Can. J. Math., Vol. XXIII, No. 1, 1971, pp. 132–150

REPRESENTATION OF CERTAIN LINEAR OPERATORS IN HILBERT SPACE

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1. In this paper we represent certain linear operators in a space with indefinite metric. Such a space may be a pair (H, B), where H is a separable Hilbert space, B is a bilinear functional on H given by $B(x, y) = [Jx, y]$, $[\ ,\ ]$ is the Hilbert inner product in H, and J is a bounded linear operator such that $J = J^*$ and $J^2 = I$. If T is a linear operator in H, then $||T||$ is the usual operator norm. The operator J above has two eigenspaces corresponding to the eigenvalues $+1$ and -1.

In case the eigenspace in which J induces a positive operator has finite dimension k, a general spectral theory is known and has been developed principally by Pontrjagin [**25**], Iohvidov and Kreĭn [**13**], Naĭmark [**20**], and others. These spaces are called Pontrjagin, or Π_k-spaces. Operators A whose domain is H, with $A^* = JAJ$, called *J-self-adjoint*, have a k-dimensional non-negative invariant subspace. By a *non-negative subspace* we mean a subspace of Π_k in which $B(x, x) \geqq 0$. Also such As as above have at most k pairs of non-real eigenvalues symmetric about the real axis.

Operators U whose domain is H, with $U^* = JU^{-1}J$, called *J-unitary*, also have a k-dimensional non-negative invariant subspace, and at most k pairs of eigenvalues symmetric about the unit circle but not on it.

All this depends on the fact that for Π_k-spaces, k is the highest dimension of any non-negative subspace and is finite.

When both eigenspaces of J are finite-dimensional, one can find the complete theory of Jordan canonical forms for these operators in a book by Mal′cev [**19**].

When both eigenspaces are infinite-dimensional, the spectral theory is largely unknown. In this paper we investigate special As and Us in this third case. All results are for "cyclic" operators, i.e., operators for which certain integral powers applied to some vector in H generate a dense subspace of H.

The methods we employ include those of Livšic [**18**] and Kalisch [**14**], using the concept of "Characteristic Function" and the theory of complex variables in the unit disc and the upper half plane [**6**; **22**; **26**].

Basically, we characterize four concrete operators abstractly. These operators are given by:

(1) $$Mf(t) = tf(t),$$

(2) $$Nf(t) = e^{it}f(t),$$

(3) $aV + r$, where $Vf(t) = \int_0^t f(s)\, ds$ and a and r are real numbers,

Received April 29, 1970 and in revised form, September 30, 1970.

(4) $$Uf(t) = e^{ib}f(t) - 2iae^{ib}\int_0^t e^{-ia(t-s)}\,ds,$$

all occurring in suitable L^2-spaces. This last is a Cayley transform of V. In the present work we exhibit operators J for which the above concrete operators are J-self-adjoint or J-unitary. Also we characterize abstractly operators similar to one of the above models in such a way that the similarity preserves the indefinite metric.

2. In this section we make a few preliminary definitions. We usually denote $B(x, y)$ by (x, y) and reserve the notation $[x, y]$ for the Hilbert inner product. With this notation, we define a *J-self-adjoint operator* to be an everywhere defined linear transformation A such that $(Ax, y) = (x, Ay)$ for all x and y in H, where (H, B) is a J-space. Also a *J-unitary operator* in a J-space (H, B) is an everywhere defined linear transformation U of H onto H such that $(Ux, Uy) = (x, y)$ for all x and y in H. We remark that J-unitary operators have a spectrum that is symmetric about the unit circle, i.e., if z is in the spectrum of a J-unitary operator U, then $\bar{z}^{-1}$ is also in the spectrum of U. This is due to the equation $((U - z)^{-1})^* = (JU^{-1}J - \bar{z})^{-1} = J(U^{-1} - \bar{z})^{-1}J$. A similar equation shows that z is in the spectrum of a J-self-adjoint operator A if and only if $\bar{z}$ is also in the spectrum of A. A space with indefinite metric will hereafter be referred to as a J-space. A subspace of a J-space is called *positive* if $(x, x) \geqq 0$ for all x in this subspace. Two elements x and y of a J-space are called *J-orthogonal* if $(x, y) = 0$ and *two subspaces are J-orthogonal* if each element of one subspace is J-orthogonal to every element of the other subspace. A subspace of a J-space H is *non-degenerate* if none of its members other than 0 is J-orthogonal to it. Note that any J-space is non-degenerate in itself, since $(x, y) = 0$ for all y in H implies that $[Jx, y] = 0$ for all y, and so $Jx = 0$ and $x = 0$. If M and N are linearly independent subspaces of a J-space H that are J-orthogonal, then we write $M \oplus N$ for the algebraic direct sum of M and N. By a *negative subspace* we mean a subspace such that $(x, x) \leqq 0$ for all x in the subspace. Let H_j $(j = 1, 2)$ be J-spaces and let the indefinite metric in H_j be given by $(\ ,\ )_j$. Then a *J-isomorphism* between H_1 and H_2 is an everywhere defined linear transformation S of H_1 onto H_2 such that $(Sx, Sy)_2 = (x, y)_1$ for all x and y in H_1. We say that *two families* T_1 and T_2 of operators in J-spaces H_1 and H_2, respectively, are *J-isomorphic* if there is a J-isomorphism S of H_1 onto H_2 such that $SW_1 = W_2S$ for all W_j in T_j $(j = 1, 2)$.

3. We now prove some preliminary facts about J-isomorphisms and J-isomorphic families of operators.

PROPOSITION 1. *Let S be a J-isomorphism of a J-space H onto a J-space K. Then S is invertible and both S and S^{-1} are continuous. In particular, J-unitary operators are bounded.*

Proof. By a well-known theorem of S. Banach, a closed operator which is defined on a complete metric space and whose range is in a complete metric space is continuous. Therefore it suffices to prove that S is closed.

Let x_n be in H for $n = 1, 2, \ldots$ and $x_n \to x$, $Sx_n \to y$ as $n \to \infty$. Now x is in H and y is in K. Let z belong to K. Then, as $n \to \infty$, $(Sx_n, z) \to (y, z)$; $(Sx_n, z) = (x_n, S^{-1}z) \to (x, S^{-1}z) = (Sx, z)$. Thus $(y, z) = (Sx, z)$, $(y - Sx, z) = 0$, i.e., the vector $y - Sx$ is J-orthogonal to all K. Since K is non-degenerate, $y = Sx$.

