentation with 2 generators can be found in Table 2.

Capdeboscq, Lubotzky and Remy connect the presentations of Chevalley groups over $\mathbb{F}_q[t,t^{-1}]$ with the profinite presentations of Chevalley groups defined over $\mathbb{F}_q[[t]]$ [5, Proposition 1.2]. An immediate consequence of their Proposition 1.2 combined with our Theorem 1.3, is the following statement.

THEOREM 1.4. *Let* $\mathbf{G}$ *be a simple simply connected Chevalley group scheme of rank at least* 2. *For* $q = p^a$, $a \geqslant 1$, p *a prime, consider a profinite group* $G = \mathbf{G}(\mathbb{F}_q[[t]])$. *Then* G *has a profinite presentation with* 2 *generators and at most* 72 *relations with the possible exceptions of* $A_2(\mathbb{F}_2[[t]])$, $B_n(\mathbb{F}_2[[t]])$, $C_n(\mathbb{F}_2[[t]])$, $G_2(\mathbb{F}_2[[t]])$, $F_4(\mathbb{F}_2[[t]])$, $A_2(\mathbb{F}_3[[t]])$ *and* $G_2(\mathbb{F}_3[[t]])$. *If* $G = A_2(\mathbb{F}_2[[t]])$ *or* $A_2(\mathbb{F}_3[[t]])$, *then* G *has a profinite presentation with at most* 3 *generators and* 31 *relations.*

Let us describe the structure of the paper. In Section 2 we outline the proof providing background results. We introduce simply connected Kac–Moody

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groups and show that a Kac–Moody subgroup of a simply connected Kac–Moody group is a simply connected Kac–Moody group itself. In Section 3 we examine the presentations of finite groups in types B_n and D_n. In Section 4 we analyse the presentations of all untwisted Kac–Moody groups giving details on the case-by-case basis. We examine the twisted Kac–Moody groups in Section 5. In Section 6 we deal with the presentations of adjoint Kac–Moody groups and classical groups over $\mathbb{F}_q[t, t^{-1}]$. We record all our findings in Tables 4 and 5.

2. Background results and outline of the proof

2.1. Presentations of finite quasisimple groups. Let us recall one of the main results of [13]. It asserts that every finite quasisimple group of Lie type admits a presentation with 2 generators and at most 51 relations. The authors achieve this in two steps. First they give a presentation for each family of groups of Lie type with very few generators and relations. Then, since every finite quasisimple group is 2-generated (see [2, Theorem B]), they use a reduction lemma [13, Lemma 2.3] to obtain the main result. We now restate this reduction lemma to include the case of infinite groups.

LEMMA 2.1 [13, Lemma 2.3]. *Let $\sigma = \langle X \mid R\rangle$ be a finite presentation of a group G, $\pi : F\langle X\rangle \to G$ the corresponding natural map from a free group. If D is a finite subset of G such that $G = \langle D\rangle$, then G also has a presentation $\langle D \mid R'\rangle$ such that $|R'| = |D| + |R| - |\pi(X) \cap D|$.*

Proof. Let $D = \{d_1, d_2, \ldots, d_l\}$. The new presentation is obtained by a sequence of Tietze transformations. First we add generators:

$$\sigma' = \langle X \cup D \mid R \cup \{d_i = \delta_i(X) \mid d_i \in D \setminus \pi(X)\} \cup \{d_j = x_{j^*} \mid d_j \in D \cap \pi(X)\}\rangle$$

where the second set in the union contains one relation for each $d_i \in D \setminus \pi(X)$ expressing d_i as a word $\delta_i(X)$ in X, while the third set in the union contains one relation for each $d_j \in D \cup \pi(X)$ where $d_j = \pi(x_{j^*})$ for some $x_j^* \in X$. We then remove X:

$$\sigma'' = \langle D \mid R\,|_{x_j=\chi_j(D)} \cup \{d_i = \delta_i(\chi_1(D), \ldots, \chi_k(D)) \text{ for } d_i \in D \setminus \pi(X)\}\rangle$$

where each $x_j \in X$ is expressed as a word $\chi_j(D)$ in D and $R\,|_{x_j=\chi_j(D)}$ is a result of this substitution in the relations R. $\square$

This result implies that, if a 2-generated group has a presentation with n generators and m relations, it also has a presentation with 2 generators and $m + 2$ relations.

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Table 3. Presentations of $\mathbf{G}(\mathbb{F}_q)$ [13]

Group	q odd				q even			
	$\|D_\sigma\|$	$\|R_\sigma\|$	Label	Contains	$\|D_\rho\|$	$\|R_\rho\|$	Label	Contains
$\mathrm{SL}(2,q)$	3	9	σ_1		3	5	ρ_1	
$\mathrm{SL}(3,q)$	4	14	σ_2	σ_1	4	10	ρ_2	ρ_1
$\mathrm{SL}(4,q)$	5	20	σ_3	σ_1	5	16	ρ_3	ρ_1
$\mathrm{SL}(4,q)$	6	21	σ_4	σ_1, σ_2	6	17	ρ_4	ρ_1, ρ_2
$\mathrm{SL}(n,q)$,	5	21	σ_5	σ_1	5	17	ρ_5	ρ_1
$\mathrm{SL}(n,q)$	6	22	σ_6	σ_1, σ_2	6	18	ρ_6	ρ_1, ρ_2
$(5 \leqslant n \leqslant 8)$								
$\mathrm{SL}(n,q)$,	6	25	σ_7	σ_1	6	21	ρ_7	ρ_1
$\mathrm{SL}(n,q)$	7	26	σ_8	σ_1, σ_2	7	22	ρ_8	ρ_1, ρ_2
$(n \geqslant 9)$								
$\mathrm{Sp}(4,q)$	5	27	σ_9	σ_1 (short)				
$\mathrm{Sp}(4,q)$	6	28	σ_{10}	σ_1 (twice)	6	20	ρ_{10}	ρ_1 (twice)
$\mathrm{Sp}(6,q)$	6	36			7	29		
$\mathrm{Spin}(7,q)$	6	36			7	29		
$\mathrm{Sp}(8,q)$	7	42			8	35		
$\mathrm{Spin}(8,q)$	6	29			6	25		
$\mathrm{Spin}(9,q)$	7	42			8	35		
$\mathrm{Sp}(2n,q)$,	7	43			8	36		
$\mathrm{Spin}(2n+1,q)$,	7	43						
$\mathrm{Spin}(2n,q)$	6	30			6	26		
$(5 \leqslant n \leqslant 8)$								
$\mathrm{Sp}(2n,q)$,	8	47	σ_{11}		9	40	ρ_{11}	
$\mathrm{Spin}(2n+1,q)$,	8	47	σ_{12}					
$\mathrm{Spin}(2n,q)$	7	34	σ_{13}	σ_1	7	30	ρ_{13}	ρ_1
$(n \geqslant 9)$								
$\mathrm{G}_2(q)$	6	31	σ_{14}	σ_1 (twice)	6	23	ρ_{14}	ρ_1 (twice)

We need some of the presentations of quasisimple groups of Lie type obtained in [13]. We exhibit them in Table 3 (cf. Table 1 of [13, page 93]).

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DEFINITION 2.2. Let B be a group and A its subgroup. Suppose further that B has a finite presentation $\sigma_B = \langle X_B \mid R_B\rangle$ and A has a presentation $\sigma_A = \langle X_A \mid R_A\rangle$ such that $X_A \subset X_B$ and $R_A \subseteq R_B$. Then we say that

$$\sigma_A \subseteq \sigma_B.$$

Let us use this definition to explain the connection between presentations of quasisimple groups of Lie type in Table 3. We call presentations σ_i if q is odd. Theorem 4.5 of [13] gives a presentation σ_1 of $\mathrm{SL}(2, q)$. Consider a group $G = \mathrm{SL}(3, q)$. If $\{\alpha_1, \alpha_2\}$ is the set of simple roots of $\mathrm{SL}(3, q)$, then G contains a subgroup $L = \langle X_{\alpha_1}, X_{-\alpha_1}\rangle \cong \mathrm{SL}(2, q)$, where X_{α_1} and $X_{-\alpha_1}$ are the root subgroups of $\mathrm{SL}(3, q)$ (cf. [9]). [13, Theorem 5.1] gives a presentation σ_2 of $\mathrm{SL}(3, q)$ that contains $\sigma_L = \sigma_1$.

Theorem 6.1 of [13] gives presentations σ_3, σ_4 of $\mathrm{SL}(4, q)$, σ_5, σ_6 of $\mathrm{SL}(n, q)$ for $5 \leqslant n \leqslant 8$, and σ_7, σ_8 of $\mathrm{SL}(n, q)$ for $n \geqslant 9$. If $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ is the set of simple roots of $\mathrm{SL}(n, q)$, the group $\mathrm{SL}(n, q)$ contains a subgroup $M = \langle X_{\alpha_1}, X_{-\alpha_1}, X_{\alpha_2}, X_{-\alpha_2}\rangle \cong \mathrm{SL}_3(q)$. From the proof of Theorem 6.1 it follows that $\sigma_2 \subseteq \sigma_i$ for $i = 4, 6$ and 8 where σ_2 is the presentation of M. This proof also shows that the shorter presentations σ_3, σ_5 and σ_7 contain σ_1. Thus we obtain that

$$\sigma_1 \subseteq \sigma_i \text{ for } i = 3, 5, 7, \quad \sigma_1 \subseteq \sigma_2 \subseteq \sigma_i \quad \text{for } i = 4, 6, 8.$$

Theorem 7.1 and Remark 7.4 of [13] give presentations σ_9 and σ_{10} of $\mathrm{Sp}(4, q)$. Let α_2 be a short root, α_1 a long root. Both presentations contain a presentation σ_1 of its short-root subgroup $L_2 = \langle X_{\alpha_2}, X_{-\alpha_2}\rangle \cong \mathrm{SL}(2, q)$. Besides σ_{10} contains a presentation σ_1 of its long-root subgroup $L_1 = \langle X_{\alpha_1}, X_{-\alpha_1}\rangle$:

$$\sigma_1(L_1) \subseteq \sigma_{10}, \quad \sigma_1(L_2) \subseteq \sigma_i \quad \text{for } i = 9, 10.$$

We also need a presentation of the family of groups $\mathrm{Spin}(2n, q)$, $n \geqslant 4$, $q = p^a$. The result of [13] says that such groups have presentations with 9 generators and 42 relations. We shorten this estimate to 7 generators and 34 relations in Section 3. We also give a slightly shorter presentation of $\mathrm{Spin}(2n + 1, q)$ in Section 3.

If q is even, all the presentations get shorter: some of the relations are no longer necessary. In one case an extra generator is required. By ρ_i we denote the presentation in characteristic 2 corresponding to the presentation σ_i in odd characteristic.

2.2. Simply connected Kac–Moody groups and their presentations.

Recall that Kac–Moody groups over arbitrary fields were defined by Tits [17]. We are only interested in the case when the group is split and the field of the definition $\mathbb{F} = \mathbb{F}_q$ is a finite field of $q = p^a$ elements ($a \geqslant 1$ and p a prime).

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Let $A = (A_{ij})_{n\times n}$ be a generalized Cartan matrix, $\mathfrak{D} = (I, A, \mathcal{X}, \mathcal{Y}, \Pi, \Pi^\vee)$ a root datum of type A. Recall that this means:

- $I = \{1, 2, \ldots, n\}$;
- $\mathcal{Y}$ is a free finitely generated abelian group;
- $\mathcal{X} = \mathcal{Y}^* = \hom(\mathcal{Y}, \mathbb{Z})$ is its dual group;
- $\Pi = \{\alpha_1, \ldots \alpha_n\}$ is a set of simple roots; where $\alpha_i \in \mathcal{X}$;
- $\Pi^\vee = \{\alpha_1^\vee, \ldots \alpha_n^\vee\}$ is a set of simple coroots; where $\alpha_i^\vee \in \mathcal{Y}$;
- for all $i, j \in I$; $\alpha_i(\alpha_j^\vee) = A_{ij}$.

From Tits' definition an explicit presentation of these groups was derived by Carter, a presentation by the explicit set of generators and relations à la Steinberg (cf. [8, page 224]). The presentation depends on the field and root datum, so the resulting group can be denoted by $G_{\mathfrak{D}}(\mathbb{F})$. If A is not of finite type (that is, A is not a Cartan matrix from classical Lie theory), the standard presentation is infinite: the number of generators and the number of relations are both infinite.

Recall that $G_{\mathfrak{D}}$ is *simply connected* if $\Pi^\vee$ is a basis of $\mathcal{Y}$ (see Section 6 for an example of a simply connected affine group), and $G_{\mathfrak{D}}(\mathbb{F})$ is 2-*spherical* if A is 2-spherical. The latter means that for each $J \subseteq I$ with $|J| = 2$, the submatrix $A_J := (A_{ij})_{i,j\in J}$ is a classical Cartan matrix. Now, for each $\alpha \in \Pi \cup -\Pi$, let X_α be a root subgroup of $G_{\mathfrak{D}}(\mathbb{F})$. Then $X_\alpha \cong (\mathbb{F}_q, +)$, and for all $i, j \in I$ with $i \neq j$, set

$$L_i := \langle X_{\alpha_i} \cup X_{-\alpha_i}\rangle \quad \text{and} \quad L_{ij} := \langle L_i \cup L_j\rangle = L_{ji}.$$

Abramenko and Muhlherr [1] (see also [7, Theorem 3.7]) proved a significant new theorem about the presentations of a large class of Kac–Moody groups. In particular, they showed that those groups were finitely presented.

