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COMPLEXES OF COUSIN TYPE AND MODULES OF
GENERALIZED FRACTIONS

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0. Introduction

Let $R$ be a commutative (Noetherian) ring, $M$ an $R$-module and let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of $\text{Spec}(R)$ which admits $M$.

A complex of $R$-modules is said to be of Cousin type if it satisfies the four conditions of ([GO], 3.2) which are reproduced below (Definition (1.5)). In ([RSZ], 3.4), Riley, Sharp and Zakeri proved that the complex, which is constructed from a chain of special triangular subsets defined in terms of $\mathcal{F}$ (Example (1.3)(3)), is of Cousin type for $M$ with respect to $\mathcal{F}$ (Corollary (3.5)(2)). Gibson and O’carroll ([GO], 3.6) showed that the complex, which is obtained by means of a chain $\mathcal{U} = (U_i)_{i \geq 1}$ of saturated triangular subsets and the filtration $\mathcal{G} = (G_i)_{i \geq 0}$ induced by $\mathcal{U}$ and $M$, is of Cousin type for $M$ with respect to $\mathcal{G}$ (Corollary (3.5)(3)).

The purpose of this paper is to show that, when the complex is defined by a chain of triangular subsets, one can give a simpler criterion, consisting of only two conditions, for being of Cousin type (Theorem (3.1) and Corollary (3.2)). In fact, we prove that, for every complex induced by a chain of triangular subsets, the first and the second conditions of the definition of Cousin type hold (Remark (2.5)).

In ([RSZ], 3.3), Riley, Sharp and Zakeri proved that every complex of Cousin type for $M$ with respect to $\mathcal{F}$ is isomorphic to the Cousin complex. Hence when we investigate the structure of a complex of Cousin type, it is useful to study the complex $C(\mathcal{U}, M)$ of Cousin type which is constructed from special modules of generalized fractions (Corollary (3.5)) whose properties are well known.

We also get a refinement of the Exactness theorem ([SZ2], 3.3 and [O], 3.1) in our Proposition (2.13).

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1. Preliminaries

Throughout this paper, $R$ is a commutative ring with identity and $M$ is an $R$-module. We use $^T$ to denote matrix transpose and $D_n(R)$ to denote the set of all $n \times n$ lower triangular matrices over $R$. For $H \in D_n(R)$, $|H|$ denotes the determinant of $H$. $N$ denotes the set of positive integers.

**Definition (1.1)** ([SZ1], 2.1). Let $n$ be a positive integer. A non-empty subset $U_n$ of $R^n$ is said to be triangular if

(i) whenever $(a_1, \ldots, a_n) \in U_n$, then $(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}) \in U_n$ for all choices of positive integers $\alpha_1, \ldots, \alpha_n$; and

(ii) whenever $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n) \in U_n$, then there exist $(c_1, \ldots, c_n) \in U_n$ and $H, K \in D_n(R)$ such that $H[a_1 \ldots a_n]^T = [c_1 \ldots c_n]^T = K[b_1 \ldots b_n]^T$.

**Definition (1.2)** ([S4], 1.1 and 1.2). Let $R$ be a ring and $M$ an $R$-module. A filtration of $\text{Spec}(R)$ is a descending sequence $\mathcal{F} = (F_i)_{i \geq 0}$ of subsets of $\text{Spec}(R)$, so that

$$\text{Spec}(R) \supseteq F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots,$$

with the property that, for each $i \geq 0$, each member of $F_i \setminus F_{i+1}$ is a minimal member of $F_i$ with respect to inclusion. We then set $\partial F_i = F_i \setminus F_{i+1}$. We say that the filtration $\mathcal{F}$ admits an $R$-module $M$ if $\text{Supp}(M) \subseteq F_0$. Let $\mathcal{F}_M = (F_{M_i})_{i \geq 0}$ be the $M$-height filtration of $\text{Spec}(R)$, i.e., $F_{M_i} = \{p \in \text{Supp}(M) : \text{ht}_M p \geq i\}$.