LEMMA 1. *Let $\{T_j: j = 1, 2, \ldots, n\}$ be a commuting family of bounded J-self-adjoint operators (J-unitary operators) in a J-space H. Assume that*

(a) *The set $\{T_j^p v: j = 1, 2, \ldots, n, p = 0, 1, 2, \ldots\}$ generates the space H for some v in H,*

(b) *There is a set of commuting bounded J-self-adjoint (J-unitary) operators $\{M_j: j = 1, 2, \ldots, n\}$ in a J-space K such that the set $\{M_j^p e: j = 1, 2, \ldots, n, p = 0, 1, 2, \ldots\}$ generates the space K for some e in K,*

(c) $(\prod_{j=1}^n (T_j - z_j)^{-1}v, v) = (\prod_{j=1}^n (M_j - z_j)^{-1}e, e)$ *is a function analytic near infinity in all variables,*

(d) *If $q_k(x_1, x_2, \ldots, x_n)$ is a polynomial in n variables for each positive integer k, then $\|q_k(T_1, T_2, \ldots, T_n)v\| \to 0$ if and only if $\|q_k(M_1, M_2, \ldots, M_n)e\| \to 0$ as k approaches infinity.*

Then we conclude that the families $\{T_j\}$ and $\{M_j\}$ are J-isomorphic.

Proof. Consider the Neumann expansion about infinity of both sides of the equation in condition (c) above. We equate the coefficients of the like powers and use the J-self-adjoint (J-unitary) character of the T_j and M_j to obtain:

$$(1)\quad (T_{j(1)}^{m(1)} \ldots T_{j(r)}^{m(r)}v, T_{i(1)}^{n(1)} \ldots T_{i(s)}^{n(s)}v) = (M_{j(1)}^{m(1)} \ldots M_{j(r)}^{m(r)}e, M_{i(1)}^{n(1)} \ldots M_{i(s)}^{n(s)}e).$$

This equation holds for all positive integers r and s between 1 and n, all sets $(m(1), m(2), \ldots, m(r))$ and $(n(1), n(2), \ldots, n(s))$ of non-negative integers, and all subsets $(i(1), i(2), \ldots, i(s))$ and $(j(1), j(2), \ldots, j(r))$ of the set of integers from 1 to n.

Define $S(\Sigma) = S(\Sigma c(m_1, \ldots, m_r)T_{j(1)}^{m(1)} \ldots T_{j(r)}^{m(r)}v)$ to be equal to the element $\Sigma c(m_1, \ldots, m_r)M_{j(1)}^{m(1)} \ldots M_{j(r)}^{m(r)}e$. The map S is well-defined, since $\Sigma = 0$ implies that $(\Sigma, \Sigma') = 0$ for any linear combination $\Sigma' = \Sigma d(n_1, \ldots, n_s)T_{i(1)}^{n(1)} \ldots T_{i(s)}^{n(s)}v$. By expanding the inner product and using (1) above we see that

$$(S(\Sigma), \Sigma d(n_1, \ldots, n_s)M_{i(1)}^{n(1)} \ldots M_{i(s)}^{n(s)}e) = 0$$

and so $(S(\Sigma), \Sigma'') = 0$ for all linear combinations of the type $\Sigma'' = \Sigma d(n_1, \ldots, n_s)M_{i(1)}^{n(1)} \ldots M_{i(s)}^{n(s)}e$. As these linear combinations are dense in the J-space K, then $S(\Sigma) = 0$. A similar calculation shows that S is one-to-one. Condition (d) of the hypothesis ensures that S and S^{-1} are continuous as linear transformations of a dense subspace of H onto a dense subspace of K. Therefore we extend S to a J-isomorphism of H onto K.

The equations $ST_j = M_jS$ for $j = 1, 2, \ldots, n$ result from the equations

$$ST_j\Sigma c(m_1, \ldots, m_r)T_{j(1)}^{m(1)} \ldots T_{j(r)}^{m(r)}v = M_jS\Sigma c(m_1, \ldots, m_r)T_{j(1)}^{m(1)} \ldots T_{j(r)}^{m(r)}v,$$

and from the fact that the linear combinations above are dense in H.

4. In this section we prove some representation theorems about J-self-adjoint operators.

First, we give the following definition. A bounded *J-self-adjoint operator A* in a J-space H will be called *regular* if $H = P \oplus N$, where P is a positive closed subspace of H and N is a negative closed subspace of H and P is invariant under A.

PROPOSITION 2. *A regular J-self-adjoint operator A in a J-space H has real spectrum. Also, $((A - z)^{-1}v, v) = \int (t - z)^{-1}\, dm(v{:}t)$ for each v in H, where m is a finite real-valued signed measure on the real line.*

We call $m(v{:}t)$ *the measure associated with A and v.*

Proof. This proposition is a trivial consequence of the spectral resolution of self-adjoint operators.

THEOREM 1. *Let A be a regular J-self-adjoint operator in a J-space H such that $\{A^nv\colon n = 0, 1, 2, \ldots\}$ generates the space H for some v in H. Assume that $\|p_k(A)v\| \to 0$ if and only if $\int |p_k(t)|^2\, d|m|$ converges to 0 as k approaches infinity, where $\{p_k(t)\}$ is a sequence of polynomials in t and m is the measure associated with A and v. Then A is J-isomorphic with the J-self-adjoint operator M in the space $L^2(|m|)$, where $Mf(t) = tf(t)$ and $Jf(t) = X(t)f(t)$ with $X = dm/d|m|$.*

Proof. By Proposition 2 we have $((A - z)^{-1}v, v) = \int (t - z)^{-1}\, dm(t)$, where m is a real-valued finite signed measure. Consider the space $L^2(|m|)$ with indefinite metric $(f, g) = \int f\bar{g}\, dm$ and Hilbert inner product $[f, g] = \int f\bar{g}\, d|m|$. Let $Jf(t) = X(t)f(t)$, where $X = dm/d|m|$. Since m is real, $X(t) = \pm 1$, and so $J^2 = I$ and $J = J^*$. Also $[Jf, g] = (f, g)$. Thus $L^2(|m|)$ is a J-space. A calculation shows that the operator M given by $Mf(t) = tf(t)$ is J-self-adjoint and that $((M - z)^{-1}e, e)$ is equal to $((A - z)^{-1}v, v)$, where e is the identity function in $L^2(|m|)$.

At this point all the conditions of Lemma 1 are satisfied except condition (d). For the operators A and M of this theorem, that condition is that $\int |p_k(t)|^2\, d|m|(t) \to 0$ if and only if $\|p_k(A)v\| \to 0$. The integral here is equal to $\|p_k(M)e\|^2$. We assumed this last condition. Thus by Lemma 1, A and M are J-isomorphic.