THEOREM (Abramenko, Muhlherr). *Let A be a 2-spherical generalized Cartan matrix and $\mathfrak{D}$ a simply connected root datum corresponding to A. Suppose that the field $\mathbb{F}$ is finite and the following condition holds:*

$$L_{ij}/Z(L_{ij}) \ncong B_2(2), G_2(2), G_2(3), {}^2F_4(2) \quad \textit{for all } i, j \in I \quad (*).$$

Let $\widetilde{G}$ be the direct limit of the inductive system formed by the L_i and L_{ij} for $i, j \in I$, with the natural inclusions. Then the canonical homomorphism $\widetilde{G} \to G_{\mathfrak{D}}(\mathbb{F})$ is an isomorphism.

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For an insightful description of 2-spherical Kac–Moody groups we refer the reader to the paper of Caprace [7].

The following observation is useful.

PROPOSITION 2.3. *Let A be a 2-spherical generalized Cartan matrix and $\mathfrak{D}$ a simply connected root datum corresponding to A. Suppose that the field $\mathbb{F}$ is finite and condition $(*)$ holds. Let $J \subseteq I$ and*

$$L_J = \langle L_i \mid i \in J \rangle.$$

Then L_J is a simply connected Kac–Moody group $G_{\mathfrak{D}(J)}(\mathbb{F})$ with a root datum of type $A_J = (A_{ij})_{i,j \in J}$.

Proof. It is clear once you construct the root datum for the L_J. If $J \subseteq I$, let $\Theta = \{\alpha_i \mid i \in J\}$. Then $\Theta^\vee = \{\alpha_i^\vee \mid i \in J\}$ is the corresponding subset of $\Pi^\vee$. Since $\mathfrak{D}$ is simply connected, the coroot lattice splits into a direct sum

$$Y = \langle \Theta^\vee \rangle \oplus \langle \Pi^\vee \setminus \Theta^\vee \rangle.$$

Hence, the root datum for L_J is

$$\mathfrak{D}(J) = (J, A_J, X/\langle \Theta^\vee \rangle^\perp, \langle \Theta^\vee \rangle, \Theta, \Theta^\vee).$$

Thus, $\mathfrak{D}(J)$ is simply connected. □

A direct consequence of the result of Abramenko and Muhlherr is that $G_{\mathfrak{D}}(\mathbb{F}_q)$ has a presentation whose generators are generators of root subgroups X_i's, $i \in I$ and with the relations that are defining relations of all L_i's and L_{ij}'s with i, $j \in I$, $i \neq j$. This allowed Capdeboscq [3, Theorem 2.1] to show that the affine Kac–Moody groups (of rank at least 3) defined over finite fields had bounded presentations.

THEOREM (Capdeboscq). *There exists $C > 0$ such that if $\widetilde{X}(q)$ is an affine Kac–Moody group of rank at least 3 defined over a field $\mathbb{F}_q$ with $q = p^a \geqslant 4$ (p a prime), then $\widetilde{X}(q)$ has a presentation $\sigma_{\widetilde{X}(q)} = \langle D_{\widetilde{X}(q)} \mid R_{\widetilde{X}(q)} \rangle$ such that*

$$|D_{\widetilde{X}(q)}| + |R_{\widetilde{X}(q)}| < C.$$

This result is based on the following observation (see the proof of [3, Theorem 2.1]) which is a direct consequence of the result of Abramenko and Muhlherr.

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PROPOSITION 2.4. *Let $\widetilde{X}(q)$ be a simply connected 2-spherical split Kac–Moody group of rank $n \geqslant 3$ over the field of q elements. Let Δ be the Dynkin diagram of $\widetilde{X}(q)$ with the vertices labelled by $\beta_1, \ldots, \beta_n$. Suppose further that if Δ contains a subdiagram of type B_2, then $q \geqslant 3$, and if it contains a subdiagram of type G_2, $q \geqslant 4$.*

Suppose that Δ contains k proper subdiagrams $\Delta_1, \Delta_2, \ldots, \Delta_k$ such that $\Delta = \bigcup_{i=1}^{k} \Delta_i$, and each pair of vertices of Δ is contained in Δ_i for some $i \in \{1, \ldots, k\}$. Let $X_i(q) := \langle L_j \mid \alpha_j \in \Delta_i \rangle$. If $\sigma_{X_i(q)} = \langle D_i \mid R_i \rangle$ is a presentation of $X_i(q)$, then $\widetilde{X}(q)$ has a presentation

$$\sigma_{\widetilde{X}(q)} = \left\langle D_1 \cup D_2 \cup \ldots \cup D_k \mid R_1 \cup \ldots \cup R_k \cup \bigcup_{1 \leqslant i < j \leqslant k} R_{ij} \right\rangle$$

where R_{ij} are the relations coming from identifying the generators of $X_{ij}(q) := \langle L_k \mid \alpha_k \in \Delta_i \cap \Delta_j \rangle$ in $X_i(q)$ and in $X_j(q)$.

Suppose now that $\Delta = \Delta_1 \cup \Delta_2$ on the level of vertices, that is, Δ may contain edges, not included in Δ_1 or Δ_2. We cannot use Proposition 2.4 immediately because it is not true that any pair of vertices is contained in Δ_i. To remedy it we may introduce Δ_3 into the picture. Let Δ_3 be a subdiagram of Δ based on vertices of $(\Delta_1 \setminus \Delta_2) \cup (\Delta_2 \setminus \Delta_1)$: now any pair of vertices of Δ is contained in some Δ_i. Proposition 2.4 gives a presentation

$$\sigma_{\widetilde{X}(q)} = \langle D_1 \cup D_2 \cup D_3 \mid R_1 \cup R_2 \cup R_3 \cup R_{12} \cup R_{13} \cup R_{23} \rangle.$$

Let $X_3^1(q)$ be the subgroup of $\widetilde{X}(q)$ corresponding to $\Delta_1 \setminus \Delta_2$, and $\sigma_{X_3^1(q)} = \langle D_3^1 \mid R_3^1 \rangle$ its presentation. *Mutatis mutandis*, $\sigma_{X_3^2(q)} = \langle D_3^2 \mid R_3^2 \rangle$ for $\Delta_2 \setminus \Delta_1$. Now we can choose a special presentation of $X_3(q)$:

$$\sigma_{X_3(q)} = \langle D_3^1 \cup D_3^2 \mid R_3^1 \cup R_3^2 \cup R_3^{**} \rangle$$

where R_3^{**} are those relations that include generators in both D_3^1 and D_3^2. Using $\sigma_{X_3(q)}$ in $\sigma_{\widetilde{X}(q)}$ allows us to eliminate generators in D_3 using Tietze transformations. Since $\Delta_1 \supseteq \Delta_1 \setminus \Delta_2$ and $\Delta_2 \supseteq \Delta_2 \setminus \Delta_1$, many relations become superfluous as summarized in the next corollary.

COROLLARY 2.5. *Let $\widetilde{X}(q)$ be a simply connected 2-spherical split Kac–Moody group of rank $n \geqslant 3$ over the field of q elements. Let Δ be the Dynkin diagram of $\widetilde{X}(q)$ with the vertices labelled by $\beta_1, \ldots, \beta_n$. Suppose further that if Δ contains a subdiagram of type B_2, then $q \geqslant 3$, and if it contains a subdiagram of type G_2, $q \geqslant 4$.*

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Suppose that Δ contains three proper subdiagrams Δ_1, Δ_2 and Δ_3 such that $\beta_1, \ldots, \beta_n \in \Delta_1 \cup \Delta_2$ and Δ_3 is a subdiagram of Δ based on vertices of $(\Delta_1 \setminus \Delta_2) \cup (\Delta_2 \setminus \Delta_1)$. If for $i = 1, 2$, $\sigma_{X_i(q)} = \langle D_i \mid R_i \rangle$ is a presentation of $X_i(q)$ and $\sigma_{X_3(q)}$ is as described before this corollary, then $\widetilde{X}(q)$ has a presentation

$$\sigma_{\widetilde{X}} = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle$$

where the relations in R_3^* are obtained from the relations in R_3^{**} by substituting generators in D_3 with their expressions via generators of D_1 or D_2.

2.3. 2-Generation. A nonabelian finite simple group G can be generated by two elements [2, Theorem B]. We are interested in $h(G)$, the largest nonnegative integer such that $G^{h(G)}$ can be generated by two elements. Maróti and Tamburini have proved that $h(G) > 2\sqrt{|G|}$ [15, Theorem 1.1].

Consider a finite group $H = G_1^{m_1} \times \ldots \times G_t^{m_t}$ where each G_i is a nonabelian simple group and G_i and G_j are not isomorphic for all $i \neq j$. It is known that a subset of H generates H if and only if its projection into $G_i^{m_i}$ generates $G_i^{m_i}$ for each i [14, Lemma 5]. We combine all this information in the following statement.

PROPOSITION 2.6. *Let $G = G_1^a \times G_2^b$ where G_1 and G_2 are finite nonabelian nonisomorphic quasisimple groups and a and b are nonnegative integers with $0 \leqslant a, b \leqslant 3$. Then G is 2-generated.*

Proof. The smallest order of a finite nonabelian simple group is 60. Hence, for any finite simple group $h(G) \geqslant 16$. The result now follows immediately from the combination of [2, Theorem B], [15, Theorem 1.1] and [14, Lemma 5] and the fact that a quasisimple finite group is a Frattini cover of a finite simple group. □

Now suppose that $\widetilde{X}(q)$ is a Kac–Moody group over the field $\mathbb{F}_q$ (where $q = p^a$, $a \geqslant 1$, p a prime). We will need the following result proved in [4] for large enough q and clarified in [6] to include the small values of p and q.

THEOREM 2.7. *Let $\widetilde{X}(q)$ be a simply connected affine Kac–Moody group of rank $n \geqslant 3$ defined over a finite field $\mathbb{F}_q$. Then $\widetilde{X}(q)$ is generated by 2 elements, with the possible exceptions of $\widetilde{A}_2(2)$ and $\widetilde{A}_2(3)$ in which case it is generated by at most 3 elements.*

2.4. In Sections 4 and 5 we are going to use Proposition 2.4 and its corollary to obtain bounded presentations of Kac–Moody groups over finite fields. We do this by looking at affine groups case by case. We then record our findings in Table 1.

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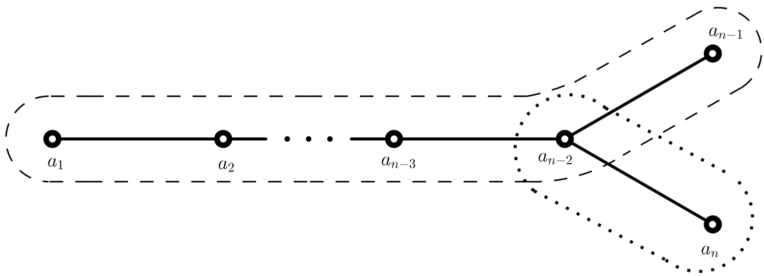
There are 7 infinite series and 7 exceptional types of affine Dynkin diagrams of rank at least 3 that are listed in the next sections. We use Dynkin labels for affine diagrams [9].

3. Presentation of Spin (n, q)

3.1. Presentation of Spin $(2n, q)$. To obtain a presentation of $G = \mathrm{Spin}\,(2n, q)$, $n \geqslant 4$, we use Corollary 2.5.



By Proposition 2.3, the groups $X_1(q)$, $X_2(q)$ and $X_3(q)$ are simply connected. Let Δ_1 be the subdiagram of Δ whose vertices are the $n-1$ nodes $a_1, \ldots, a_{n-1}$. It has type A_{n-1} and so $X_1(q) \cong \mathrm{SL}\,(n, q)$. Let Δ_2 be the subdiagram of Δ whose vertices are the nodes a_{n-2} and a_n. It is of type A_2 and so $X_2(q) \cong \mathrm{SL}\,(3, q)$. Then Δ_3 is the subdiagram of Δ based on all vertices but a_{n-2}, thus of type $A_{n-3} \times A_1 \times A_1$. Hence, $X_3(q) \cong \mathrm{SL}\,(n-2, q) \times \mathrm{SL}\,(2, q) \times \mathrm{SL}\,(2, q)$. Clearly, $\Delta = \bigcup_{i=1}^{2} \Delta_i$. Therefore G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle$$

as described in Corollary 2.5.

Take a presentation $\sigma_{X_1(q)}$. If $n \geqslant 9$, $\sigma_{X_1(q)} = \sigma_7$, if $5 \leqslant n \leqslant 8$, $\sigma_{X_1(q)} = \sigma_5$, and if $n = 4$, $\sigma_{X_1(q)} = \sigma_3$. Consider a subgroup $X = L_{n-2}$ of G. Its Dynkin diagram is of type A_1 and so by Proposition 2.3, $X \cong \mathrm{SL}\,(2, q)$. From Table 3 we know that X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ with $|D_X| = 3$ and $|R_X| = 9$. Now $X \leqslant X_i(q)$ for $i = 1, 2$. The group $X_2(q)$ has a presentation $\sigma_{X_2(q)} = \sigma_2$ with $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, obviously, $D_X \subseteq X_1(q)$. Thus elements of D_X can be expressed in terms of elements of D_1. Moreover, the relations R_X hold, as they hold in $X_1(q)$. We use Tietze transformations to eliminate D_X, R_X and R_{12} to obtain:

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

If $n \geqslant 9$, we use σ_7 that requires 6 generators and 25 relations, so that $|D_1 \cup (D_2 \setminus D_X)| = 6 + (4-3) = 7$ and $|R_1 \cup (R_2 \setminus R_X)| = 25 + (14-9) = 30$.