We say that a sequence of elements $a_1, \ldots, a_n$ of $R$ is a poor $M$-sequence if $a_i$ is not a zerodivisor on $M/(a_1, \ldots, a_{i-1})M$ for each $i = 1, \ldots, n$; it is an $M$-sequence if, in addition, $M \neq (a_1, \ldots, a_n)M$.

**Example (1.3)**. Let $R$ be a Noetherian ring. Then the following five non-empty sets are triangular subsets of $R^n$.

1. ([SZ1], 3.10) Let $M$ be a finitely generated $R$-module. $$(U_1)_n = \{(a_1, \ldots, a_n) \in R^n : a_1, \ldots, a_n \text{ forms a poor } M\text{-sequence}\}.$$

2. (cf. [SZ2], 5.2) Suppose that $M$ is a finitely generated $R$-module. $$(U_2)_n = \{(a_1, \ldots, a_n) \in R^n : \text{ht}_M(a_1, \ldots, a_i)R \geq i \quad (1 \leq i \leq n)\}.$$ 

3. ([RSZ], 2.3) Assume that $M$ is an $R$-module such that $\text{Ass}(M)$ contains only finitely many minimal members. $$(U_3)_n = \{(a_1, \ldots, a_n) \in R^n : \text{ for each } i = 1, \ldots, n, \quad (a_1, \ldots, a_i)R \not\subseteq p \text{ for all } p \in \partial F_{i-1} \cap \text{Supp}(M)\}.$$ 

4. ([C], 1.1) Suppose that $M$ is a finitely generated $R$-module of dimension $d$. 

(U,) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : \dim \mathbb{M}/(a_1, \ldots, a_i) = d - i \ (1 \leq i \leq n)\}.

(5) ([C], 1.2) Suppose that (R, m) is a local ring and \( \mathbb{M} \) is a finitely generated \( R \)-module.

(U,) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : a_1, \ldots, a_n \text{ is an } f\text{-regular sequence (See [SV], p. 252) with respect to } \mathbb{M}\}.

\((a_1, \ldots, a_n) \in \mathbb{R}^n\) forms an \( \mathbb{M} \)-sequence for all \( p \in \text{Supp}(\mathbb{M}) \setminus \{m\} \) such that \( (a_1, \ldots, a_n) \mathbb{R} \subset p\).

For a given triangular subset \( U_n \) of \( \mathbb{R}^n \), let \( \tilde{U}_n = ((a_1, \ldots, a_i, 1, \ldots, 1) \in \mathbb{R}^n : \forall i \ (0 \leq i \leq n), \exists a_{i+1}, \ldots, a_n \in \mathbb{R} \text{ s.t. } (a_1, \ldots, a_{i+1}, \ldots, a_n) \in U_n\).

This is a triangular subset of \( \mathbb{R}^n \) and is called the expansion of \( U_n \) ([SZ1], p. 38). Then, by ([SZ1], 3.2), we may assume without loss of the generality that \( U_n \) is expanded, i.e., \( U_n = \tilde{U}_n \), when we consider the module of generalized fractions for \( \mathbb{M} \) with respect to \( U_n \). So, from now on, we assume that every triangular subset is expanded by means of the expansion of itself.

For a fixed non-negative integer \( n \), \( U_{n+1} \mathbb{M} \) denotes the module of generalized fractions of \( \mathbb{M} \) with respect to \( U_{n+1} \) ([SZ1]). The other notation and terminology about the module of generalized fractions follow ([SZ1]).

**Definition** (1.4) ([RSZ], p. 52). Let \( \mathbb{R} \) be a ring. A family \( \mathcal{U} = (U_i)_{i \geq 1} \) is called a chain of triangular subsets on \( \mathbb{R} \) if the following conditions are satisfied:

(i) \( U_i \) is a triangular subset of \( \mathbb{R}^i \) for all \( i \in \mathbb{N} \);

(ii) \((1) \in U_1 \);

(iii) whenever \((a_1, \ldots, a_i) \in U_i \) with \( i \in \mathbb{N} \), then \((a_1, \ldots, a_i, 1) \in U_{i+1} \); and

(iv) whenever \((a_1, \ldots, a_i) \in U_i \) with \( 1 < i \in \mathbb{N} \), then \((a_1, \ldots, a_{i-1}) \in U_{i-1} \).