PROPOSITION 3. *Let A be a bounded J-self-adjoint operator in a J-space H with spectrum on the unit circle such that $\|p(A)\| \leqq c\|p(A)\|_{\mathrm{sp}}$, where $\|p(A)\|_{\mathrm{sp}}$ is the spectral norm of $p(A)$, p is any trigonometric polynomial, and c is a real constant* *independent of* p. *Then*

$$((A - z)^{-1}x, y) = \int_0^{2\pi} (e^{it} - z)^{-1} dm(t{:}x, y)$$

for all x *and* y *in* H *where* $m(t{:}x, y)$ *is a finite function of bounded variation on* $[0, 2\pi]$. *Also* m *is unique if properly normalized.*

Proof. By [**30**], since A is power bounded (take $p(t) = t^n$ above to obtain $||A^n|| \leqq c$ for $n = 0, 1, 2, \ldots$), there is a self-adjoint bounded invertible linear operator Q such that $A = Q^{-1}TQ$, where T is a unitary operator in H. From the spectral resolution of T we know that

$$((T - z)^{-1}x, y) = \int_0^{2\pi} (e^{it} - z)^{-1} d\hat{m}(t{:}x, y)$$

for all x and y in H, where $\hat{m}(t{:}x, y)$ is a finite function of bounded variation on $[0, 2\pi]$. Also, if $\hat{m}$ is normalized so that $\hat{m}(0) = 0$, and $\hat{m}(t) = \hat{m}(t + 0)$ for $0 \leqq t \leqq 2\pi$, then $\hat{m}$ is unique. Now, for any x and y in H we have

$$\begin{aligned} ((A - z)^{-1}x, y) &= (Q(T - z)^{-1}Q^{-1}x, y) \\ &= ((T - z)^{-1}Q^{-1}x, Q'y) \\ &= \int_0^{2\pi} (e^{it} - z)^{-1} d\hat{m}(t{:}Q^{-1}x, Q'y) \\ &= \int_0^{2\pi} (e^{it} - z)^{-1} dm(t{:}x, y), \end{aligned}$$

where $m(t{:}x, y) = \hat{m}(t{:}Q^{-1}x, Q'y)$. Here $Q' = JQ^*J$.

In the following, a J-self-adjoint operator A will be called *J-complex-self-adjoint* if its spectrum lies on the unit circle and if $||p(A)|| \leqq c||p(A)||_{sp}$ for all trigonometric polynomials p, where c is a real constant independent of p. The measure m of the above proposition will be called the *measure associated with* A, x, *and* y.

PROPOSITION 4. *Let*

$$f(z) = \int_0^{2\pi} (e^{it} + z)(e^{it} - z)^{-1} dm(t),$$

where m *is a real-valued function of bounded variation in* $[0, 2\pi]$. *Assume that the equation* $f(\bar{z}) = \overline{f(z)}$ *holds. Then, for all integers* n, *we have*

$$\int_0^{2\pi} e^{int} d|m|(t) = \int_0^{2\pi} e^{-int} d|m|(t),$$

where $|m|$ *is the total variation measure of* m.

Proof. From [**22**], we know that

$$m(t) = \lim_{r \to 1} \int_0^t u(re^{is})\, ds,$$

where u is the real part of f. From [**6**] we also know that

$$|m|(t) = \lim_{r\to 1} \int_0^t |u(re^{is})|\, ds.$$

Both of these formulas are valid almost everywhere. Now $f(\bar{z}) = \overline{f(z)}$ implies that $u(\bar{z}) = u(z)$.

Choose a sequence of numbers $r_q \uparrow 1$. Let

$$|m_q|(t) = \int_0^t |u(r_q e^{is})|\, ds.$$

Then $|m_q|(0) = 0$ and there is an M such that

$$\int_0^{2\pi} d|m_q| \leqq M$$

for all q, since

$$\int_0^{2\pi} |u(re^{is})|\, ds \leqq M$$

for some constant $m > 0$ and for $0 < r < 1$. Each $|m_q|$ is of bounded variation on $[0, 2\pi]$. Since for each q, the function $|u(r_q e^{is})|$ is continuous, then $d|m_q|(t) = |u(r_q e^{it})|\, dt$.

Now

$$\int_0^{2\pi} e^{int}\, d|m_q|(t) = \int_0^{2\pi} e^{int} |u(r_q e^{it})|\, dt.$$

The change of variables from t to $2\pi - t$ implies that

$$\int_0^{2\pi} e^{int}\, d|m_q|(t) = \int_0^{2\pi} e^{-int} |u(r_q e^{-it})|\, dt$$

and this equals

$$\int_0^{2\pi} e^{-int} |u(r_q e^{it})|\, dt$$

since $u(\bar{z}) = u(z)$. Therefore we have

$$\int_0^{2\pi} e^{int}\, d|m_q|(t) = \int_0^{2\pi} e^{-int}\, d|m_q|(t)$$

for each q. Now we apply the Helly theorem [**26**] to each side. There is a subsequence (q_j) of the sequence (q) such that $|m_{q_j}|(t) \to |m|(t)$ as $j \to \infty$ for each t in the closed interval $[0, 2\pi]$ and

$$\int_0^{2\pi} g(t)\, d|m_{q_j}|(t) \to \int_0^{2\pi} g(t)\, d|m|(t)$$

for all continuous functions g. The result now follows.

THEOREM 2. *Let A be a bounded J-complex-self-adjoint operator in a J-space H*

such that the set $\{A^n v\colon n = 0, \pm 1, \pm 2, \ldots\}$ *generates* H *for some* v *in* H. *Assume that* $\|p_k(A)v\| \to 0$ *if and only if*

$$\int_0^{2\pi} |p_k(e^{it})|^2 \, d|m|(t) \to 0,$$

where m *is the measure associated with* A, v, *and* v, *and each* $p_n(t)$ *is a polynomial in* t *and* t^{-1}. *Then* A *is* J*-isomorphic with* J*-self-adjoint operator* M *acting in the space* $L^2(|m|)$, *where* $Mf(t) = e^{it}f(t)$ *and* $Jf(t) = X(t)f(2\pi - t)$ *with* $X = dm/d|m|$.

Proof. We have

$$((A - z)^{-1}v, v) = \int_0^{2\pi} (e^{it} - z)^{-1} \, dm(t)$$

since A is J-complex-self-adjoint by Proposition 3. Also m is a real-valued function of bounded variation that gives rise to a real finite signed Borel measure in $[0, 2\pi]$ again denoted by m. From the Neumann expansions of each side of this equation about 0 and ∞ and the identity principle of complex variables, we have that for all integers n, the equation

$$(A^n v, v) = \int_0^{2\pi} e^{int} \, dm(t)$$

holds. Since A is J-self-adjoint, we have $(A^n v, v) = (v, A^n v) = \overline{(A^n v, v)}$ and this is equal to

$$\int_0^{2\pi} e^{-int} \, dm(t).$$

Therefore

$$\int_0^{2\pi} e^{int} \, dm(t) = \int_0^{2\pi} e^{-int} \, dm(t)$$

for all integers n. Now consider $L^2(|m|)$ and $Mf(t) = e^{it}f(t)$. Let $Jf(t) = X(t)f(2\pi - t)$ where $X = dm/d|m|$. Let the Hilbert inner product be given by

$$[f, g] = \int_0^{2\pi} f(t)\overline{g(t)} \, d|m|(t).$$

Then

$$[Jf, g] = \int_0^{2\pi} f(2\pi - t)\overline{g(t)} \, dm(t)$$

and

$$[f, Jg] = \int_0^{2\pi} f(t)\overline{g(2\pi - t)} \, dm(t).$$

Let $f(t) = e^{ikt}$ and $g(t) = e^{iqt}$ where k and q are integers. Then $[Jf, g] = [f, Jg]$. Since the set $\{e^{int}\colon n = 0, \pm 1, \pm 2, \ldots\}$ generates $L^2(|m|)$ [**28**], $[Jf, g] = [f, Jg]$ for all pairs f and g in $L^2(|m|)$. Therefore $J = J^*$.