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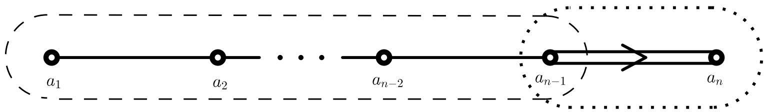
Consider $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q))$ where $X_3(q) \cap X_1(q) \cong \mathrm{SL}(n-2, q) \times \mathrm{SL}(2, q)$ and $X_3(q) \cap X_2(q) \cong \mathrm{SL}(2, q)$. Each factor has two generators (Proposition 2.6). Denote them by a_1, a_2 and b_1, b_2, respectively. Then $R_3^* = \{[a_1, b_1] = [a_1, b_2] = [a_2, b_1] = [a_2, b_2] = 1\}$ and $|R_3^*| = 4$.

Therefore if $n \geqslant 9$, G has a presentation with 7 generators and 34 relations. We call it σ_{13}. If $n \leqslant 8$, we get shorter presentations (see Table 3). Note that $\sigma_1 \subseteq \sigma_{13}$.

If q is even, the corresponding presentations have 4 fewer relations each. For instance, for ρ_{13} the corresponding calculation is $|R_1 \cup (R_2 \setminus R_X) \cup R_3^*| = 21 + (10 - 5) + 4 = 30$. This is typical for simply laced groups: all of them will get 4 fewer relations in characteristic 2.

3.2. Presentation of Spin $(2n + 1, q)$.

To obtain a presentation of $G = \mathrm{Spin}(2n+1, q)$, $n \geqslant 3$, we use Corollary 2.5.



By Proposition 2.3, the groups $X_1(q)$, $X_2(q)$ and $X_3(q)$ are simply connected. Let Δ_1 be the subdiagram of Δ whose vertices are the $n-1$ nodes $a_1, \ldots, a_{n-1}$. It has type A_{n-1} and so $X_1(q) \cong \mathrm{SL}(n, q)$. Let Δ_2 be the subdiagram of Δ whose vertices are the nodes a_{n-1} and a_n. It is of type C_2 and so $X_2(q) \cong \mathrm{Sp}(4, q)$. Then Δ_3 is the subdiagram of Δ based on all vertices but a_{n-1}, thus of type $A_{n-2} \times A_1$. Hence, $X_3(q) \cong \mathrm{SL}(n-1, q) \times \mathrm{SL}(2, q)$. Clearly, $\Delta = \cup_{i=1}^{2} \Delta_i$. Therefore G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle$$

as described in Corollary 2.5. Take presentations $\sigma_{X_1(q)} = \sigma_7$ and $\sigma_{X_2(q)} = \sigma_9$. Consider a subgroup $X = L_{n-1}$ of G. Its Dynkin diagram is of type A_1 and so by Proposition 2.3, $X \cong \mathrm{SL}(2, q)$. From Table 3 we know that X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ with $|D_X| = 3$ and $|R_X| = 9$. Observe that $\sigma_X \subseteq \sigma_{X_1(q)}$. Since $X \leqslant X_2(q)$, obviously, $D_X \subseteq X_2(q)$. Thus, elements of D_X can be expressed in terms of elements of D_2. Moreover, the relations R_X hold, as they hold in $X_2(q)$. We use Tietze transformations to eliminate D_X, R_X and R_{12} to obtain:

$$\sigma_G' = \langle (D_1 \setminus D_X) \cup D_2 \mid (R_1 \setminus R_X) \cup R_2 \cup R_3^* \rangle.$$

Observe that $|(D_1 \setminus D_X) \cup D_2| = (6 - 3) + 5 = 8$ and $|(R_1 \setminus R_X) \cup R_2)| = (25 - 9) + 27 = 43$.

Consider $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q))$ where $X_3(q) \cap X_1(q) \cong \mathrm{SL}(n-1, q)$ and $X_3(q) \cap X_2(q) \cong \mathrm{SL}(2, q)$. Each factor has two generators

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(Proposition 2.6). Denote them by a_1, a_2 and b_1, b_2, respectively. Then $R_3^* = \{[a_1, b_1] = [a_1, b_2] = [a_2, b_1] = [a_2, b_2] = 1\}$ and $|R_3^*| = 4$.

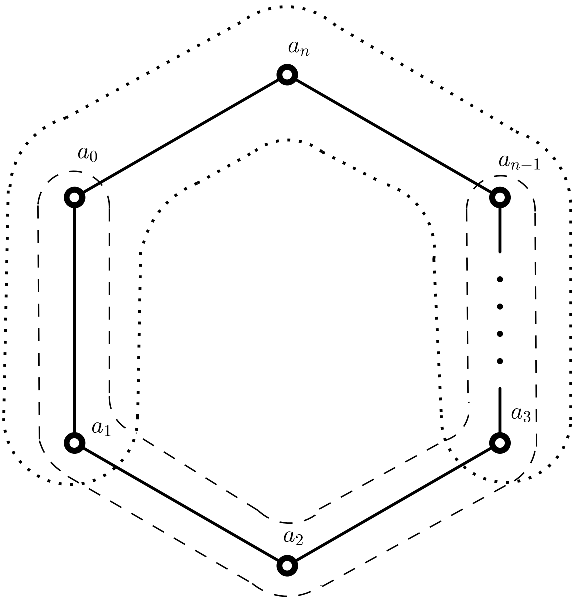
We call this presentation σ_{12}. Notice that the same method gives a presentation σ_{11} of $\mathrm{Sp}\,(2n, q)$ of the same size. If $n \leqslant 8$, we use a shorter presentation of $\mathrm{SL}\,(n, q)$ to obtain a shorter presentation of $\mathrm{Spin}\,(2n+1, q)$ (or $\mathrm{Sp}\,(2n, q)$). We do not give them names but record their lengths in Table 3. If q is even, $\mathrm{Spin}\,(2n+1, q) \cong \mathrm{Sp}\,(2n, q)$ and we call the corresponding presentation ρ_{11}.

4. Untwisted affine Kac–Moody groups

We now go through the calculations for the 4 infinite series and 5 exceptional types of affine untwisted simply connected Kac–Moody groups of rank at least 3.

4.1. $\widetilde{A}_n(q), n \geqslant 4$.

The Dynkin diagram Δ of $G = \widetilde{A}_n(q)$ consists of $n+1$ vertices:



We now use Corollary 2.5. Notice, that by Proposition 2.3, the groups $X_1(q)$, $X_2(q)$ and $X_3(q)$ are simply connected. Let Δ_1 be the subdiagram of Δ whose vertices are the n nodes $a_0, a_1, a_2, \ldots, a_{n-1}$. It has type A_n and so

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$X_1(q) \cong \mathrm{SL}(n+1,q)$. Let Δ_2 be the subdiagram of Δ whose vertices are the n nodes $a_1, a_0, a_n, a_{n-1}\ldots, a_3$. It is of type A_n and so $X_2(q) \cong \mathrm{SL}(n+1,q)$. Then Δ_3 is the subdiagram of Δ whose vertices are the two nodes a_2 and a_n. This diagram is of type $A_1 \times A_1$ and thus corresponds to a subgroup $X_3(q) \cong \mathrm{SL}(2,q) \times \mathrm{SL}(2,q)$. Clearly, $\Delta = \cup_{i=1}^{2}\Delta_i$. Therefore G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12}\rangle$$

as described in Corollary 2.5.

Let us find presentations of $X_i(q)$ for $i = 1, 2, 3$. Consider a subgroup X of G that is generated by L_0 and L_1. Its Dynkin diagram is of type A_2 and so by Proposition 2.3, $X \cong \mathrm{SL}(3,q)$. From Table 3 we know that X has a presentation $\sigma_X = \langle D_X \mid R_X\rangle = \sigma_2$ with 4 generators and 14 relations. Now $X \leqslant X_i(q)$ for $i = 1, 2$. Moreover, $X_1(q)$ has a presentation $\sigma_{X_1(q)} = \sigma_7$ that requires 6 generators and 25 relations. Now $X_2(q)$ has a presentation $\sigma_{X_2(q)} = \sigma_8$ with $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, $D_X \subseteq X_1(q)$ and relations R_X already hold. We use Tietze transformations to eliminate D_X and R_X:

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \cup R_{12}\rangle.$$

Now σ_8 requires 7 generators and 26 relations, and hence $|D_1 \cup (D_2 \setminus D_X)| = 6 + (7-4) = 9$ and $|R_1 \cup (R_2 \setminus R_X)| = 25 + (26-14) = 37$.

Consider $X_3(q) = L_2 \times L_n$. Since $L_2 \cong L_n \cong \mathrm{SL}(2,q)$, each factor has 2 generators (Proposition 2.6). Denote them by a_1, a_2 and b_1, b_2, respectively. Then $R_3^* := \{[a_1,b_1] = [a_1,b_2] = [a_2,b_1] = [a_2,b_2] = 1\}$ and so $|R_3^*| = 4$.

Finally, $\Delta_1 \cap \Delta_2$ is of type $A_2 \times A_{n-3}$. The corresponding group $\mathrm{SL}(3,q) \times \mathrm{SL}(n-2,q)$ has 2 generators (Proposition 2.6). We call them c_1, d_1 as elements of $X_1(q)$ and c_2, d_2 as elements of $X_2(q)$. Then $R_{12} = \{c_1 = c_2, d_1 = d_2\}$ and so $|R_{12}| = 2$.

It follows that G has a presentation with 9 generators and $37 + 4 + 2 = 43$ relations. If $4 \leqslant n \leqslant 7$, we may use σ_5 and σ_6 instead of σ_7 and σ_8. Then G has a presentation with 7 generators and 35 relations.

If q is even, we use ρ_2, ρ_7 and ρ_8 instead to get 9 generators and $|R_1 \cup (R_2 \setminus R_X) \cup R_3^* \cup R_{12}| = 21 + (22-10) + 4 + 2 = 39$ relations for $n \geqslant 8$, and 7 generators and 31 relations for $4 \leqslant n \leqslant 7$.

4.2. $\widetilde{A}_2(q)$. To obtain a presentation of $G = \widetilde{A}_2(q)$, we start with Proposition 2.4.

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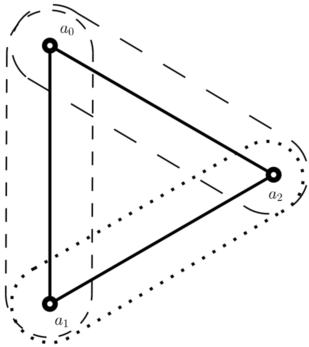


Take $\Delta_1 = A_2$ based on a_0 and a_1, $\Delta_2 = A_2$ based on a_0 and a_2, and $\Delta_3 = A_2$ based on a_1 and a_2. Clearly, $\Delta = \cup_{i=1}^{3} \Delta_i$ and every pair of vertices is contained in at least one of Δ_i. Thus G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \cup D_3 \mid R_1 \cup R_2 \cup R_3 \cup R_{12} \cup R_{13} \cup R_{23} \rangle.$$

Since $X_i(q) \cong A_2(q)$, $X_i(q) \cong \mathrm{SL}(3, q)$ by Proposition 2.3, and $X_i(q)$ has a presentation $\sigma_{X_i(q)} = \sigma_2$ $(i = 1, 2, 3)$. Take $X = L_0$. Then for $i = 1, 2$, $X \leqslant X_i(q)$ and $\sigma_X \subseteq \sigma_{X_i(q)}$. It follows that $|D_1 \cup D_2| = 4+4-3$, $|R_1 \cup R_2| = 14+14-9 = 19$ and the relations R_{12} are superfluous.