Each \( U_i \) leads to a module of generalized fractions \( U_i \mathbb{M} \) and we can obtain a complex

\[
0 \rightarrow \mathbb{M} \rightarrow U_1^{-1} \mathbb{M} \rightarrow U_2^{-1} \mathbb{M} \rightarrow \cdots \rightarrow U_i^{-1} \mathbb{M} \rightarrow U_{i+1}^{-1} \mathbb{M} \rightarrow \cdots
\]

denoted by \( C(\mathcal{U}, \mathbb{M}) \), for which \( e^0(m) = \frac{m}{(1)} \) for all \( m \in \mathbb{M} \) and

\[
e^i \left( \frac{x}{(a_1, \ldots, a_i)} \right) = \frac{x}{(a_1, \ldots, a_i, 1)}
\]

for all \( i \in \mathbb{N}, x \in \mathbb{M} \) and \((a_1, \ldots, a_i) \in U_i \).

\( H^i_V(\mathbb{M}) \) denotes the \( i \)-th cohomology group of \( C(\mathcal{U}, \mathbb{M}) \). That is \( H^i_V(\mathbb{M}) = \text{Ker } e^i / \text{Im } e^{i-1} \).
DEFINITION (1.5) ([GO], 3.2). Let \( R \) be a Noetherian ring and \( M \) an \( R \)-module. Let \( \mathcal{F} = (F_i)_{i \geq 0} \) be a filtration of \( \text{Spec}(R) \) that admits \( M \). A complex \( X' = (X^i : i \geq -2) \) of \( R \)-modules and \( R \)-homomorphisms is said to be of Cousin type for \( M \) with respect to \( \mathcal{F} \) if it has the form

\[
0 \to M \to X^0 \to X^1 \to \cdots \to X^i \to X^{i+1} \to \cdots
\]

and satisfies the following, for each \( n \in \mathbb{N} \cup \{0\} \),

(i) \( \text{Supp}(X^n) \subseteq F_n \);

(ii) \( \text{Supp}(\text{Coker } d^{n-1}) \subseteq F_{n+1} \);

(iii) \( \text{Supp}(\text{Ker } d^{n-1}/\text{Im } d^{n-2}) \subseteq F_{n+1} \); and

(iv) The natural \( R \)-homomorphism \( \xi(X^n) : X^n \to \bigoplus_{p \in \partial F_n} (X^n)_p \) such that, for \( x \in X^n \) and \( p \in \partial F_n \), the component of \( \xi(X^n)(x) \) in the summand \( (X^n)_p \) is \( x/1 \), is an isomorphism.

LEMMA (1.6). Let \( R \) be a ring and \( M \) an \( R \)-module. Let \( U_n \) be an expanded triangular subset of \( R^n \). Let \( (a_1, \ldots, a_n) \) and \( (b_1, \ldots, b_n) \) be elements of \( U_n \) such that \( H[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T \) for some \( H \in D_\mathbb{N}(R) \). Then we have

(1) ([SZ1], 2.8 and 3.3(i)) \( \frac{m}{(a_1, \ldots, a_n)} = \frac{|H| m}{(b_1, \ldots, b_n)} \) and \( \frac{a_m}{(a_1, \ldots, a_n)} = \frac{m}{(a_1, \ldots, a_{n-1}, 1)} \) in \( U_n^{-n} M \).

(2) ([SZ1, 3.3(ii)] and [SY, 2.2]) If \( m \in (a_1, \ldots, a_{n-1})M \) then \( \frac{m}{(a_1, \ldots, a_n)} = 0 \) in \( U_n^{-n} M \). In particular, if each element of \( U_n \) is a poor \( M \)-sequence, then the converse holds.

(3) ([SZ2], 5.1 and [SZ3], 2.1) \( \text{Ann}_R\left(\frac{m}{(a_1, \ldots, a_n)}\right) = \text{Ann}_R\left(\frac{m}{(a_1, \ldots, a_{n-1}, 1)}\right) \).