To show that $J^2 = I$, it suffices to show that $[Jf, Jg] = [f, g]$ for all f and g in $L^2(|m|)$. Again we need only show it when $f(t) = e^{ikt}$ and $g(t) = e^{iqt}$ where k and q are integers. We have

$$[Jf, Jg] = \int_0^{2\pi} e^{-i(k-q)t}\, d|m|(t) = \int_0^{2\pi} e^{i(k-q)t}\, d|m|(t) = [f, g]$$

by Proposition 4. This now implies that $X(2\pi - t) = X(t)$ and that $L^2(|m|)$ with the indefinite inner product $(f, g) = [Jf, g]$ is a J-space.

A calculation shows that M is J-self-adjoint and that

$$((M - z)^{-1}e, e) = \int_0^{2\pi} (e^{it} - z)^{-1}\, dm(t) = ((A - z)^{-1}v, v),$$

where e is the identity function in $L^2(|m|)$. Also, $\{M^n e\colon n = 0, \pm 1, \pm 2, \ldots\}$ generates the space $L^2(|m|)$. By hypothesis, $||p_k(A)v|| \to 0$ if and only if

$$\int_0^{2\pi} |p_k(e^{it})|^2\, d|m|(t) = ||p_k(M)e||^2 \to 0.$$

Therefore by Lemma 1, the sets $\{M, M^{-1}\}$ and $\{A, A^{-1}\}$ are J-isomorphic and the theorem follows.

In what follows we shall call a bounded J-self-adjoint operator in a J-space H a *J-Volterra-self-adjoint operator* if it has a one-point spectrum r and if $A + A^* - 2rI$ has one-dimensional range spanned by a vector v with $Jv = v$ and $||v|| = 1$. (I is the identity operator in H.) We will refer to the operator A as a *J-Volterra-self-adjoint operator with spectrum r and vector v.*

THEOREM 3. *Let A be a J-Volterra-self-adjoint operator with spectrum r and vector v in a J-space H such that the set $\{A^n v\colon n = 0, 1, 2, \ldots\}$ generates the space H. Then A is J-isomorphic with the operator $aV + rI$, operating in the Lebesgue measure space $L^2(0, 1)$, where*

$$Vf(t) = \int_0^t f(s)\, ds,$$

and $Jf(t) = f(1 - t)$. The real number a is given by $(A + A^*)v = (a + 2r)v$.

Proof. Let $B = A - rI$. Then B has a one-point spectrum 0, is a bounded J-self-adjoint operator, and $B + B^*$ has one-dimensional range spanned by v. Also $(B + B^*)v = av$. Let $F(z) = 1 - a((B - z^{-1})^{-1}v, v)$. Since $(u, v) = [u, Jv]$ for all u and v in H and $Jv = v$, then $F(z) = 1 - a[(B - z^{-1})^{-1}v, v]$. We now refer the reader to [**14**]. From [**14**, the proof of Theorem 8], $F(z) = e^{az}$. Furthermore,

$$(z_1 + z_2) \sum_{p,q=0} z_1{}^p z_2{}^q [B^p v, B^q v] = (1/a)(F(z_1)\overline{F(z_2)} - 1),$$

where z_1 and z_2 are complex variables in the finite plane. Here F is analytic in the finite complex plane.

Consider aV and J as above in $L^2(0, 1)$. The Hilbert inner product is given by

$$[f, g] = \int_0^1 f(t)\overline{g(t)}\,dt.$$

Thus

$$[Jf, g] = \int_0^1 f(1 - t)\overline{g(t)}\,dt.$$

By a change of variables t to $1 - t$, the last integral becomes

$$\int_0^1 f(t)\overline{g(1 - t)}\,dt$$

which is $[f, Jg]$. Therefore $J = J^*$. Also $J^2 = I$. Thus $L^2(0, 1)$ is a J-space with indefinite metric $(f, g) = [Jf, g]$. Let e be the identity function in $L^2(0, 1)$. Then since aV satisfies all the conditions that B does, we have

$$((aV - z^{-1})^{-1}e, e) = F(z).$$

And also we have

$$(z_1 + z_2) \sum_{p,q=0} z_1{}^p z_2{}^q[(aV)^p e, (aV)^q e] = (z_1 + z_2) \sum_{p,q=0} z_1{}^p z_2{}^q [B^p v, B^q v].$$

By the identity principle of complex variables, $[(aV)^p e, (aV)^q e] = [B^p v, B^q v]$ for all non-negative integers p and q. Therefore $\|p_k(B)v\| \to 0$ if and only if $\|p_k(aV)e\| \to 0$ where $\{p_k\}$ is a sequence of polynomials. Also the set $\{(aV)^n e\colon n = 0, 1, 2, \ldots\}$ generates the space $L^2(0, 1)$ and the set $\{B^n v\colon n = 0, 1, 2, \ldots\}$ generates the space H. A calculation shows that aV is J-self-adjoint.

Therefore by Lemma 1, B and aV are J-isomorphic, and so are A and $aV + rI$.

Note that the above isomorphism is also a Hilbert isomorphism, i.e. the two operators are unitarily equivalent.

5. In this section we prove some theorems about J-unitary operators.

First, we give the following definition. A *J-unitary operator* U in a J-space H will be called *regular* if $H = P \oplus N$, where P is a positive closed subspace of H, N is a negative closed subspace of H, and P is invariant under U and U^{-1}.

PROPOSITION 5. *A regular J-unitary operator U in a J-space H has spectrum on the unit circle. Also*

$$((U - z)^{-1}v, v) = \int_0^{2\pi} (e^{it} - z)^{-1}\,dm(v{:}t)$$

for each v in H, where $m(v{:}t)$ is a real-valued finite signed measure on the interval $[0, 2\pi]$.

We call $m(v{:}t)$ *the measure associated with U and v.*

Proof. This proposition is a trivial consequence of the spectral resolution of unitary operators.

We remark that regular J-unitary operators are the stable operators in the terminology of Kreĭn [**16**], and were first considered by Phillips [**24**] in 1961.

THEOREM 4. *Let U be a regular J-unitary operator in a J-space H such that $\{U^n v: n = 0, \pm 1, \pm 2, \ldots\}$ generates H for some v in H. Assume that $||p_k(U)v|| \to 0$ if and only if*

$$\int_0^{2\pi} |p_k(e^{it})|^2 \, d|m(v:t)| \to 0 \qquad \text{as } k \to \infty,$$

where $\{p_k(t)\}$ is a sequence of polynomials in t and t^{-1} and $m(v:t)$ is the measure associated with U and v. Then U is J-isomorphic with the J-unitary operator M in the space $L^2(|m|)$, where $Mf(t) = e^{it}f(t)$ and $Jf(t) = X(t)f(t)$ with $X = dm/d|m|$.