Take now $Y = L_1$. Without loss of generality we may assume that $\sigma_Y \subseteq \sigma_{X_3(q)}$ with $\sigma_Y = \sigma_1$. Then by [13, Theorem 5.1], $\sigma_{X_3(q)} = \langle D_Y \cup \{c\} \mid R_Y \cup R_3^c \rangle$ where $D_Y = \{u, t, h\}$ (in the notations of [13, Theorem 5.1]) and R_3^c is a set of 5 relations involving c and elements of D_Y. Since $Y \leqslant X_1(q)$, obviously, $D_Y \subseteq X_1(q)$. Thus u, t and h can be expressed in terms of elements of D_1. This makes the relations R_{13} superfluous. Moreover, the relations R_Y hold, as they hold in $X_1(q)$. We use Tietze transformations to eliminate D_Y, R_Y and R_{13}:

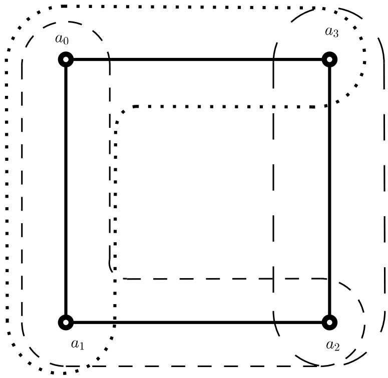
$$\sigma'_G = \langle D_1 \cup D_2 \cup \{c\} \mid R_1 \cup R_2 \cup R_3^c \cup R_{23} \rangle.$$

Notice that $X_3(q) \cong \mathrm{SL}(3, q)$ is generated by its subgroups L_1 and L_2 (cf. [4, Lemma 2.1]). Hence, c can be expressed in terms of elements of $D_1 \cup D_2$. Finally, since $\Delta_2 \cap \Delta_3 = A_1$ and $A_1(q)$ is 2-generated (Proposition 2.6), $|R_{23}| = 2$. Therefore

$$\sigma^*_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^c \cup R_{23} \rangle,$$

and so G has a presentation with 5 generators and 26 relations. If q is even, we use ρ_2 instead and the corresponding calculation gives 5 generators and $(10 + 10 - 5) + 5 + 2 = 22$ relations.

4.3. $\widetilde{A}_3(q)$. To obtain a presentation of $G = \widetilde{A}_3(q)$, we start with Proposition 2.4.



Take $\Delta_1 = A_3$ based on a_0, a_1 and a_2, $\Delta_2 = A_3$ based on a_0, a_1 and a_3, and $\Delta_3 = A_2$ based on a_2 and a_3. Thus G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \cup D_3 \mid R_1 \cup R_2 \cup R_3 \cup R_{12} \cup R_{13} \cup R_{23} \rangle.$$

Since $X_i(q) \cong A_3(q)$, $X_i(q) \cong \mathrm{SL}(4, q)$ by Proposition 2.3 for $i = 1, 2$. Now $X_1(q)$ has a presentation $\sigma_{X_1(q)} = \sigma_3$ with 5 generators and 20 relations.

Consider a subgroup X of G generated by L_0 and L_1. Its Dynkin diagram is of type A_2 and so by Proposition 2.3, $X \cong \mathrm{SL}(3, q)$. From Table 3 we know that X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_2$ with 4 generators and 14 relations. Now $X \leqslant X_i(q)$ for $i = 1, 2$. Moreover $X_2(q)$ has a presentation $\sigma_{X_2(q)} = \sigma_4$ with $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, $D_X \subseteq X_1(q)$ and the relations R_X already hold. We use Tietze transformations to eliminate D_X, R_X and R_{12}:

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \cup D_3 \mid R_1 \cup (R_2 \setminus R_X) \cup R_3 \cup R_{13} \cup R_{23} \rangle.$$

Since σ_4 requires 6 generators and 21 relations, $|D_1 \cup (D_2 \setminus D_X)| = 5 + (6-4) = 7$ and $|R_1 \cup (R_2 \setminus R_X)| = 20 + (21 - 14) = 27$.

Take now $Y = L_3$. Without loss of generality we may assume that $\sigma_Y \subseteq \sigma_{X_3(q)}$ with $\sigma_Y = \sigma_1$. Then by [13, Theorem 5.1], $\sigma_{X_3(q)} = \langle D_Y \cup \{c\} \mid R_Y \cup R_3^c \rangle$ where $D_Y = \{u, t, h\}$ (in the notations of [13, Theorem 5.1]) and R_3^c is a set of 5 relations involving c and elements of D_Y. Since $Y \leqslant X_2(q)$, obviously, $D_Y \subseteq X_2(q)$. Thus u, t and h can be expressed in terms of elements of $D_1 \cup (D_2 \setminus D_X)$. This makes the relations R_{23} superfluous. Moreover, the relations R_Y hold, as they hold in $X_2(q)$. We use Tietze transformations to eliminate D_Y, R_Y and R_{23}:

$$\sigma''_G = \langle D_1 \cup (D_2 \setminus D_X) \cup \{c\} \mid R_1 \cup (R_2 \setminus D_X) \cup R_3^c \cup R_{13} \rangle.$$

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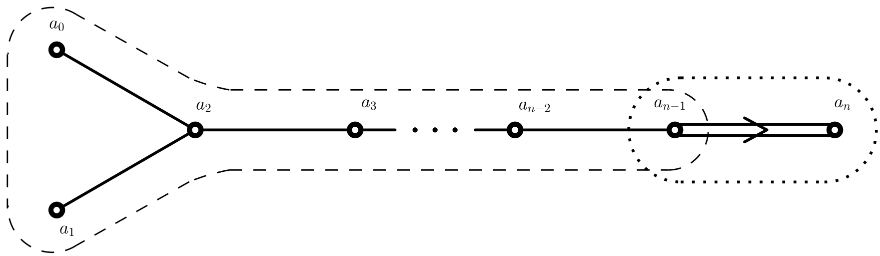
Notice that $X_3(q) \cong \mathrm{SL}(3, q)$ is generated by its subgroups L_2 and L_3 (cf. [4, Lemma 2.1]). Hence, c can be expressed in terms of elements of $D_1 \cup (D_2 \setminus D_X)$. Finally, since $\Delta_1 \cap \Delta_3 = A_1$ and $A_1(q)$ is 2-generated (Proposition 2.6), $|R_{13}| = 2$. Therefore

$$\sigma_G^* = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus D_X) \cup R_3^c \cup R_{13} \rangle,$$

and so G has a presentation with 7 generators and 34 relations. For q even, we use ρ_2, ρ_3 or ρ_4 instead to get $16 + (17 - 10) + 5 + 2 = 30$ relations.

4.4. $\widetilde{B}_n(q)$, $n \geqslant 9$. To obtain a presentation of $G = \widetilde{B}_n(q)$, $n \geqslant 9$, we use Corollary 2.5.



By Proposition 2.3, the groups $X_1(q)$, $X_2(q)$ and $X_3(q)$ are simply connected. Let Δ_1 be the subdiagram of Δ whose vertices are the n nodes $a_0, a_1, \ldots, a_{n-1}$. It has type D_n and so $X_1(q) \cong \mathrm{Spin}(2n, q)$. Let Δ_2 be the subdiagram of Δ whose vertices are the nodes a_{n-1} and a_n. It is of type C_2 and so $X_2(q) \cong \mathrm{Sp}(4, q)$. Then Δ_3 is the subdiagram of Δ based on all vertices but a_{n-1}, thus of type $D_{n-1} \times A_1$. Hence, $X_3(q) \cong \mathrm{Spin}(2n - 2, q) \times \mathrm{SL}(2, q)$. Clearly, $\Delta = \bigcup_{i=1}^{2} \Delta_i$. Therefore G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle$$

as described in Corollary 2.5.

Consider a subgroup $X = L_{n-1}$ of G. Its Dynkin diagram is of type A_1 and so by Proposition 2.3, $X \cong \mathrm{SL}(2, q)$. From Table 3 we know that X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ with $|D_X| = 3$ and $|R_X| = 9$. Now $X \leqslant X_i(q)$ for $i = 1$, 2. The group $X_2(q)$ has a presentation $\sigma_{X_2(q)} = \sigma_9$ if q is odd ($\sigma_{X_2(q)} = \rho_{10}$ if q is even). By the results of Section 3, $X_1(q)$ has a presentation $\sigma_{X_1(q)} = \sigma_{13}$ and $\sigma_X \subseteq \sigma_{X_1(q)}$.

Since $X \leqslant X_2(q)$, obviously, $D_X \subseteq X_2(q)$ and the relations R_X hold, as they hold in $X_2(q)$. We use Tietze transformations to eliminate D_X, R_X and R_{12} to obtain:

$$\sigma_G' = \langle (D_1 \setminus D_X) \cup D_2 \mid (R_1 \setminus R_X) \cup R_2 \cup R_3^* \rangle.$$

Now σ_{13} requires 7 generators and 34 relations, so that $|(D_1 \setminus D_X) \cup D_2| = (7-3) + 5 = 9$ and $|(R_1 \setminus R_X) \cup R_2| = (34-9) + 27 = 52$ if q is odd. The corresponding calculation for even q uses ρ_{10} and ρ_{13} instead: $|(D_1 \setminus D_X) \cup D_2| = (7-3)+6 = 10$ and $|(R_1 \setminus R_X) \cup R_2| = (30-5) + 20 = 45$.

Consider $X_3(q) \cong \mathrm{Spin}\,(2n-2, q) \times \mathrm{SL}\,(2, q)$. Each factor has two generators (Proposition 2.6). Denote them by a_1, a_2 and b_1, b_2, respectively. Then $R_3^* = \{[a_i, b_j] = 1, 1 \leqslant i, j \leqslant 2\}$ and so $|R_3^*| = 4$.

Therefore G has a presentation with 9 generators and 56 relations if q is odd, and 10 generators and 49 relations if q is even.

4.5. $\widetilde{B}_n(q)$, $3 \leqslant n \leqslant 8$. To obtain a presentation of $G = \widetilde{B}_3(q)$, we use Corollary 2.5 similar to the case $\widetilde{B}_n(q)$ with $n \geqslant 9$. However, in this case we take $\Delta_1 = A_3$ and $\Delta_2 = C_2$. Then $X_1(q) \cong \mathrm{SL}\,(4, q)$ has a presentation $\sigma_{X_2(q)} = \sigma_3$ and $X_2(q) \cong \mathrm{Sp}\,(4, q)$ has a presentation $\sigma_{X_2(q)} = \sigma_9$. Taking $X = L_2$ (as in the previous case), we have that $X \leqslant X_i(q)$ for $i = 1, 2$, X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ and $\sigma_X \subseteq \sigma_{X_1(q)}$. This is possible because L_1 and L_2 are conjugate inside $X_1(q)$.

Since $X \leqslant X_2(q)$, $D_X \subseteq X_2(q)$ and R_X hold, as they hold in $X_2(q)$. Finally, $X_3(q) = (X_1(q) \cap X_3(q)) \times (X_2(q) \cap X_3(q)) = (\mathrm{SL}\,(2, q) \times \mathrm{SL}\,(2, q)) \times \mathrm{SL}\,(2, q)$, and as before (using Proposition 2.6) we obtain that $|R_3^*| = 4$. Using Tietze transformations we obtain that G has a presentation

$$\sigma_G' = \langle (D_1 \setminus D_X) \cup D_2 \mid (R_1 \setminus R_X) \cup R_2 \cup R_3^* \rangle$$

with $(5-3) + 5 = 7$ generators and $(20-9) + 27 + 4 = 42$ relations if q is odd. If q is even the corresponding calculation gives $(5-3) + 6 = 8$ generators and $(16-5) + 20 + 4 = 35$ relations.

To obtain a presentation of $G = \widetilde{B}_n(q)$ for $4 \leqslant n \leqslant 8$, we use Corollary 2.5 just as in the case $\widetilde{B}_n(q)$ with $n \geqslant 9$. If $G = \widetilde{B}_4(q)$, Δ_1 is of type D_4, and so $X_1(q) \cong \mathrm{Spin}\,(8, q)$ and has a presentation $\sigma_{X_1(q)}$ that is the reduced σ_{13} with 6 generators and 29 relations. Thus G has a presentation with $(6-3) + 5 = 8$ generators and $(29-9) + 27 + 4 = 51$ relations if q is odd. If q is even we get $(6-3) + 6 = 9$ generators and $(25-5) + 20 + 4 = 44$ relations.

Finally, if $G = \widetilde{B}_n(q)$ with $5 \leqslant n \leqslant 8$, Δ_1 is of type D_n, and so $X_1(q) \cong \mathrm{Spin}\,(2n, q)$ and has a presentation $\sigma_{X_1(q)}$ that is the reduced σ_{13} with 6 generators and 30 relations. Thus G has a presentation with $(6-3) + 5 = 8$ generators and $(30-9) + 27 + 4 = 52$ relations if q is odd. If q is even we get $(6-3) + 6 = 9$ generators and $(26-5) + 20 + 4 = 45$ relations.

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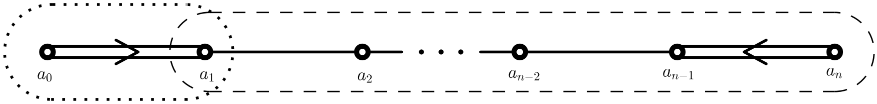
4.6. $\widetilde{C}_n(q)$, $n \geqslant 3$. To obtain a presentation of $G = \widetilde{C}_n(q)$, $n \geqslant 3$, we use Corollary 2.5.



By Proposition 2.3, the groups $X_1(q)$, $X_2(q)$ and $X_3(q)$ are simply connected. Let Δ_1 be the subdiagram of Δ whose vertices are the n nodes $a_1, \ldots, a_n$. It has type C_n and so $X_1(q) \cong \mathrm{Sp}\,(2n, q)$. Let Δ_2 be the subdiagram of Δ whose vertices are the nodes a_0 and a_1. It is of type C_2 and so $X_2(q) \cong \mathrm{Sp}\,(4, q)$. Then Δ_3 is the subdiagram of Δ based on all vertices but a_1, thus of type $A_1 \times C_{n-1}$. Hence, $X_3(q) \cong \mathrm{SL}\,(2, q) \times \mathrm{Sp}\,(2n-2, q)$. Clearly, $\Delta = \Delta_1 \cup \Delta_2$. Therefore G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle$$

as described in Corollary 2.5.