LEMMA (1.7) ([C], 2.4). Let \( (R, m) \) be a Noetherian local ring and let \( M \) be a finitely generated \( R \)-module of dimension \( d \). Let \( (U_i)_{i \geq 1} \) be the expansion of the triangular subset \( \{(a_1, \ldots, a_d, 1) \in R^{d+1} : \dim M/(a_1, \ldots, a_d)M = 0\} \). Let \( \{x_1, \ldots, x_d\} \) be a fixed system of parameters for \( M \). Then we have

\[
(U_i)_{i \geq 1}^{-d+1} M \cong U(x)_d[1]^{-d-1} M \cong H_{m}^d (M),
\]

where \( U(x)_d[1] = \{(x_1^{\alpha_1}, \ldots, x_d^{\alpha_d}, 1) \in R^{d+1} : \text{ there is } i (0 \leq i \leq d) \text{ such that } \alpha_i, \ldots, \alpha_i \in \mathbb{N} \text{ and } \alpha_{i+1} = \cdots = \alpha_d = 0\} \).
**Lemma (1.8) ([GO], 3.4).** Let $R$ be a ring. For a positive integer $n$, suppose that 
\[ \frac{m}{(a_1, \ldots, a_n, 1)} = 0 \text{ in } U_{n+1}^{-1} M. \]
Then there exist $(b_1, \ldots, b_{n+1}) \in U_{n+1}$ and $H \in D_n(R)$ such that $H[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T$ and $b_{n+1} | H | m \in (b_1, \ldots, b_n)M.$

**Lemma (1.9) ([GO], 3.3 and [SY], 2.7).** Let $R$ be a ring and $M$ an $R$-module. Let $\mathcal{U} = (U_i)_{i \geq 1}$ be a chain of triangular subsets on $R$. Then in $C(\mathcal{U}, M)$, for all $n \in \mathbb{N}$
\[ \text{Coker } e^{n-1} \cong U_n^{-1} M / \text{Im } e^{n-1} \cong U_n[1]^{-1} M, \]
where $U_n[1] = \{(a_1, \ldots, a_n, 1) \in R^{n+1} : (a_1, \ldots, a_n) \in U_n\}$.

**2. Associated prime ideals of modules of generalized fractions**

**Lemma (2.1).** Let $R$ be a ring and $M$ an $R$-module. Fix a positive integer $n$. Let $U_n$ be a triangular subsets of $R^n$. Let $0 \neq \frac{m}{(a_1, \ldots, a_n)} \in U_n^{-1} M$. Then we have, for all $(b_1, \ldots, b_n) \in U_n$,
\[ (b_1, \ldots, b_n) R \not\subset \left(0 : \frac{m}{(a_1, \ldots, a_n)}\right). \]

**Proof.** Suppose that for some $(b_1, \ldots, b_n) \in U_n$
\[ (b_1, \ldots, b_n) R \subset \left(0 : \frac{m}{(a_1, \ldots, a_n)}\right). \]
Then by the definition of triangular subset there are $(c_1, \ldots, c_n) \in U_n$ and $H, K \in D_n(R)$ such that $H[a_1 \ldots a_n]^T = [c_1 \ldots c_n]^T = K[b_1 \ldots b_n]^T$. Hence we get $(c_1, \ldots, c_n) R \subset (b_1, \ldots, b_n) R$.

On the other hand, by Lemma (1.6)(1)(3) we have
\[ \left(0 : \frac{m}{(a_1, \ldots, a_n)}\right) = \left(0 : \frac{|H| m}{(c_1, \ldots, c_n)}\right) = \left(0 : \frac{|H| m}{(c_1, \ldots, c_{n-1}, 1)}\right) \supset (b_1, \ldots, b_n) R \supset (c_1, \ldots, c_n) R. \]
Therefore we have the following contradiction.
From now on, we suppose that \( U_0[1]^{-1} M = M, U_0^0 M = M \) and \( n \) is a non-negative integer.