Proof. By Proposition 5, *we have*

$$((U - z)^{-1}v, v) = \int_0^{2\pi} (e^{it} - z)^{-1} \, dm(v:t),$$

where m is a real signed Borel measure in $[0, 2\pi]$.

Consider the space $L^2(|m|)$ with indefinite metric

$$(f, g) = \int_0^{2\pi} f\bar{g} \, dm$$

and Hilbert inner product

$$[f, g] = \int_0^{2\pi} f\bar{g} \, d|m|.$$

Let $Jf(t) = X(t)f(t)$, where $X = dm/d|m|$. Since m is real, $X(t) = \pm 1$ and so $J^2 = I$ and $J = J^*$. Also $[Jf, g] = (f, g)$. Thus $L^2(|m|)$ is a J-space. A calculation shows that the operator M given by $Mf(t) = e^{it}f(t)$ is J-unitary and $((M - z)^{-1}e, e) = ((U - z)^{-1}v, v)$, where e is the identity function in $L^2(|m|)$. Also the set $\{M^n e: n = 0, \pm 1, \pm 2, \ldots\}$ generates this L^2-space. Since

$$||p_k(M)e||^2 = \int_0^{2\pi} |p_k(e^{it})|^2 \, d|m|(t),$$

then our last assumption implies that $||p_k(U)v|| \to 0$ if and only if $||p_k(M)e|| \to 0$. Thus by Lemma 1, the families $\{U, U^{-1}\}$ and $\{M, M^{-1}\}$ are J-isomorphic.

PROPOSITION 6. *Let U be a J-unitary operator in a J-space H with positive real spectrum and such that for some real c we have $||p(U)|| \leqq c||p(U)||_{\mathrm{sp}}$ for all polynomials p, where $||p(U)||_{\mathrm{sp}}$ is the spectral norm of $p(U)$. Then*

$$((U - z)^{-1}x, y) = \int (t - z)^{-1} \, dm(t:x, y)$$

for each pair (x, y) of members of H where $m(t:x, y)$ is a real finite function of bounded variation in some interval $[1/a, a]$ $(a > 1)$ containing the spectrum of U. Also m is unique if properly normalized.

Proof. Let $a > 1$ be chosen so that $[1/a, a]$ contains the spectrum of U. We may do this since the spectrum of U is symmetric about the unit circle. Let P be the set of all polynomials on $[1/a, a]$ and C the continuous functions on the same interval. Then P is dense in C in the sup norm topology where the norm is given by the formula $\|f\|_\infty = \operatorname{Sup}\{|f(t)|: 1/a \leqq t \leqq a\}$ for f in C. Now we consider for each pair (x, y) of members of H the linear functional on P given by $L(p) = (p(U)x, y)$. We have the inequality

$$|L(p)| = |(p(U)x, y)| \leqq \|p(U)\| \, \|x\| \, \|y\|$$

and thus we have $|L(p)| \leqq c\|p(U)\|_{\mathrm{sp}} \|x\| \, \|y\|$. Since the spectrum of U is contained in $[1/a, a]$, we have $\|p(U)\|_{\mathrm{sp}} \leqq \|p\|_\infty$ [**31**].

Therefore $|L(p)| \leqq c\|x\| \, \|y\| \, \|p\|_\infty$ and L is bounded. Thus there is a complex-valued function of bounded variation $V(t{:}x, y)$ such that

$$(p(U)x, y) = \int_{1/a}^{a} p(t) \, dV(t{:}x, y)$$

for all p in P. Since Riesz's theorem is valid for C, we have

$$(f(U)x, y) = \int_{1/a}^{a} f(t) \, dV(t{:}x, y)$$

for all f in C. In particular,

$$(U^n x, y) = \int_{1/a}^{a} t^n \, dV(t{:}x, y)$$

for all integers n. If $|z| < 1/a$, then

$$(t - z)^{-1} = \sum_{n=0}^{\infty} t^{-(n+1)} z^n$$

and if $|z| > a$, then

$$(t - z)^{-1} = -\sum_{n=0}^{\infty} t^n z^{-n-1}.$$

Therefore we have

$$((U - z)^{-1}x, y) = \left(-\sum_{n=0}^{\infty} U^n z^{-n-1} x, y\right) = -\sum_{n=0}^{\infty} z^{-n-1} (U^n x, y).$$

The last expression above is equal to

$$-\sum_{n=0}^{\infty} z^{-n-1} \int_{1/a}^{a} t^n \, dV(t{:}x, y) = -\int_{1/a}^{a} \left(\sum_{n=0}^{\infty} t^n z^{-n-1}\right) dV(t{:}x, y)$$
$$= \int_{1/a}^{a} (t - z)^{-1} \, dV(t{:}x, y),$$

whenever $|z| > a$. As the singular points of $((U - z)^{-1}x, y)$ occur in the interval $[1/a, a]$, by analytic continuation we have

$$F(z) = ((U - z)^{-1}x, y) = \int_{1/a}^{a} (t - z)^{-1} \, dV(t{:}x, y)$$

for z not in $[1/a, a]$. Let $V(t{:}x, y) = V'(t) + iV''(t)$, where V' and V'' are real-valued functions of bounded variation. Then $((U - z)^{-1}x, y) = F'(z) + iF''(z)$, where

$$F'(z) = \int_{1/a}^{a} (t - z)^{-1}\, dV'(t) \quad \text{and} \quad F''(z) = \int_{1/a}^{a} (t - z)^{-1}\, dV''(t).$$

By [**26**],

$$\int_{-\infty}^{\infty} |F'(r + is)|\, dr \quad \text{and} \quad \int_{-\infty}^{\infty} |F''(r + is)|\, dr$$

are both uniformly bounded for $s > 0$. Therefore

$$\int_{-\infty}^{\infty} |F(r + is)|\, dr$$

is uniformly bounded for $s > 0$ where $F(z) = ((U - z)^{-1}x, y)$. Hence there is a real-valued function of bounded variation $m(t{:}x, y)$ with support in $[1/a, a]$ such that

$$((U - z)^{-1}x, y) = \int_{1/a}^{a} (t - z)^{-1}\, dm(t{:}x, y).$$

Moreover,

$$m(t{:}x, y) = \lim_{s \to 0} \int_{1/a}^{t} j(r + is)\, ds$$

except on an at most countable point set where $j(z)$ is the imaginary part of $F(z)$.

PROPOSITION 7. *Let*

$$F(z) = \int_{1/a}^{a} (t - z)^{-1}\, dm(t),$$

where $a > 1$ and m is a real-valued function of bounded variation with support on the interval $[1/a, a]$ and such that $m(t) + m(1/t)$ is constant there. Then

$$\int_{1/a}^{a} t^{k}\, d|m|(t) = \int_{1/a}^{a} t^{-k}\, d|m|(t)$$

for all integers k where $|m|$ is the total variation measure of m.