Take a presentation $\sigma_{X_1(q)} = \sigma_{11}$. Consider a subgroup $X = L_1$ of G. Its Dynkin diagram is of type A_1 and so by Proposition 2.3, $X \cong \mathrm{SL}\,(2, q)$. From Table 3 we know that X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ with $|D_X| = 3$ and $|R_X| = 9$. Now $X \leqslant X_i(q)$ for $i = 1, 2$. The group $X_2(q)$ has a presentation $\sigma_{X_2(q)} = \sigma_9$ (or $\sigma_{X_2(q)} = \rho_{10}$ if q is even). By [13, Theorem 7.1], $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, obviously, $D_X \subseteq X_1(q)$. Thus elements of D_X can be expressed in terms of elements of D_1. Moreover, the relations R_X hold, as they hold in $X_1(q)$. We use Tietze transformations to eliminate D_X, R_X and R_{12} to obtain:

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Therefore $|D_1 \cup (D_2 \setminus D_X)| = 8 + (5-3) = 10$ and $|R_1 \cup (R_2 \setminus R_X)| = 47 + (27-9) = 65$ if q is odd. For even q the corresponding calculations are $|D_1 \cup (D_2 \setminus D_X)| = 9 + (6-3) = 12$ and $|R_1 \cup (R_2 \setminus R_X)| = 40 + (20-5) = 55$.

Finally, consider $X_3(q) \cong \mathrm{Sp}\,(2n-2, q) \times \mathrm{SL}\,(2, q)$. Each factor has two generators (Proposition 2.6). Thus as in the previous case we obtain $|R_3^*| = 4$.

Therefore G has a presentation with 10 generators and 69 relations if q is odd, and 12 generators and 59 relations if q is even. For $3 \leqslant n \leqslant 8$ we obtain shorter presentations, see Table 1.

4.7. $\widetilde{C}_2(q)$. The only difference with the previous case is that $\Delta_1 = C_2$, and thus $X_1(q)$ has a presentation $\sigma_{X_1(q)} = \sigma_9$ if q is odd and $\sigma_{X_1(q)} = \rho_{10}$ if q is even. Replacing σ_{11} and ρ_{11} by these, we obtain that $G = \widetilde{C}_2(q)$ has a presentation with $5 + (5-3) = 7$ generators and $27 + (27-9) + 4 = 49$ relations if q is odd, and $6 + (6-3) = 9$ generators and $20 + (20-5) + 4 = 39$ relations if q is even.

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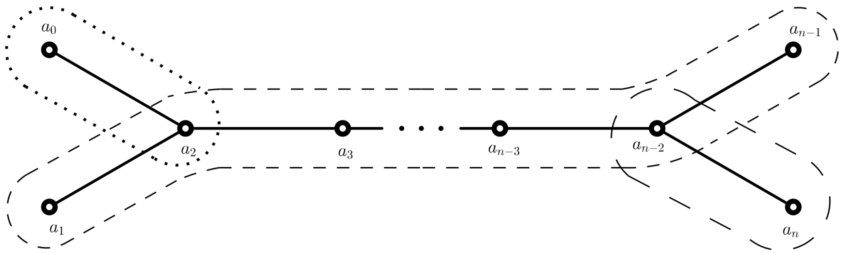
4.8. $\widetilde{D}_n(q), n \geqslant 6.$ Let us assume that $n \geqslant 9$. We use Corollary 2.5. Let $\Delta_1 = A_{n-1}$ on vertices $a_1, a_2, \ldots, a_{n-1}$, $\Delta_2 = A_2 \times A_2$ on vertices a_0, a_2, a_{n-2} and a_n, and $\Delta_3 = (A_1)^4 \times A_{n-5}$ on all vertices but a_2 and a_{n-2}.



By Proposition 2.3, $X_i(q)$ are simply connected for $1 \leqslant i \leqslant 3$, and so $X_1(q) \cong \mathrm{SL}(n, q)$, $X_2(q) \cong \mathrm{SL}(3, q) \times \mathrm{SL}(3, q)$ and $X_3(q) \cong (\mathrm{SL}(2, q))^4 \times \mathrm{SL}(n-4, q)$. Thus G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle.$$

Now $X_1(q)$ has a presentation $\sigma_{X_1(q)} = \sigma_7$ with 6 generators and 25 relations.

Consider a subgroup X of G generated by L_2 and L_{n-2}. Its Dynkin diagram is of type $A_1 \times A_1$ and so by Proposition 2.3, $X \cong \mathrm{SL}(2, q) \times \mathrm{SL}(2, q)$. From Table 3 it follows that X has a presentation

$$\sigma_X = \langle D_X^2 \cup D_X^{n-2} \mid R_X^2 \cup R_X^{n-2} \cup R_X^* \rangle$$

where $\sigma_X^i = \langle D_X^i \mid R_X^i \rangle$ is a presentation of $L_i \cong \mathrm{SL}(2, q)$ for $i = 2$ and $n-2$, $\sigma_X^i = \sigma_1$ and R_X^* are the relations that ensure that $[L_2, L_{n-2}] = 1$. Since both L_2 and L_{n-2} are 2-generated (Proposition 2.6), we may choose $c_1, c_2 \in L_2$ and $d_1, d_2 \in L_{n-2}$ so that $R_X^* = \{[c_i, d_j] = 1, 1 \leqslant i, j \leqslant 2\}$. Hence, $|R_X^*| = 4$.

Now $X \leqslant X_i(q)$ for $i = 1, 2$. Moreover $X_2(q)$ has a presentation $\sigma_{X_2(q)}$ with $\sigma_X \subseteq \sigma_{X_2(q)}$. Indeed, take $\sigma_{X_2(q)} = \langle D_2 \mid R_2 \rangle$ such that $D_2 = D_2^2 \cup D_2^{n-2}$ and $R_2 = R_2^2 \cup R_2^{n-2} \cup R_2^*$ where for $i = 2$ and $n-2$, $\sigma_{X_2(q)}^i = \langle D_2^i \mid R_2^i \rangle$ is a presentation of a subgroup $L_{0,2}$ of $X_2(q)$ if $i = 2$, and $L_{n-2,n}$ if $i = n-2$, $\sigma_{X_2(q)}^i = \langle D_2^i \mid R_2^i \rangle = \sigma_2$ and R_2^* are the relations that ensure that $[L_{0,2}, L_{n-2,n}] = 1$. Using [2] and [10, page 745, Corollary], we may take $c_1', c_2' \in L_{0,2}$ and $d_1', d_2' \in L_{n-2,n}$ with $c_1' = c_1$, $d_1' = d_1$. Then let $R_2^* = R_X^* \cup \{[c_1', d_2'] = [c_2', d_1'] = [c_2', d_2'] = 1\}$. Then $R_X^* \subseteq R_2^*$ and $|R_2^* \setminus R_X^*| = 3$.

Since $X \leqslant X_1(q)$, $D_X \subseteq X_1(q)$ and relations R_X already hold in $X_1(q)$. We use Tietze transformations to eliminate D_X, R_X and R_{12}:

$$\sigma_G' = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

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Notice that $|D_1 \cup (D_2 \setminus D_X)| = 6 + (4-3) + (4-3) = 8$ and $|R_1 \cup (R_2 \setminus R_X)| = 25 + (14-9) + (14-9) + 3 = 38$.

Finally, $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q))$ where $X_3(q) \cap X_1(q) = L_1 \times \langle L_3, \ldots, L_{n-3} \rangle \times L_{n-1} \cong \mathrm{SL}(2,q) \times \mathrm{SL}(n-4,q) \times \mathrm{SL}(2,q)$ and $X_3(q) \cap X_2(q) = L_0 \times L_n \cong \mathrm{SL}(2,q) \times \mathrm{SL}(2,q)$. Since each of the factors requires only two generators (Proposition 2.6), we obtain that $|R_3^*| = 4$. Therefore G has a presentation with 8 generators and 42 relations if q is odd. For even q the corresponding calculation gives 8 generators and $21+((10-5)+(10-5)+3)+4 = 38$ relations.

If $6 \leqslant n \leqslant 8$, then the above argument works with little variation. The subgroup $X_1(q) \cong \mathrm{SL}(n,q)$ has a presentation $\sigma_{X_1(q)} = \sigma_5$ with 5 generators and 21 relations. The rest of the argument does not change, producing a presentation of G with 7 generators and 38 relations if q is odd and 7 generators and 34 relations if q is even.

4.9. $\widetilde{D}_n(q), n = 4, 5$. This time we use Corollary 2.5 with $\Delta_1 = D_n$ based on all vertices but a_0, and $\Delta_2 = A_2$ based on vertices a_0 and a_2. Then $X_1(q) \cong \mathrm{Spin}(2n,q)$ and $X_2(q) \cong \mathrm{SL}(3,q)$. Thus $\Delta_3 = A_1^4$ if $n = 4$, and $\Delta_3 = A_1^2 \times A_3$ if $n = 5$, giving $X_3(q) \cong \mathrm{SL}(2,q)^4$ and $X_3(q) \cong \mathrm{SL}(2,q)^2 \times \mathrm{SL}(4,q)$, respectively.

Consider a subgroup $X = L_2$ of G. Then $X \leqslant X_i(q)$ for $i = 1, 2$, $X \cong \mathrm{SL}(2,q)$ and X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$. Now $X_1(q)$ has a presentation $\sigma_{X_1(q)}$ that is the reduced σ_{13}. The group $X_2(q)$ has a presentation $\sigma_{X_2(q)} = \sigma_2$ and $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, $D_X \subseteq X_1(q)$ and R_X hold as they hold in $X_1(q)$. We use Tietze transformations to eliminate D_X, R_X and R_{12} to obtain a presentation

$$\sigma_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle$$

where as usual $|R_3^*| = 4$ (using Proposition 2.6). Thus G has a presentation with $6 + (4-3) = 7$ generators and $29 + (14-9) + 4 = 38$ relations if $n = 4$, and $6 + (4-3) = 7$ generators and $30 + (14-9) + 4 = 39$ relations if $n = 5$. For even q the corresponding calculations give $25 + (10-5) + 4 = 34$ relations if $n = 4$ and $26 + (10-5) + 4 = 35$ relations if $n = 5$.

4.10. $\widetilde{E}_6(q)$. This time we use Corollary 2.5. Take $\Delta_1 = A_5$ on vertices a_1, a_2, a_3, a_5 and a_6, $\Delta_2 = A_3$ on vertices a_0, a_4 and a_3. Then $\Delta_3 = A_2 \times A_2 \times A_2$ is based on all vertices but a_3.

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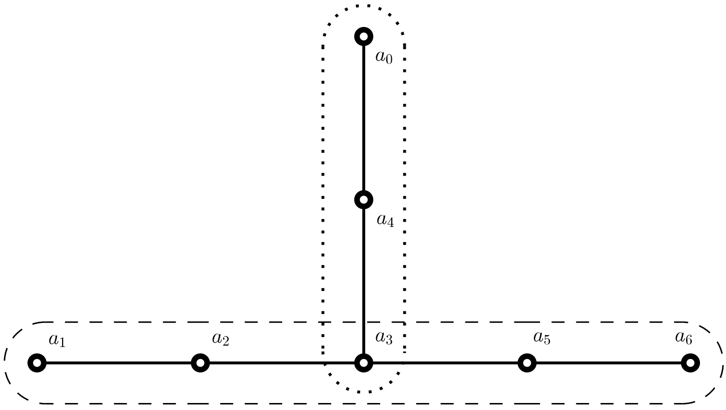


Hence, $G = \widetilde{E}_6(q)$ has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle.$$

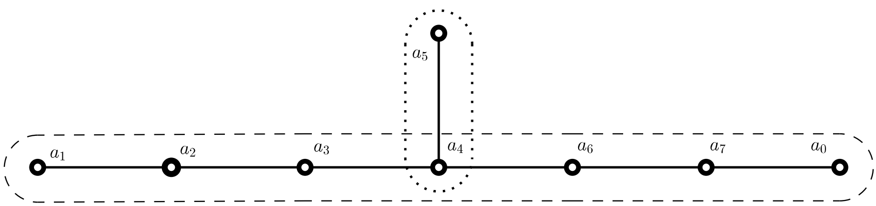
Using Proposition 2.3 and Table 3 we have that $X_1 \cong \mathrm{SL}(6, q)$ has a presentation $\sigma_{X_1(q)} = \sigma_5$, and $X_2 \cong \mathrm{SL}(4, q)$ has a presentation $\sigma_{X_2(q)} = \sigma_3$.

Consider a subgroup $X = L_3 \cong \mathrm{SL}(2, q)$ of G. Then X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$. Now $X \leqslant X_i(q)$ for $i = 1, 2$, and [13, Theorem 6.1] implies that $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, $D_X \subseteq X_1(q)$ and the relations R_X hold (as they hold in $X_1(q)$). We use Tietze transformations to eliminate D_X, R_X and R_{12}. Hence, G has a presentation

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Finally, $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q)) \cong (\mathrm{SL}(3, q) \times \mathrm{SL}(3, q)) \times \mathrm{SL}(3, q)$. Both factors require 2 generators (Proposition 2.6), implying $|R_3^*| = 4$. Therefore G has a presentation with $5 + (5 - 3) = 7$ generators and $21 + (20 - 9) + 4 = 36$ relations if q is odd. For even q the corresponding calculation gives 7 generators and $17 + (16 - 5) + 4 = 32$ relations.