**Lemma (2.2).** Let \( R \) and \( M \) be as above. Then in \( C(\mathcal{U}, M) \) we have

\[
\text{Supp}(U_{n+1}^{-1} M) \subset \text{Supp}(U_n[1]^{-n-1} M) \subset \text{Supp}(U_n^{-n} M).
\]

**Proof.** For the first half, this follows from the following short exact sequence

\[
0 \rightarrow \text{Ker} e^n / \text{Im} e^{n+1} \rightarrow U_n^{-n} M / \text{Im} e^{n+1} \rightarrow U_n^{-n} M / \text{Ker} e^n \rightarrow 0,
\]

since \( \text{Supp}(U_{n+1}^{-1} M) = \text{Supp}(\text{Im} e^n) \) by Lemma (1.6)(3).

For the second inclusion, it follows from Lemma (1.9) that

\[
\text{Supp}(U_1[1]^{-1} M) = \text{Supp}(U_1^{-1} M) \subset \text{Supp}(U_1^{-n} M).
\]

**Example (2.3).** In general, \( \text{Supp}(U_{n+1}^{-1} M) \neq \text{Supp}(U_1[1]^{-n-1} M) \). Let \((R, m)\) be a Noetherian local ring. Suppose that \( M \) is an \( f \)-module (see [SZ4], 1.8(ii)) of dimension \( d \). Then \( \text{Supp}((U_\delta)_d[1]^{-d-1} M) = \text{Supp}((U_\delta)_d^{-d-1} M) = \{m\} \). But \( \text{Supp}((U_\delta)_d^{-1} M) = \emptyset \) by ([C], 2.3).

**Lemma (2.4).** Let \( R \) and \( M \) be as above. Then in \( C(\mathcal{U}, M) \) we have

\[
\text{Supp}(U_{n+1}^{-1} M) \subset \text{Supp}(U_n[1]^{-n-1} M) \subset F_{Mn} \subset F_n.
\]

**Proof.** This follows from Lemma (2.2), ([HS], 3.1) and ([C], 2.7).

**Remark (2.5).** Lemma (2.4) shows that, for every complex \( C(\mathcal{U}, M) \), the first and the second conditions of the definition of Cousin type hold by Lemma (1.9).

**Lemma (2.6).** Let \( R \) and \( M \) be as above. Then in \( C(\mathcal{U}, M) \) we have the following.

(1) \( \partial F_n \cap \text{Supp}(M) = (\bigcup_{i=0}^n \partial F_{M^i}) \cap \partial F_n. \)

(2) (cf. [ST], 2.7) \( \partial F_n \cap \text{Supp}(U_{n+1}^{-1} M) \subset \partial F_n \cap \text{Supp}(U_n[1]^{-n-1} M) \subset \partial F_n \cap \partial F_{Mn}. \)

(3) \( \partial F_n \cap \partial F_{Mn} = \bigcup_{q \in \partial F_{M^{n-1}} \cap \partial F_{M^{n-1}}} (V(q) \cap \partial F_n \cap \partial F_{Mn}). \)
Proof. (1) Let \( p \in \partial F_n \cap \text{Supp}(M) \setminus \bigcup_{i=0}^{n} \partial F_{M_i} \). Hence \( \text{ht}_M p > n \). Therefore there is \( q \in \partial F_{M_n} \subset F_n \) such that \( q \preceq p \). That is, \( p \) is not minimal in \( F_n \).

(2) Since \( \text{Supp}(U_{n+1}[n]^{-n-1} M) \subset \text{Supp}(U_n[1]^{-n-1} M) \subset \partial F_n \cap \text{Supp}(M) \cap F_{M_n} \)

\[ \subset (\cup_{i=0}^{n} \partial F_{M_i}) \cap \partial F_n \cap F_{M_n} = \partial F_n \cap \partial F_{M_n} \]

by (1).

(3) Let \( p \in \partial F_n \) and \( \text{ht}_M p = n \). Suppose that \( q \not\in \partial F_{n-1} \) for some \( q \in \text{Supp}(M) \) such that \( \text{ht}_M q = n - 1 \) and \( q \preceq p \). Hence \( q \in F_n \), since \( \partial F_{n-1} = F_n \setminus F_{n-1} \) and \( F_{M(n-1)} \subset F_{n-1} \). This contradicts that \( p \) is a minimal element in \( F_n \).