Proof. We can consider the above function $F(z)$ to be equal to

$$\int_{-\infty}^{\infty} (t - z)^{-1}\, dm(t)$$

if m is defined to be constant off $[1/a, a]$. From the general theory, the inversion formulas for m and $|m|$ are

$$m(t) = \lim_{s \to 0} \int_{1/a}^{t} j(r + is)\, dr \quad \text{and} \quad |m|(t) = \lim_{s \to 0} \int_{1/a}^{t} |j(r + is)|\, dr,$$

respectively, where $j(z)$ is the imaginary part of $F(z)$. Also we have

$$\lim_{s\to 0} \int_{1/a}^{a} |j(r+is) - m'(r)|\, dr = 0$$

[**27**; **26**]. (Here m' means the derivative of m.) Thus

$$\lim_{s\to 0} \int_{1/a}^{a} ||j(r+is)| - |m'(r)||\, dr = 0.$$

By a change of variables r to $1/r$ we obtain

$$\lim_{s\to 0} \int_{1/a}^{a} ||r^{-2}j(r^{-1}+is)| - |r^{-2}m'(r^{-1})||\, dr = 0.$$

Let $\{s_n\}$ be a strictly monotonically decreasing sequence of real numbers that converges to 0. Let

$$g_n(r) = |r^{-2}j(r^{-1}+is_n)| - |j(r+is_n)|.$$

Then by the triangle inequality, $|g_n(r)|$ is less than or equal to the sum of

$$||r^{-2}j(r^{-1}+is_n)| - |r^{-2}m'(r^{-1})||$$

and

$$||j(r+is_n)| - |m'(r)|| + ||r^{-2}m'(r^{-1})| - |m'(r)||.$$

Let $m(t) + m(1/t) = c$, where c is a constant. Now m' exists except on a countable set, and so $dm/dr + (dm/du)(du/dr) = 0$ wherever m' exists if $u = 1/r$. Therefore $m'(r) = r^{-2}m'(r^{-1})$ almost everywhere and so

$$\int_{1/a}^{a} ||r^{-2}m'(r^{-1})| - |m'(r)||\, dr = 0.$$

Thus

$$\lim_{n\to\infty} \int_{1/a}^{a} |g_n(r)|\, dr = 0.$$

Now define

$$|m_n|(t) = \int_{1/a}^{t} |j(r+is_n)|\, dr$$

in the interval $[1/a, a]$. Then $|m_n|(1/a) = 0$ for all n, and

$$\int_{1/a}^{a} d|m_n| = \int_{1/a}^{a} |j(r+is_n)|\, dr$$

is uniformly bounded in n. Therefore, by the Helly theorem, there is a subsequence $\{n_p\}$ of $\{n\}$ such that $|m_{n_p}| \to |m|$ as $p\to\infty$, and we also have

$$\int_{1/a}^{a} f(t)\, d|m_{n_p}|(t) \to \int_{1/a}^{a} f(t)\, d|m|(t)$$

for all continuous functions f on the interval $[1/a, a]$.

Let k be any integer. Then t^k is continuous and bounded in $[1/a, a]$ and

$$\int_{1/a}^{a} t^k \, d|m_n|(t) = \int_{1/a}^{a} t^k |j(t + is_n)| \, dt.$$

Now let $u = 1/t$. Then

$$\int_{1/a}^{a} t^k \, d|m_n|(t) = \int_{1/a}^{a} u^{-k} |u^{-2} j(u^{-1} + is_n)| \, du.$$

But by definition, $|u^{-2} j(u^{-1} + is_n)| = g_n(u) + |j(u + is_n)|$ and thus

$$\int_{1/a}^{a} t^k \, d|m_n|(t) = \int_{1/a}^{a} u^{-k} |j(u + is_n)| \, du + \int_{1/a}^{a} u^{-k} g_n(u) \, du$$

and so

$$\int_{1/a}^{a} t^k \, d|m_{n_p}|(t) = \int_{1/a}^{a} t^{-k} \, d|m_{n_p}|(t) + \int_{1/a}^{a} t^{-k} g_{n_p}(t) \, dt.$$

Taking limits as p approaches infinity, the result now follows.

Henceforth, a J-unitary operator U will be called *J-real-unitary* if it has positive real spectrum and if $\|p(U)\| \leqq c \|p(U)\|_{sp}$ for all polynomials $p(t)$ where c is a real constant. The measure $m(t{:}x, y)$ of Proposition 6 will be called the measure associated with U, x, and y.

THEOREM 5. *Let U be a J-real-unitary operator in a J-space H such that the set $\{U^n v: n = 0, 1, 2, \ldots\}$ generates the space H for some v in H. Assume that $\|p_n(U)v\| \to 0$ if and only if $\int |p_n(t)|^2 \, d|m|(t) \to 0$, where m is the measure associated with U, v, and v, and $\{p_n(t)\}$ is a sequence of polynomials. Then U is J-isomorphic with the J-unitary operator M in $L^2(|m|)$, where $Mf(t) = tf(t)$ and $Jf(t) = X(t) f(1/t)$ with $X = dm/d|m|$.*

Proof. Since U is J-real,

$$((U - z)^{-1} v, v) = \int_{1/a}^{a} (t - z)^{-1} \, dm(t)$$

by Proposition 6, where m may be considered a real finite signed Borel measure on the interval $[1/a, a]$ which contains the spectrum of U. From the Neumann expansions of both sides about 0 and ∞ and the identity principle of complex variables we have

$$\int_{1/a}^{a} t^n \, dm(t) = (U^n v, v)$$

for all integers n. Therefore

$$\int_{1/a}^{a} t^n \, dm(t) = (U^n v, v) = (v, U^{-n} v) = \overline{(U^{-n} v, v)} = \int_{1/a}^{a} t^{-n} \, dm(t).$$

Thus we have, by changing t to $1/t$ in the last integral,

$$\int_{1/a}^{a} t^n \, d(m(t) + m(1/t)) = 0$$

for all integers n. Thus $m(t) + m(1/t)$ is a constant. Therefore

$$\int_{1/a}^{a} t^n \, d|m|(t) = \int_{1/a}^{a} t^{-n} \, d|m|(t)$$

for all n by Proposition 7.

Now consider $L^2(|m|)$ with $Mf(t) = tf(t)$ and $Jf(t) = X(t)f(1/t)$, where $X = dm/d|m|$. The Hilbert inner product is

$$[f, g] = \int_{1/a}^{a} f(t)\overline{g(t)} \, d|m|(t).$$

Therefore

$$[Jf, g] = \int_{1/a}^{a} f(1/t)\overline{g(t)} \, dm(t) \quad \text{and} \quad [f, Jg] = \int_{1/a}^{a} f(t)\overline{g(1/t)} \, dm(t).$$

Since the polynomials are dense in $L^2(|m|)$ [**1**], it suffices to show that $[Jf, g] = [f, Jg]$ for $f(t) = t^w$ and $g(t) = t^q$. For this pair of functions,

$$[Jf, g] = \int_{1/a}^{a} t^{q-w} \, dm(t) \qquad \text{and} \qquad [f, Jg] = \int_{1/a}^{a} t^{-(q-w)} \, dm(t),$$

and these are equal by the above. Thus $J = J^*$.