4.11. $\widetilde{E}_7(q)$. Again we use Corollary 2.5. Take $\Delta_1 = A_7$ based on all vertices but a_5, and $\Delta_2 = A_3$ based on vertices a_4 and a_5. Then $\Delta_3 = A_3 \times A_3 \times A_1$ is based on all vertices but a_4.



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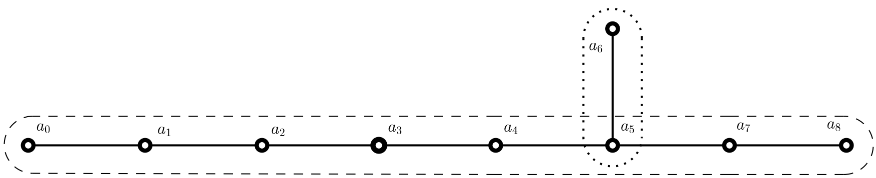
Hence, $G = \widetilde{E}_7(q)$ has a presentation as described in Corollary 2.5. Using Proposition 2.3 and Table 3 we have that $X_1 \cong \mathrm{SL}(8, q)$ has a presentation $\sigma_{X_1(q)} = \sigma_5$, $X_2 \cong \mathrm{SL}(3, q)$ has a presentation $\sigma_{X_2(q)} = \sigma_2$. Consider a subgroup $X = L_4 \cong \mathrm{SL}(2, q)$ of G. Then X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$. Now $X \leqslant X_i(q)$ for $i = 1, 2$, and $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, $D_X \subseteq X_1(q)$ and the relations R_X hold (as they hold in $X_1(q)$). We use Tietze transformations to eliminate D_X, R_X and R_{12}. Hence, G has a presentation

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Finally, $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q)) \cong (\mathrm{SL}(4, q) \times \mathrm{SL}(4, q)) \times \mathrm{SL}(2, q)$. Both factors require 2 generators (Proposition 2.6), implying $|R_3^*| = 4$. Therefore G has a presentation with $5 + (4 - 3) = 6$ generators and $21 + (14 - 9) + 4 = 30$ relations if q is odd. If q is even the corresponding calculation gives 6 generators and $17 + (10 - 5) + 4 = 26$ relations.

4.12. $\widetilde{E}_8(q)$. We use Corollary 2.5. Take $\Delta_1 = A_8$ based on all vertices but a_6, and $\Delta_2 = A_2$ based on vertices a_5 and a_6. Then $\Delta_3 = A_5 \times A_2 \times A_1$ is based on all vertices but a_5.



Hence, $G = \widetilde{E}_8(q)$ has a presentation as described in Corollary 2.5. Using Proposition 2.3 and Table 3 we have that $X_1 \cong \mathrm{SL}(9, q)$ has a presentation $\sigma_{X_1(q)} = \sigma_7$, $X_2 \cong \mathrm{SL}(3, q)$ has a presentation $\sigma_{X_2(q)} = \sigma_2$. Consider a subgroup $X = L_5 \cong \mathrm{SL}(2, q)$ of G. Then X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$. Now $X \leqslant X_i(q)$ for $i = 1, 2$, and $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, $D_X \subseteq X_1(q)$ and the relations R_X hold (as they hold in $X_1(q)$). We use Tietze transformations to eliminate D_X, R_X and R_{12}. Hence, G has a presentation

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Finally, $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q)) \cong (\mathrm{SL}(6, q) \times \mathrm{SL}(3, q)) \times \mathrm{SL}(2, q)$. Both factors require 2 generators (Proposition 2.6), implying $|R_3^*| = 4$. Therefore G has a presentation with $6 + (4 - 3) = 7$ generators and $25 + (14 - 9) + 4 = 34$ relations if q is odd. If q is even we get 7 generators and $21 + (10 - 5) + 4 = 30$ relations.

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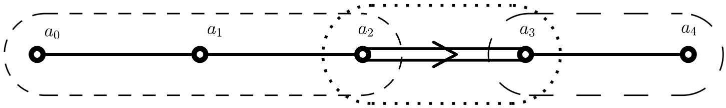
4.13. $\widetilde{F}_4(q)$. This time we use Proposition 2.4 with $k = 5$. Take $\Delta_1 = A_3$ based on vertices a_0, a_1 and a_2, $\Delta_2 = C_2$ based on vertices a_2 and a_3, $\Delta_3 = A_2$ based on vertices a_3 and a_4, $\Delta_4 = A_2 \times A_2$ based on all vertices but a_2, and finally $\Delta_5 = A_3 \times A_1$ based on all vertices but a_3.



Taking subgroups $X_i(q)$ corresponding to Δ_i for $1 \leqslant i \leqslant 5$, we obtain a presentation of G

$$\sigma_G = \left\langle D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \mid R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup \bigcup_{i<j} R_{ij} \right\rangle$$

as described in Proposition 2.4. Notice that $R_{13} = \emptyset$.

Using Proposition 2.3 and Table 3 we have that $X_1(q) \cong \mathrm{SL}(4, q)$ has a presentation $\sigma_{X_1(q)} = \sigma_3$, $X_2(q) \cong \mathrm{Sp}(4, q)$ has a presentation $\sigma_{X_2(q)} = \sigma_9$ (or ρ_{10} if q is even), and $X_3(q) \cong \mathrm{SL}(3, q)$ has a presentation $\sigma_{X_3(q)} = \sigma_2$.

Take $X = L_2 \cong \mathrm{SL}(2, q)$. Then $X \leqslant X_i(q)$ for $i = 1, 2$ and [**13**, Theorem 6.1] implies that $\sigma_X \subseteq \sigma_{X_1(q)}$. Since $X \leqslant X_2(q)$, $D_X \subseteq X_2(q)$ and R_X hold as they hold in $X_2(q)$. We use Tietze transformations to eliminate D_X, R_X and R_{12} to obtain a presentation

$$\sigma_G^{(1)} = \Big\langle (D_1 \setminus D_X) \cup D_2 \cup D_3 \cup D_4 \cup D_5 \mid (R_1 \setminus R_X) \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup \bigcup_{i<j} R_{ij} \setminus R_{12} \Big\rangle.$$

Take $Y = L_3 \cong \mathrm{SL}(2, q)$. Then $Y \leqslant X_i(q)$ for $i = 2, 3$ and $\sigma_Y \subseteq \sigma_{X_3(q)}$. Again we use Tietze transformations. This time we eliminate D_Y, R_Y and R_{23} to obtain

$$\sigma_G^{(2)} = \Big\langle (D_1 \setminus D_X) \cup D_2 \cup (D_3 \setminus D_Y) \cup D_4 \cup D_5 \mid \mid (R_1 \setminus R_X) \cup R_2 \cup (R_3 \setminus R_Y) \cup R_4 \cup R_5 \cup \bigcup_{i<j} R_{ij} \setminus (R_{12} \cup R_{23}) \Big\rangle.$$

By Proposition 2.3, $X_4(q) = (X_4(q) \cap X_1(q)) \times X_3(q) \cong \mathrm{SL}(3, q) \times \mathrm{SL}(3, q)$. Each factor is 2-generated (Proposition 2.6). Let us denote these pairs of generators by c_1, c_2 and d_1, d_2, respectively. In fact, [**10**, page 745, Corollary]

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allows us to choose $c_1 \in L_0 \leqslant X_4(q) \cap X_1(q)$ and $d_1 \in L_4 \leqslant X_3(q)$. Then $X_4(q)$ has a presentation $\sigma_{X_4(q)} = \langle c_1, c_2, d_1, d_2 \mid R_{c_1,c_2} \cup R_{d_1,d_2} \cup R_4^* \rangle$ where $\langle c_1, c_2 \mid R_{c_1,c_2} \rangle$ is a presentation of $X_4(q) \cap X_1(q) \cong \mathrm{SL}(3, q)$, $\langle d_1, d_2 \mid R_{d_1,d_2} \rangle$ is a presentation of $X_4(q) \cap X_3(q) = X_3(q) \cong \mathrm{SL}(3, q)$, and $R_4^* = \{[c_1, d_1] = [c_1, d_2] = [c_2, d_1] = [c_2, d_2] = 1\}$. Since $X_4(q) \cap X_1(q) \leqslant X_1(q)$, $c_1, c_2 \in X_1(q)$ and the relations R_{c_1,c_2} hold as they hold in $X_1(q)$. Similarly, $d_1, d_2 \in X_3(q)$ and relations R_{d_1,d_2} hold as they hold in $X_3(q)$. We now use Tietze transformations to eliminate c_1, c_2, d_1, d_2, $R_{c_1,c_2} \cup R_{d_1,d_2}$ and $R_{14} \cup R_{34}$. Now we may eliminate relations R_{24}: R_{24} identify $X_2(q) \cap X_4(q)$. Note that $X_2(q) \cap X_4(q) = (X_2(q) \cap X_3(q)) \cap (X_3(q) \cap X_4(q))$, and we have already identified $X_2(q) \cap X_3(q)$ and $X_3(q) \cap X_4(q)$. Thus G has a presentation

$$\sigma_G^{(3)} = \Big\langle (D_1 \setminus D_X) \cup D_2 \cup (D_3 \setminus D_Y) \cup D_5 \mid \\ (R_1 \setminus R_X) \cup R_2 \cup (R_3 \setminus R_Y) \cup R_4^* \cup R_5 \cup \bigcup_{i=1}^{5} R_{i5} \Big\rangle.$$

Finally, by Proposition 2.3, $X_5(q) = X_1(q) \times (X_3(q) \cap X_5(q)) \cong \mathrm{SL}(4, q) \times \mathrm{SL}(2, q)$. Each factor is 2-generated (Proposition 2.6). Let us denote these pairs of generators by c_1', c_2' and d_1', d_2', respectively. Notice that [10, page 745, Corollary] implies that we may choose $c_1' = c_1$ and $d_1' = d_1$. Then $X_5(q)$ has a presentation $\sigma_{X_5(q)} = \langle c_1, c_2', d_1, d_2' \mid R_{c_1,c_2'} \cup R_{d_1,d_2'} \cup R_5^* \rangle$ where $\langle c_1, c_2' \mid R_{c_1,c_2'} \rangle$ is a presentation of $X_5(q) \cap X_1(q) = X_1(q) \cong \mathrm{SL}(4, q)$, $\langle d_1, d_2' \mid R_{d_1,d_2'} \rangle$ is a presentation of $X_3(q) \cap X_5(q) \cong \mathrm{SL}(2, q)$, and $R_5^* = \{[c_1, d_1] = [c_1, d_2'] = [c_2', d_1] = [c_2', d_2'] = 1\}$. Since $X_5(q) \cap X_1(q) = X_1(q)$, $c_1, c_2' \in X_1(q)$ and relations $R_{c_1,c_2'}$ hold as they hold in $X_1(q)$. Similarly, $d_1, d_2' \in X_3(q)$ and relations $R_{d_1,d_2'}$ hold as they hold in $X_3(q)$. We use Tietze transformations to eliminate c_1, c_2', d_1, d_2', $R_{c_1,c_2'} \cup R_{d_1,d_2'}$ and $R_{15} \cup R_{35}$ to obtain

$$\sigma_G^{(4)} = \langle (D_1 \setminus D_X) \cup D_2 \cup (D_3 \setminus D_Y) \mid \\ (R_1 \setminus R_X) \cup R_2 \cup (R_3 \setminus R_Y) \cup R_4^* \cup R_5^* \cup R_{25} \cup R_{45} \rangle.$$

Now we may eliminate relations R_{25}: R_{25} identify $X_2(q) \cap X_5(q)$. Note that $X_2(q) \cap X_5(q) = (X_1(q) \cap X_2(q)) \cap (X_1(q) \cap X_5(q))$, and we have already identified $X_1(q) \cap X_2(q)$ and $X_1(q) \cap X_5(q)$. We may also eliminate relations R_{45}. Relations R_{45} identify $X_4(q) \cap X_5(q)$ which is a direct product of two components: $L_{01} = (X_1(q) \cap X_4(q)) \cap (X_1(q) \cap X_5(q))$ and $L_4 = (X_3(q) \cap X_4(q)) \cap (X_3(q) \cap X_5(q))$, and we have already identified those.

Notice that $|R_4^* \cap R_5^*| = 1$ and so $|R_4^* \cup R_5^*| = 7$. Thus we have obtained a presentation

$$\sigma_G^{(5)} = \langle (D_1 \setminus D_X) \cup D_2 \cup (D_3 \setminus D_Y) \mid (R_1 \setminus R_X) \cup R_2 \cup (R_3 \setminus R_Y) \cup R_4^* \cup R_5^* \rangle$$

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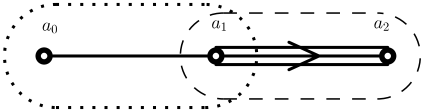
with $(5-3)+5+(4-3)=8$ generators and $(20-9)+27+(14-9)+7=50$ relations if q is odd. If q is even the corresponding calculation gives $(5-3)+6+(4-3)=9$ generators and $(16-5)+20+(10-5)+7=43$ relations.