**Lemma (2.7).** Let \( R \) be a ring and \( M \) an \( R \)-module. Then in \( C(U, M) \), for each

\[ \frac{m}{(a_1, \ldots, a_n)} + \text{Im } e^{-1} \in H_U^n(M) \]

there are \( (b_1, \ldots, b_{n+1}) \in U_{n+1} \) and \( H \in D_n(R) \) such that \( H[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T \) and

\[ (b_1, \ldots, b_{n+1}) R \subset \left( \text{Im } e^{-1}, \frac{m}{(a_1, \ldots, a_n)} \right) \]

**Proof.** Since \( \frac{m}{(a_1, \ldots, a_n)} \in \text{Ker } e^n \), we have \( \frac{m}{(a_1, \ldots, a_n, 1)} = 0 \) in \( U_{n+1}[n]^{-n-1} M \). Hence by Lemma (1.8) there are \( (b_1, \ldots, b_{n+1}) \in U_{n+1} \) and \( H \in D_n(R) \) such that \( H[a_1 \ldots a_n]^T = [b_1 \ldots b_{n+1}]^T \) and \( b_{n+1} | H | m \in (b_1, \ldots, b_{n+1})M \). Therefore we have

\[ (b_1, \ldots, b_{n+1}) R \subset \left( \text{Im } e^{-1}, \frac{m}{(a_1, \ldots, a_n)} \right) \]

**Lemma (2.8).** Let \( R \) be a ring and \( M \) an \( R \)-module. Let \( U = \{ U_i \}_{i \geq 1} \) be a chain of triangular subsets on \( R \). Then in \( C(U, M) \), for a fixed non-negative integer \( n \), we have the following.

1. \( \text{Ass}(U_{n+1}[n]^{-n-1} M) \cap \text{Supp}(U_{n+2}[n]^{-n-2-i} M) = \emptyset \) for all \( i \geq 0 \).
2. \( \text{Ass}(U_{n+1}[n]^{-n-1} M) \cap \text{Supp}(U_{n+1}[n]^{-n-2-i} M) = \emptyset \) for all \( i \geq 0 \).
3. \( \text{Ass}(U_{n+1}[n]^{-n-1} M) = \text{Ass}(\text{Im } e^n) = \text{Ass}(\text{Ker } e^n) \).
4. \( \text{Ass}(U_{n+1}[n]^{-n-1} M) \cap \text{Supp}(H_U^n(M)) = \emptyset \) for all \( i \geq 0 \).
5. \( \text{Ass}(H_U^n(M)) \subset \text{Ass}(U_n[1]^{-n-1} M) \subset \text{Ass}(H_U^n(M)) \cup \text{Ass}(U_{n+1}[n]^{-n-1} M) \).
6. If \( R \) is Noetherian, then

\[ \partial F_n \cap \text{Ass}(U_n[1]^{-n-1} M) = (\partial F_n \cap \text{Ass}(H_U^n(M))) \cup (\partial F_n \cap \text{Ass}(U_{n+1}[n]^{-n-1} M)) \].
7. \( \text{Ass}(U_n[1]^{-n-1} M) \cap \text{Ass}(U_{n+1}[n]^{-n-2} M) \subset \text{Ass}(H_U^n(M)) \).
Proof. (1) and (2) follow from Lemma (2.1) and Lemma (1.6)(2).

(3) Since \( \text{Im} e^n \subset \text{Ker} e^{n+1} \subset U_{n+1}^{-n-1} M \), this follows from Lemma (1.6)(3).

(4) This follows from Lemma (2.1), Lemma (2.7) and Lemma (1.6)(2).

(5) The following short exact sequence and (3) complete the proof.

\[
0 \to \text{Ker} e^n / \text{Im} e^{n-1} \to U_n^{-n} M / \text{Im} e^{n-1} \to U_n^{-n} M / \text{Ker} e^n \to 0.
\]

\[\begin{array}{ccc}
\text{Ker} e^n & \text{Im} e^{n-1} & U_n^{-n} M / \text{Ker} e^n \\
H^p_n(M) & U_n[1]^{-n-1} M & \text{Im} e^n
\end{array}\]

(6) By Lemma (2.4), we have

\[\partial F_n \cap \text{Supp}(U_{n+1}^{-n-1} M) = \partial F_n \cap \text{Ass}(U_{n+1}^{-n-1} M) \subset \partial F_n \cap \text{Ass}(U_n[1]^{-n-1} M).\]

Hence the assertion follows from (5).