To show that $J^2 = I$, we need only show that $[Jf, Jg] = [f, g]$ for all pairs f and g where $f(t) = t^w$ and $g(t) = t^q$. For this pair of functions,

$$[Jf, Jg] = \int_{1/a}^{a} t^{-(q+w)} \, d|m|(t) = \int_{1/a}^{a} t^{q+w} \, d|m|(t) = [f, g],$$

again by the above. From the fact that $|X|$ is identically one [**28**] and all the preceding, we conclude that J is invertible, self-adjoint, and unitary. This implies that $X(t) = X(1/t)$. The space $L^2(|m|)$, as above, is a J-space and the indefinite metric is given by $(f, g) = [Jf, g]$.

A calculation shows that M is J-unitary and that

$$((M - z)^{-1}e, e) = \int_{1/a}^{a} (t - z)^{-1} \, dm(t) = ((U - z)^{-1}v, v),$$

where $e(t) \equiv 1$. By assumption, $||p_n(U)v|| \to 0$ if and only if

$$\int |p_n(t)|^2 \, d|m|(t) = ||p_n(M)e||^2$$

converges to zero. Finally, the set $\{M^n e \colon n = 0, 1, 2, \ldots\}$ generates $L^2(|m|)$ since this is the set of all powers of t. Therefore by Lemma 1, the operators U and M are J-isomorphic.

Now we make the following definition. Let U be a J-unitary operator in a J-space H with one-point spectrum e^{ib} for real b. If

$$i((U - e^{ib})(U + e^{ib})^{-1} - (U^* - e^{-ib})(U^* + e^{-ib})^{-1})$$

has one-dimensional range spanned by a vector v in H with $Jv = v$ and $||v|| = 1$, then we call U a *J-Volterra-unitary operator with vector v and spectrum e^{ib}.*

THEOREM 6. *Let U be a J-Volterra-unitary operator with vector v and spectrum e^{ib} in a J-space H such that the set $\{U^n v\colon n = 0, 1, 2, \ldots\}$ generates H. Then U is J-isomorphic to $-e^{-ib}M$ in the Lebesgue measure space $L^2(0, 1)$, where*

$$Mf(t) = -f(t) + 2ia \int_0^t e^{-ia(t-s)} f(s)\, ds$$

and $Jf(t) = f(1-t)$. The real number a is determined by the equation

$$i((U - e^{ib})(U + e^{ib})^{-1} - (U^* - e^{-ib})(U^* + e^{-ib})^{-1})v = av.$$

Proof. Let $V = -e^{-ib}U$. Then $\{V^n v\colon n = 0, 1, 2, \ldots\}$ generates H, and V is J-unitary with one-point spectrum -1. Also

$$i((V + I)(V - I)^{-1} - (V^* + I)(V^* - I)^{-1})$$

has one-dimensional range spanned by v. Let $A = i(V + I)(V - I)^{-1}$. Then A is J-self-adjoint with one-point spectrum 0 and $A + A^*$ has one-dimensional range spanned by v. Consider the function $1 - a((A - z^{-1})^{-1}v, v) = F(z)$. Since $Jv = v$, then $F(z) = 1 - a[(A - z^{-1})^{-1}v, v]$ due to the fact that $(u, Jv) = [u, v]$ for all u and v in H.

The operator A here satisfies [**14**, Theorem 8 (i) and (ii)], and so $F(z) = e^{az}$. Furthermore,

$$(z_1 + z_2) \sum_{p,q=0} z_1^p z_2^q [A^p v, A^q v] = (1/a)(F(z_1)\overline{F(z_2)} - 1).$$

Also the operator aW given by

$$(aW)f(t) = \int_0^t af(s)\, ds$$

satisfies the same conditions that A does and so

$$1 - a((aW - z^{-1})^{-1}e, e) = F(z),$$

where e is the identity function in $L^2(0, 1)$. Also

$$(z_1 + z_2) \sum_{p,q=0} z_1^p z_2^q [(aW)^p e, (aW)^q e] = (1/a)(F(z_1)\overline{F(z_2)} - 1).$$

Equating the coefficients of $z_1^p z_2^q$ in the two equal summations above, we have $[A^p v, A^q v] = [(aW)^p e, (aW)^q e]$ for all non-negative integers p and q. Recall that $A = i(V + I)(V - I)^{-1}$. Thus

$$V = -I - 2 \sum_{n=1}^{\infty} (-iA)^n$$

by a power series expansion. Thus

$$[V^p v, V^q v] = \left[\left(-I - 2 \sum_{n=1}^{\infty} (-iA)^n\right)^p v,\ \left(-I - 2 \sum_{n=1}^{\infty} (-iA)^n\right)^q v\right].$$

The right-hand side of this last equality is clearly a sum of terms of type

$$\beta_{rs}[A^r v, A^s v] = \beta_{rs}[(aW)^r e, (aW)^s e].$$

Adding up we have

$$[V^p v, V^q v] = \left[\left(-I - 2\sum_{n=1}^{\infty} (-iaW)^n\right)^p e, \left(-I - 2\sum_{n=1}^{\infty} (-iaW)^n\right)^q e\right].$$

However,

$$-I - 2\sum_{n=1}^{\infty} (-iaW)^n = M$$

by definition. Therefore $[V^p v, V^q v] = [M^p e, M^q e]$ for $p, q = 0, 1, 2, \ldots$. In particular, $\|p_k(V)v\| \to 0$ if and only if $\|p_k(M)e\| \to 0$, where $\{p_k(t)\}$ is a sequence of polynomials in t.

Recall that $((A - z^{-1})^{-1}v, v) = (1/a)(1 - e^{az})$ and $(A + A^*)v = av$. But $(A - z^{-1})^{-1} = (1/2i)(1 - w)(V - w)^{-1}(V - I)$ by the definition of A where $w = (1 + iz)(1 - iz)^{-1}$.