4.14. $\widetilde{G}_2(q)$. We now use Corollary 2.5. Take Δ_1 based on a_0 and a_1, Δ_2 based on a_1 and a_2, and Δ_3 based on a_0 and a_2.



Then $X_1(q) \cong \mathrm{SL}(3, q)$ has a presentation $\sigma_{X_1(q)} = \sigma_2$, $X_2(q) \cong G_2(q)$ has a presentation $\sigma_{X_2(q)} = \sigma_{14}$. Take $X = L_1 \cong \mathrm{SL}(2, q)$. Then $X \leqslant X_i(q)$ for $i = 1, 2$. Since $X \leqslant X_2(q)$ and $\sigma_X \subseteq \sigma_{X_1(q)}$, we may use Tietze transformations to remove D_X, R_X and R_{12}, thus obtaining

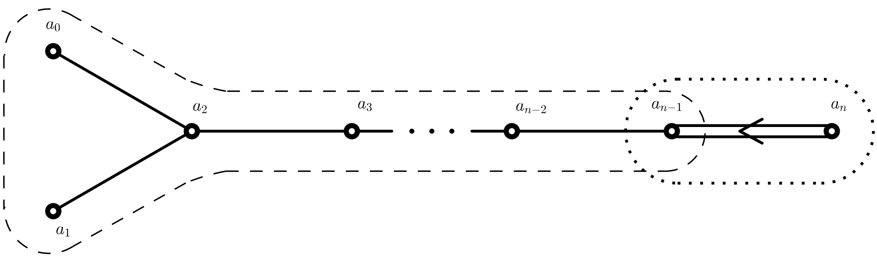
$$\sigma_G = \langle (D_1 \setminus D_X) \cup D_2 \mid (R_1 \setminus R_X) \cup R_2 \cup R_3^* \rangle.$$

Since $X_3(q) \cong \mathrm{SL}(2, q) \times \mathrm{SL}(2, q)$ and $\mathrm{SL}(2, q)$ is 2-generated (Proposition 2.6), we obtain $|R_3^*| = 4$, and so σ_G has $(4-3)+6=7$ generators and $(14-9)+31+4=40$ relations if q is odd. If q is even we get 7 generators and $(10-5)+23+4=32$ relations.

5. Twisted affine Kac–Moody groups

We briefly go through the calculations for the remaining 2 infinite series and 3 exceptional types of twisted affine Kac–Moody groups.

5.1. $\widetilde{B}_n^t(q)$.



For $n \neq 3$, the proof is line by line repetition of the case $G = \widetilde{B}_n(q)$ giving us the same result: a presentation of G with 9 generators and 56 relations when q is

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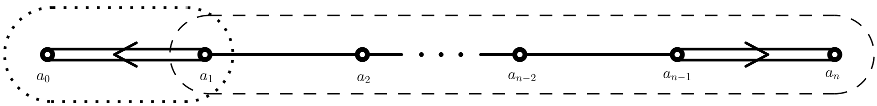
odd, and with 10 generators and 49 relations when q is even, for $n \geqslant 9$, and the same results as for $\widetilde{B}_n(q)$ for $4 \leqslant n \leqslant 8$.

Now for $G = \widetilde{B}^t_3(q)$, we repeat the proof of the case $\widetilde{B}_3(q)$ line by line with one change: in the case when q is odd, we take $\sigma_{X_2(q)} = \sigma_9$, thus obtaining a presentation of G with $5 + (5 - 3) = 7$ generators and $20 + (27 - 9) + 4 = 42$ relations if q is odd, and 8 generators and 35 relations if q is even.

5.2. $\widetilde{C}^t_n(q)$. To obtain a presentation of $G = \widetilde{C}^t_n(q)$, $n \geqslant 3$, we use Corollary 2.5.



By Proposition 2.3, the groups $X_1(q)$, $X_2(q)$ and $X_3(q)$ are simply connected. Let $\Delta_1 = B_n$ be the subdiagram of Δ whose vertices are the n nodes $a_1, \ldots,$ a_n, and $\Delta_2 = C_2$ the subdiagram of Δ whose vertices are the nodes a_0 and a_1. Then $X_1(q) \cong \mathrm{Spin}\,(2n+1, q)$ and $X_2(q) \cong \mathrm{Sp}\,(4, q)$. It follows that Δ_3 is the subdiagram of Δ based on all vertices but a_1, thus of type $A_1 \times B_{n-1}$. Hence, $X_3(q) \cong \mathrm{SL}\,(2, q) \times \mathrm{Spin}\,(2n-1, q)$. Clearly, $\Delta = \Delta_1 \cup \Delta_2$. Therefore G has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R^*_3 \cup R_{12} \rangle$$

as described in Corollary 2.5.

Take a presentation $\sigma_{X_1(q)} = \sigma_{12}$ if q is odd and $\sigma_{X_1(q)} = \rho_{11}$ if q is even (notice that $B_m(2^a) \cong C_m(2^a)$). Consider a subgroup $X = L_1$ of G. Its Dynkin diagram is of type A_1 and so by Proposition 2.3, $X \cong \mathrm{SL}\,(2, q)$. From Table 3 we know that X has a presentation $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ with $|D_X| = 3$ and $|R_X| = 9$. Now $X \leqslant X_i(q)$ for $i = 1, 2$. The group $X_2(q)$ has a presentation $\sigma_{X_2(q)} = \sigma_{10}$. By [13, Theorem 7.1], $\sigma_X \subseteq \sigma_{X_2(q)}$. Since $X \leqslant X_1(q)$, obviously, $D_X \subset X_1(q)$. Thus elements of D_X can be expressed in terms of elements of D_1. Moreover, the relations R_X hold, as they hold in $X_1(q)$. We use Tietze transformations to eliminate D_X, R_X and R_{12} to obtain:

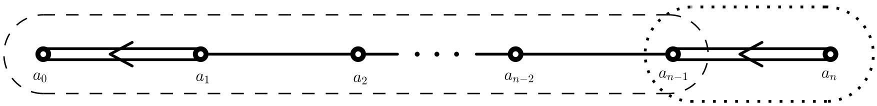
$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R^*_3 \rangle.$$

Finally, consider $X_3(q) \cong \mathrm{SL}\,(2, q) \times \mathrm{Spin}\,(2n-1, q)$. Each factor has two generators (Proposition 2.6). Thus as before we obtain $|R^*_3| = 4$.

Therefore G has a presentation with $8+(6-3) = 11$ generators and $47+(28-9)+4 = 70$ relations if q is odd. For even q the corresponding calculation gives $9+(6-3) = 12$ generators and $40+(20-5)+4 = 59$ relations. For $3 \leqslant n \leqslant 8$ we get shorter presentations, see Table 1.

If $n = 2$, we use $\sigma_{X_1(q)} = \sigma_9$ if q is odd and $\sigma_{X_1(q)} = \rho_{10}$ if q is even, thus obtaining a presentation with $5+(6-3) = 8$ generators and $27+(28-9)+4 = 50$ relations if q is odd, and $6 + (6 - 3) = 9$ generators and $20 + (20 - 5) + 4 = 39$ relations if q is even.

5.3. $\widetilde{C}'_n(q)$.



For $n \geqslant 3$, the proof is very similar to the case $G = \widetilde{C}_n(q)$. Here we take $\Delta_1 = B_n$ based on the n vertices $a_0, \ldots a_{n-1}$ and $\Delta_2 = C_2$ based on a_{n-1} and a_n. Then Δ_3 is based on all vertices but a_{n-1}, thus has type $B_{n-1} \times A_1$.

$X_2(q) \cong \mathrm{Sp}\,(4, q)$ and we again take $\sigma_{X_2(q)} = \sigma_9$. $X_1(q) \cong \mathrm{Spin}\,(2n + 1, q)$ so we take $\sigma_{X_1(q)} = \sigma_{12}$, which has the same size as σ_{11}.

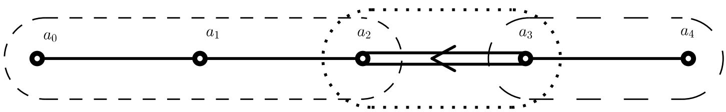
The rest of the proof is exactly like the $G = \widetilde{C}_n(q)$ case. We get a presentation with $8 + (5 - 3) = 10$ generators and $47 + (27 - 9) + 4 = 69$ relations if q is odd, and $9 + (6 - 3) = 12$ generators and $40 + (20 - 5) + 4 = 59$ relations if q is even. For $3 \leqslant n \leqslant 8$ we get shorter presentations, see Table 1.

For $n = 2$ we repeat the proof of the $G = \widetilde{C}_2(q)$ case, but now $\sigma_X \subseteq \sigma_{X_1(q)}$, so our presentation becomes:

$$\sigma'_G = \langle (D_1 \setminus D_X) \cup D_2 \mid (R_1 \setminus R_X) \cup R_2 \cup R_3^* \rangle.$$

This does not change the calculation, G still has a presentation with 7 generators and 49 relations if q is odd and 9 generators and 39 relations if q is even.

5.4. $\widetilde{F}_4^t(q)$.



The proof is line by line repetition of the case $G = \widetilde{F}_4(q)$. The outcome is the same: a presentation with 8 generators and 50 relations if q is odd and 9 generators and 43 relations if q is even.

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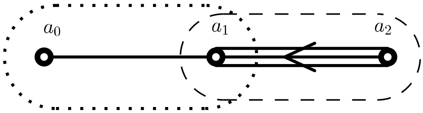
5.5. $\widetilde{G}_2^t(q)$.



The argument follows the proof in the case $G = \widetilde{G}_2(q)$. The outcome is the same: a presentation of G with 7 generators and 40 relations if q is odd and 7 generators and 32 relations if q is even.

6. Adjoint and classical groups

So far we have worked with presentations of a simply connected Kac–Moody group $X(q)$ defined over a finite field $\mathbb{F} = \mathbb{F}_q$. Our method can be used to derive a presentation of a Kac–Moody group that is not necessarily simply connected. In this section we deal with adjoint and classical groups. A reader can use our approach to derive a presentation of an arbitrary Kac–Moody group of his choosing.

There are two different meanings of the term *adjoint group*. For a group $X(q)$, its adjoint group is its image under the natural homomorphism $X(q) \to \mathrm{Aut}(X(q))$ given by the adjoint action: $X(q)_{ad} := X(q)/Z(X(q))$.

Besides the adjoint group $X(q)_{ad}$, there is also a group of points for an adjoint root datum. A convenient language to discuss this is a language of *group* $\mathbb{F}$-*functors*, the functors from the category of commutative $\mathbb{F}$-algebras to groups. A Kac–Moody group $G_{\mathfrak{D}}(\mathbb{F})$ with a root datum $\mathfrak{D} = (I, A, \mathcal{X}, \mathcal{Y}, \Pi, \Pi^{\vee})$ is the result of applying a group functor $G_{\mathfrak{D}}$ to the field $\mathbb{F}$. The Kac–Moody group $G_{\mathfrak{D}}(\mathbb{F})$ is called *adjoint* if Π is a basis of $\mathcal{X}$. We denote the adjoint Kac–Moody group $G_{\mathfrak{D}}(\mathbb{F})$ by $X_{ad}(q)$.

A homomorphism of the root data induces a homomorphism of the group $\mathbb{F}$-functors $\pi : X \to X_{ad}$. Taking points over $\mathbb{F}$ yields a group homomorphism $\pi(q) : X(q) \to X_{ad}(q)$. The kernel of $\pi(q)$ is the centre $Z(X(q))$ of $X(q)$. Hence, we have an exact sequence of groups

$$1 \to Z(X(q)) \to X(q) \xrightarrow{\pi(q)} X_{ad}(q)$$

and $X(q)_{ad}$ is observed in this sequence as the image of $\pi(q)$. For instance, if $X = A_{n-1}$, it is

$$1 \to \mu_n(\mathbb{F}_q) \to A_{n-1}(q) = \mathrm{SL}_n(q) \xrightarrow{\pi(q)} (A_{n-1})_{ad}(q) = \mathrm{PGL}_n(q)$$

where μ_n is the group scheme of the n-th roots of unity and $A_{n-1}(q)_{ad} = \mathrm{PSL}_n(q)$ is the image of $\pi(q)$ in $\mathrm{PGL}_n(q)$.

Another insightful example is $X = \widetilde{A}_{n-1}$. The key exact sequence is

$$1 \to \mu_n(\mathbb{F}_q) \times \mathbb{F}_q^\times \to \widetilde{A}_{n-1}(q) = \widetilde{\mathrm{SL}}_n(\mathbb{F}_q[t, t^{-1}]) \xrightarrow{\pi(q)} \\ \xrightarrow{\pi(q)} (\widetilde{A}_{n-1})_{ad}(q) = \mathbb{F}_q^\times \ltimes \mathrm{PGL}_n(\mathbb{F}_q[t, t^{-1}])$$

where the simply connected group $\widetilde{A}_{n-1}(q)$ is the Steinberg central extension of $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$ by $\mathbb{F}_q^\times$ and the adjoint group $(\widetilde{A}_{n-1})_{ad}(q)$ is the semidirect product where the action of $\mathbb{F}_q^\times$ is given by $\alpha \cdot \sum_k P_k t^k = \sum_k \alpha^k P_k t^k$.