(7) This follows from (1), (4) and (5).

Remark (2.9). If we also change associated prime to weakly associated in the sense of ([B], p. 289 ex. 17), then we can omit the Noetherian condition of Proposition (2.8)(6).

PROPOSITION (2.10). Let \( R \) and \( M \) be as above. Assume that \( p \in \text{Spec}(R) \). In \( C(U, M) \), consider the following statements:

(i) For all \((a_1, \ldots, a_{n+1}) \in U_{n+1}, (a_1, \ldots, a_{n+1}) R \not\subset p;\)

(ii) \((U_{n+1}^{-n-1} M)_p \equiv (U_n[1]^{-n-1} M)_p;\)

(iii) \((H^p_n(M))_p = 0 \) and \((U_n[1]^{-n-2} M)_p = 0;\)

(iii') \((U_{n+1}^{-n-1} M)_p \equiv (\text{Im} e^n)_p;\)

(iii'') \((H^p_n(M))_p = 0 \) and \((U_{n+2}^{-n-2} M)_p = 0;\)

(iv) \((\text{Ker} e^{n+1})_p \equiv (\text{Im} e^n)_p;\)

(iv') \((U_{n+1}^{-n-1} M)_p \equiv (\text{Im} e^{n+1})_p.\)

Then we have the following.

(1) (ii) \(\Leftrightarrow\) (ii').

(2) (iii) \(\Leftrightarrow\) (iii') \(\Rightarrow\) (iii'').

(3) (iv) \(\Leftrightarrow\) (iv').

(4) (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (iv). That is, if (i) holds, then

\[(U_{n+1}^{-n-1} M)_p \equiv (U_n[1]^{-n-1} M)_p \equiv (\text{Im} e^n)_p \equiv (\text{Ker} e^{n+1})_p.\]

(5) If \( p \in \text{Ass}(U_{n+1}^{-n-1} M), \) then the above four modules are isomorphic.

(6) If \( p \not\in \text{Supp}(U_{n+2}^{-n-2} M), \) then (iv) \(\Rightarrow\) (iii).
Proof. (1) Using the short exact exact sequence (*), we prove as follows.

(⇒) Assume that \((U[1]^{-n-1} M)_p \cong (U[1]^{-n-1} M)_p\). Then, from the following short exact sequence

\[
0 \rightarrow \text{Im} e^{-1} \rightarrow U_n^{-n} M \rightarrow U_n^{-n} M / \text{Im} e^{-1} \rightarrow 0,
\]

we have a commutative diagram with exact rows.

\[
\begin{array}{ccc}
0 & \rightarrow & (\text{Im} e^{-1})_p \\
& & \| \\
& & (U_n^{-n} M)_p \rightarrow (U_n[1]^{-n-1} M)_p \rightarrow 0
\end{array}
\]

Therefore we get

\[
(\text{Ker } e^n)_p = (\text{Im } e^{-1})_p.
\]

Hence, from the following short exact sequence

\[
0 \rightarrow (H^{n}_{U}(M))_p \rightarrow (U_n[1]^{-n-1} M)_p \rightarrow (\text{Im } e^n)_p \rightarrow 0
\]

induced from the short exact sequence (*), we have

\[
(U_{n+1}^{-n-1} M)_p \cong (U_n[1]^{-n-1} M)_p \cong (\text{Im } e^n)_p.
\]

Therefore from the following short exact sequence

\[
(*) \quad 0 \rightarrow \text{Im } e^n \rightarrow U_{n+1}^{-n-1} M \rightarrow U_{n+1}[1]^{-n-2} M \rightarrow 0
\]

we have

\[
(U_{n+1}[1]^{-n-2} M)_p = 0.
\]

(⇐) By the assumption and the short exact sequences (*)(*), we have

\[
(U_{n}[1]^{-n-1} M)_p \cong (\text{Im } e^n)_p \cong (U_{n+1}^{-n-1} M)_p.
\]

(2) The first equivalence follows immediately from the above short exact sequence (*). For the second half, this follows from

\[
\text{Supp}(U_{n+1}[1]^{-n-2} M) = \text{Supp}(H^{n+1}_{U}(M)) \cup \text{Supp}(U_{n+2}^{-n-2} M)
\]

induced by the short exact sequence (*) with \(n+1\) instead of \(n\) and Lemma (2.8) (3).
(3) This follows similarly from the short exact sequence (\(*\)) with \(n\) replaced by \(n + 1\).