Therefore we have

$$(2i)^{-1}(1 - w)((V - w)^{-1}(V - I)v, v) = a^{-1}(1 - \exp(ia(1 - w)(1 + w)^{-1})).$$

Since $(V - w)^{-1}(V - I) = I - (1 - w)(V - w)^{-1}$, we have

$$(2i)^{-1}(1 - w)((v, v) - (1 - w)((V - w)^{-1}v, v)) = a^{-1}(1 - \exp(ia(1 - w)(1 + w)^{-1})),$$

and therefore

$$((V - w)^{-1}v, v) = (1 - w)^{-1} - 2ia^{-1}(1 - w)^{-2}(1 - \exp(ia(1 - w)(1 + w)^{-1})).$$

Both sides of this last equation are analytic except at $w = 1$. Let $L^2(0, 1)$, M, and J be as in the conclusion of the theorem. Then $L^2(0, 1)$ is a J-space with indefinite metric $(f, g) = [Jf, g]$ and M is J-unitary. Since for each $n = 0, 1, 2, \ldots$ we have

$$(M + I)^{n+1}e(t) - 2(M + I)^n e(t) = -2(2ia)^n e^{-iat}t^n/n!,$$

the set $\{(M + I)^n e: n = 0, 1, 2, \ldots\}$ generates $L^2(0, 1)$, where e is the identity function in that space. Therefore the set $\{M^n e: n = 0, 1, 2, \ldots\}$ generates $L^2(0, 1)$ [**1**]. Also,

$$((M + I - Z)^{-1}e, e) = -\sum_{n=0}^{\infty} Z^{-n-1}((M + I)^n e, e)$$
$$= -Z^{-1} - \sum_{n=1}^{\infty} Z^{-n-1}\int_0^1 (M + 1)^n e(t)\overline{e(1 - t)}\, dt$$

and this equals

$$-Z^{-1} - \sum_{n=1}^{\infty} Z^{-n-1}(2ia)^n \int_0^1 \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} e^{-ia(t-s)}\, ds\, dt.$$

Thus

$$((M+I-Z)^{-1}e,e) = -Z^{-1} - 2iaZ^{-2}\int_0^1\int_0^t \sum_{n=1}^{\infty} (2ia)^{n-1}Z^{-(n-1)} \times \frac{(t-s)^{n-1}}{(n-1)!}e^{-ia(t-s)}\,ds\,dt$$

$$= -Z^{-1} - 2iaZ^{-2}\int_0^1\int_0^t e^{ia(t-s)(2Z^{-1}-1)}\,ds\,dt$$

$$= (2-Z)^{-1} - 2ia^{-1}(2-Z)^{-2}(1-\exp(ia(2Z^{-1}-1))),$$

and so

$$((M-w)^{-1}e,e) = (1-w)^{-1} - 2ia^{-1}(1-w)^{-2}(1-\exp(ia(1-w)(1+w)^{-1})),$$

where $w = Z-1$. The last expression is equal to $((V-w)^{-1}v, v)$. Therefore $((M-w)^{-1}e,e) = ((V-w)^{-1}v,v)$.

Thus by Lemma 1, the operators M and V are J-isomorphic and hence $-e^{-ib}M$ and U are J-isomorphic.

Note that these two operators are also Hilbert isomorphic, i.e. unitarily equivalent.

References

1. N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Vols. I and II, translated from the Russian by Merlynd Nestell (Ungar, New York, 1961/63).
2. A. Brown and C. Pearcy, *Spectra of tensor products of operators*, Proc. Amer. Math. Soc. *17* (1966), 162–166.
3. N. Dunford, *Spectral operators*, Pacific J. Math. *4* (1954), 321–354.
4. ——— *Spectral theory.* II: *Resolutions of the identity*, Pacific J. Math. *2* (1952), 559–614.
5. M. M. Džrbašjan, *Integral transforms and representations of functions in the complex domain* (Izdat. "Nauka", Moscow, 1966).
6. G. C. Evans, *The logarithmic potential: Discontinuous Dirichlet and Neumann problems*, Amer. Math. Soc. Colloq. Publ., Vol. 6 (Amer. Math. Soc., New York, 1927).
7. S. R. Foguel, *The relations between a spectral operator and its scalar part*, Pacific J. Math. *8* (1958), 51–65.
8. P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity* (Chelsea, New York, 1957).
9. ——— *Measure theory* (Van Nostrand, Princeton, N. J., c1950, reprinted 1966).
10. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge, at the University Press, 1967).
11. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Series in Modern Analysis (Prentice-Hall, Englewood Cliffs, N. J., 1962).
12. I. S. Iohvidov, *G-isometric and J-semiunitary operators in Hilbert space*, Uspehi Mat. Nauk *20* (1965), no. 3 (123), 175–181. (Russian)
13. I. S. Iohvidov and M. G. Kreĭn, *Spectral theory of operators in spaces with indefinite metric.* I, II, Amer. Math. Soc. Transl. (2) *13* (1960), 105–175; *ibid.* (2) *34* (1963), 283–373.
14. G. K. Kalisch, *Direct proofs of spectral representation theorems*, J. Math. Anal. Appl. *8* (1964), 351–363.
15. ——— *Characterizations of direct sums and commuting sets of Volterra operators*, Pacific J. Math. *18* (1966), 545–552.

16. M. G. Kreĭn, *Introduction to the geometry of indefinite J-spaces and to the theory of operators in those spaces*, Second Mathematical Summer School, Part I, pp. 15–92 (Naukova Dumka, Kiev, 1965). (Russian)

17. M. G. Kreĭn and G. K. Langer, *On the spectral function of a self-adjoint operator in a space with indefinite metric*, Dokl. Akad. Nauk SSSR *152* (1963), 39–42. (Russian)

18. M. S. Livšic, *On spectral decomposition of linear non-selfadjoint operators*, Mat. Sb. (N. S.) *34* (*76*) (1954), 145–199. (Russian)

19. A. I. Mal′cev, *Foundations of linear algebra*, Translated from the Russian by Thomas Craig Brown; edited by J. B. Robarts (Freeman, San Francisco, Calif.–London, 1963).

20. M. A. Naĭmark, *Unitary permutation operators in the space* Π_k, Dokl. Akad. Nauk SSSR *149* (1963), 1261–1263. (Russian)

21. M. A. Naĭmark and S. V. Fomin, *Continuous direct sums of Hilbert spaces and some of their applications*, Amer. Math. Soc. Transl. (2) *5* (1957), 35–65.

22. R. Nevanlinna, *Eindeutige analytische Funktionen*, 2te Aufl., Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. 66 (Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953).

23. ——— *Über metrische lineare Raüme*. I, II, III, IV, V, Ann. Acad. Sci. Fenn. Ser. AI Math.-Phys. no. 108 (1952), no. 113 (1952), no. 115 (1952), no. 163 (1954), no. 222 (1956).

24. R. S. Phillips, *The extension of dual subspaces invariant under an algebra*, Proc. Internat. Sympos. Linear Spaces, Jerusalem, 1960, pp. 366–398 (Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961).

25. L. S. Pontrjagin, *Hermitian operators in spaces with indefinite metric*, Bull. Acad. Sci. URSS Sér. Math. [Izv. Akad. Nauk SSSR] *8* (1944), 243–280. (Russian)

26. P. Porcelli, *Linear spaces of analytic functions* (Rand McNally, Chicago, 1966).

27. F. Riesz and B. Sz.-Nagy, *Functional analysis* (Ungar, New York, 1965).

28. W. Rudin, *Real and complex analysis* (McGraw-Hill, New York, 1966).

29. H. H. Schaefer, *Topological vector spaces* (Macmillan, New York; Collier-Macmillan, London, 1966).

30. B. Sz.-Nagy, *On uniformly bounded transformations in Hilbert space*, Acta Univ. Szeged Sect. Sci. Math. *11* (1947), 152–157.

31. A. E. Taylor, *Introduction to functional analysis* (Wiley, New York, 1967).

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