Let $\mathcal{P}$ be the weight lattice, $\mathcal{Q}$ the root lattice of the corresponding Kac–Moody Lie algebra. The weight lattice $\mathcal{P}$ is the root lattice $\mathcal{X}$ for a simply connected root datum. *Mutatis mutandis*, $\mathcal{Q}$ for an adjoint root datum. The natural map $p : \mathcal{Q} \to \mathcal{P}$ is given by the Cartan matrix (or its transpose, depending on conventions). It is a part of an exact sequence

$$\mathcal{Q} \xrightarrow{p} \mathcal{P} \to \mathcal{Z} \to 1$$

where $\mathcal{Z} = \operatorname{coker} p$. The Cartan matrix pinpoints all the tori (of the corresponding Kac–Moody groups) of interest for us:

$$Z(q) = Z(X(q)) = \hom(\mathcal{Z}, \mathbb{F}_q^\times), \quad T(q) = \hom(\mathcal{P}, \mathbb{F}_q^\times), \\ T_{ad}(q) = \hom(\mathcal{Q}, \mathbb{F}_q^\times), \quad \pi(q)(\mathbf{x}) = \mathbf{x} \circ p.$$

Let us examine the corresponding (not exact) sequence of tori

$$1 \to Z(X(q)) \to T(q) \xrightarrow{\pi(q)} \overline{T}(q) \hookrightarrow T_{ad}(q)$$

where $\overline{T}(q) = T(q)/Z(X(q))$ can be thought of as a torus of $X(q)_{ad}$.

PROPOSITION 6.1. *Let $X(q)$ be a simply connected irreducible Kac–Moody group over a finite field $\mathbb{F} = \mathbb{F}_q$ (finite, affine or indefinite). Let $H(q) := \mathrm{Ext}^1(\mathcal{Z}, \mathbb{F}_q^\times)$ (in the category of abelian groups) in the finite or indefinite case and $H(q) := \mathrm{Ext}^1(\mathcal{Z}, \mathbb{F}_q^\times) \times \mathbb{F}_q^\times$ in the affine case. Then there exists a short exact sequence*

$$1 \to X(q)_{ad} \to X_{ad}(q) \to H(q) \to 1. \tag{6.1}$$

Proof. Let us assume that X is of finite or indefinite type. Then $p : \mathcal{Q} \to \mathcal{P}$ is injective and $\mathcal{Z}$ is finite. The long exact sequence in cohomology

$$1 \to \hom(\mathcal{Z}, \mathbb{F}_q^\times) \to \hom(\mathcal{P}, \mathbb{F}_q^\times) \xrightarrow{\pi(q)} \hom(\mathcal{Q}, \mathbb{F}_q^\times) \to \mathrm{Ext}^1(\mathcal{Z}, \mathbb{F}_q^\times) \to 1$$

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reduces to a short exact sequence connecting the adjoint tori

$$1 \to \overline{T}(q) \to T_{ad}(q) \to H(q) \to 1. \tag{6.2}$$

This implies the existence of the short exact sequence (6.1).

If X is affine, the map $p : \mathcal{Q} \to \mathcal{P}$ is no longer injective. We can decompose $\mathcal{Q} = \mathcal{Q}' \times \mathbb{Z}$ where $\mathbb{Z} = \ker p$ and $p : \mathcal{Q}' \to \mathcal{P}$ is injective. The long exact sequence in cohomology is

$$1 \to \hom(\mathcal{Z}, \mathbb{F}_q^\times) \to \hom(\mathcal{P}, \mathbb{F}_q^\times) \xrightarrow{\pi(q)} \hom(\mathcal{Q}', \mathbb{F}_q^\times) \to \mathrm{Ext}^1(\mathcal{Z}, \mathbb{F}_q^\times) \to 1.$$

It gives a description of the tori using an auxiliary group $T'(q) = \hom(\mathcal{Q}', \mathbb{F}_q^\times)$. The sequence

$$1 \to \overline{T}(q) \to T'(q) \to \mathrm{Ext}^1(\mathcal{Z}, \mathbb{F}_q^\times) \to 1$$

is exact. Since $T_{ad}(q) = T'(q) \times \mathbb{F}_q^\times$, a direct product with $\mathbb{F}_q^\times$ establishes exact sequence (6.2) in the affine case. This proves an existence of exact sequence (6.1) in all cases. □

Proposition 6.1 gives presentations of both $X(q)_{ad}$ and $X_{ad}(q)$. Since $X(q)_{ad} = X(q)/Z(q)$, one gets $X(q)_{ad}$ from $X(q)$ by 'killing' generators of $Z(q)$. The presentation of $X_{ad}(q)$ is obtained from presentations of $X(q)_{ad}$ and $H(q)$ by P. Hall's Lemma [5, Lemma 2.2]. Observe that the right conjugations in P. Hall's Lemma are superfluous. One usually adds them for convenience.

COROLLARY 6.2. *Suppose we have a presentation of $X(q)$, $Z(q)$ and $H(q)$:*

$$\sigma_{X(q)} = \langle D \mid R \rangle, \quad \sigma_{Z(q)} = \langle D_1 \mid R_1 \rangle, \quad \sigma_{H(q)} = \langle D_2 \mid R_2 \rangle.$$

Then we have presentations of adjoint groups

$$\sigma_{X(q)_{ad}} = \langle D \mid R \cup D_1^\sharp \rangle \quad \textit{and} \quad \sigma_{X_{ad}(q)} = \langle D \cup D_2 \mid R \cup D_1^\sharp \cup R_2^\sharp \cup D_2^{act} \rangle$$

where $D_1^\sharp = \{x^\sharp = 1 \mid x \in D_1,\ x^\sharp$ *is an expression of* x *in* $D\}$,

$R_2^\sharp = \{w = w^\sharp \mid w \in R_2,\ w^\sharp \in X(q)_{ad}$ *is an expression of* $w(D_2)$ *in* $D\}$ *and*

$D_2^{act} = \{xax^{-1} = {}^x a(D) \mid x \in D_2,$ *a is a generator of* $X(q)_{ad}$, ${}^x a(D)$ *is an expression of* xax^{-1} *in* $D\}$.

The group $\mathcal{P}/\mathcal{Q}$ is computed by calculating the integral Smith normal forms of Cartan matrices. We summarize these calculations in Table 4. We list Dynkin's labels for affine groups and Cartan labels for finite groups in the first column. The second column contains the group $\mathcal{P}/\mathcal{Q}$: by $(a_1, \ldots, a_k)$ we mean the group $\mathbb{Z}/a_1 \times \ldots \times \mathbb{Z}/a_k$.

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Table 4. Extra generators and relations for $\widetilde{X}(q)_{ad}$ and $\widetilde{X}_{ad}(q)$.

X_n	$\mathcal{P}/\mathcal{Q}$	$\lvert D_1\rvert = \lvert D_2\rvert$	max $\lvert D_1\rvert$	max $\lvert D_1^\sharp R_2^\sharp D_2^{act}\rvert$
A_n	$(n+1)$	$\mathfrak{A}(n+1)$	1	4
B_n, C_n, E_7	(2)	$\mathfrak{A}(2)$	1	4
D_{2n}	(2, 2)	$2\mathfrak{A}(2)$	2	8
D_{2n+1}	(4)	$\mathfrak{A}(2)$	1	4
G_2, F_4, E_8	()	0	0	0
E_6	(3)	$\mathfrak{A}(3)$	1	4
$\widetilde{A}_{n-1}$	$(0, n)$	$1 + \mathfrak{A}(n)$	2	8
$\widetilde{B}_n, \widetilde{C}_n, \widetilde{E}_7, \widetilde{B}_n^t, \widetilde{C}_n^t$	(0, 2)	$1 + \mathfrak{A}(2)$	2	8
$\widetilde{D}_{2n}$	(0, 2, 2)	$1 + 2\mathfrak{A}(2)$	3	12
$\widetilde{D}_{2n+1}$	(0, 4)	$1 + \mathfrak{A}(2)$	2	8
$\widetilde{G}_2, \widetilde{F}_4, \widetilde{E}_8, \widetilde{C}_n', \widetilde{F}_4^t, \widetilde{G}_2^t$	(0)	1	1	4
$\widetilde{E}_6$	(0, 3)	$1 + \mathfrak{A}(3)$	2	8

The next two columns are related to $X(q)_{ad}$. The third column lists the minimal number of generators for $Z(q)$. We get a generator, if there is a nontrivial homomorphism from a cyclic direct summand $\mathbb{Z}/k$ of $\mathcal{P}/\mathcal{Q}$ to $\mathbb{F}_q^\times$. We introduce a symbol $\mathfrak{A}(k)$, equal to 1, if $gcd(k, q-1) > 1$ and 0 if $gcd(k, q-1) = 1$. The fourth column is a maximal possible value of $\mathfrak{A}(k)$ taken over all q: this is the number of extra relations to describe $X(q)_{ad}$ for generic q.

The right three columns are related to $X_{ad}(q)$. The third column lists the minimal number of generators for $H(q)$. We get a generator if there is a nontrivial quotient by the k-th powers, where $\mathbb{Z}/k$ is a direct summand of $\mathcal{P}/\mathcal{Q}$: $\mathrm{Ext}^1(\mathbb{Z}/k, \mathbb{F}_q^\times) \cong \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^k$. This is controlled by the symbol $\mathfrak{A}(k)$. No generator arises from the infinite cyclic group: $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{F}_q^\times) = 0$, yet the infinite cyclic group appears only in the affine types where $H(q)$ has an extra generator. Hence, $|D_1| = |D_2|$. The fourth column uses a maximal possible value of $\mathfrak{A}(k)$: this is a number of extra generators needed to describe $X_{ad}(q)$ for generic q. The last column is the maximal cardinality of $D_1^\sharp R_2^\sharp D_2^{act} := D_1^\sharp \cup R_2^\sharp \cup D_2^{act}$, the number of extra relations needed to describe $X_{ad}(q)$. In our computation we use the estimates $|D_1^\sharp| = |D_1| = |D_2|$, $|R_2^\sharp| = |R_2| = |D_2|$ and $|D_2^{act}| = 2|D_2|$. The latter holds because $X(q)$ is generated by 2 elements (with few exceptions, see Theorem 2.7). Hence, $|D_1^\sharp \cup R_2^\sharp \cup D_2^{act}| = 4|D_1|$.

As an application of our techniques we write down the numbers of generators and relations of the remaining classical groups over $\mathbb{F}_q[t, t^{-1}]$ in Table 5

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Table 5. Generators and relations of classical $\mathbf{G}(\mathbb{F}_q[t,t^{-1}])$.

$\mathbf{G}$	D_σ	R_σ q odd	R_σ q even	$\mathbf{G}$	D_σ q odd	R_σ q odd	D_σ q even	R_σ q even
PSL_3	5	28	24	SO_7	9	47	8	36
PSL_n $(4 \leqslant n \leqslant 8)$	7	37	33	SO_9	9	56	9	45
PSL_n $(n \geqslant 9)$	9	45	41	SO_{2n+1} $(5 \leqslant n \leqslant 8)$	9	57	9	46
PGL_3	6	31	27	SO_{2n+1} $(n \geqslant 9)$	10	61	10	50
PGL_n $(4 \leqslant n \leqslant 8)$	8	40	36	SO_8 or SO_{2n} $(6 \leqslant n \leqslant 8)$	8	43	7	35
PGL_n $(n \geqslant 9)$	10	48	44	SO_{10}	8	44	7	36
				SO_{2n} $(n \geqslant 9)$	9	47	8	39

(for sufficiently large q). The groups SL_n, Spin_n and Sp_{2n} are simply connected, so they are already in Tables 1 and 2. The group

$$\mathrm{PSL}_n(\mathbb{F}_q[t,t^{-1}]) = \widetilde{A}_{n-1}(q)_{ad}$$

is adjoint, hence its presentation follows from Tables 1 and 4. The groups

$$\mathrm{PGL}_n(\mathbb{F}_q[t,t^{-1}]) \lhd (\widetilde{A}_{n-1})_{ad}(q), \qquad \mathrm{SO}_{2n+1}(\mathbb{F}_q[t,t^{-1}]) \lhd (\widetilde{B}_n)_{ad}(q)$$

are normal subgroups in the adjoint groups (before the semidirect product). Similar to Proposition 6.1 they appear in an exact sequence

$$1 \to X(q)_{ad} \to \mathbf{G}(\mathbb{F}_q[t,t^{-1}]) \to \mathrm{Ext}^1(\mathcal{Z}, \mathbb{F}_q^\times) \to 1,$$

hence they get a presentation as in Corollary 6.2 but with $\mathrm{Ext}^1(\mathcal{Z}, \mathbb{F}_q^\times)$ instead of $H(q)$.

Finally, SO_{2n} is not related to the adjoint group. It is an intermediate quotient fitting into exact sequence of group schemes

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}_{2n} \to \mathrm{SO}_{2n} \to 1.$$

Using our arguments, we fit the group into an exact sequence

$$1 \to \widetilde{D}_n(q)/Z \to \mathrm{SO}_{2n}(\mathbb{F}_q[t,t^{-1}]) \to \mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{F}_q^\times) \to 1$$

where the central subgroup Z is isomorphic to $\hom(\mathbb{Z}/2, \mathbb{F}_q^\times)$. We get a presentation as in Corollary 6.2 where the result depends on whether q is even or odd.

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