(4) Suppose that (i) holds. By the hypothesis and Lemma (1.6)(2) we have 

\((U_{n+1}[1]^{-n-2}M)_p = 0\).

On the other hand, from the assumption and Lemma (2.7), we have 

\((H^n_U(M))_p = 0\).

The other assertions are obvious.

(5) This follows from the hypothesis, Lemma (2.1) and (4).

(6) This follows easily from (2), since 

\((H^n_{n+1}(M))_p = 0\).

**Example (2.11).** (1) In Proposition (2.10), (ii) does not imply (i). Let 

\(R = k[[X, Y]]\). Let \(M\) be the quotient field of \(R\). Let \(U_1 = R \setminus \{X\}\) and \(p = (X, Y)\). Then 

\((U^{-1}_1 M)_p = M = (U_0[1]^{-1} M)_p = (\text{Im } e^0)_p\) but \(U_1 \cap p \neq \emptyset\).

(2) ((iii) \(\Rightarrow\) (ii)) is not the case. See Example (2.3) and note that 

\((U^{-1}_1 M)_p \neq 0\). When \(p \in \text{Supp}(U^{-n-1}_n M)\), we don’t know whether this holds or not.

(3) If \(p \in \text{Supp}(U^{-n-2}_n M)\), then ((iv) \(\Rightarrow\) (iii)) does not hold. Let \((R, m)\) be a Buchsbaum ring of dimension \(d \geq 3\) such that 

\(H^1_m(R) \neq 0\) and \(H^d_m(R) = 0\) for 

\(n \neq 1, d\). Let \(U_i = ((U_p^i))_{i \geq 1}\) be the chain of triangular subsets on \(R\) in the following Proposition (2.15) (when \(M = R\)). Then by Proposition (2.15) we have 

\(\text{Ker } f^1/\text{Im } f^0 = H^1_m(R) \neq 0\) and \(\text{Ker } f^n/\text{Im } f^{n-1} = H^n_m(R) = 0\) for 

\(n \neq 1, d\). Hence by the short exact sequence (\(*\)) we have 

\((U_p^1 R) = (R^p) = (R^1)_q = (U_q[1]^{-1} R) = (\text{Im } e^0)_q \neq 0\) and 

\(U_1 \cap q = \emptyset\).

But \(q \notin \text{Ass}(R_p)\).

**Corollary (2.12).** Let \(R\) be a Noetherian ring and \(M\) an \(R\)-module. Then we have the following.

(1) \(\text{Ass}(U^{-n-1}_n M) \subset \text{Ass}(U^{-1}_n M)\).

(2) \(\text{Ass}(U^{-n-1}_n M) = \text{Ass}(H^n_U(M)) \cup \text{Ass}(U^{-n-1}_n M)\).

**Proof.** (1) Let \(p \in \text{Ass}_R(U^{-n-1}_n M)\). Then \(pR_p \in \text{Ass}_{R_p}(U^{-n-1}_n M)_p\) by ([M], p. 38 Corollary). Hence \(pR_p \in \text{Ass}_{R_p}(U^{-n-1}_n M)_p\) by Proposition (2.10)(5).
Therefore $p \in \text{Ass}_R(U_n[1]^{-n-1}M)$ again by ([M], p. 38 Corollary).

(2) This follows from (1) and Lemma (2.8)(5).

**Proposition (2.13).** Let $R$ be a ring and $M$ an $R$-module. Fix a non-negative integer $t$. Then in $C(U, M)$, the following four conditions are equivalent.

1. $H^0_n(M) = 0$ for all $n = 0, \ldots, t$.
2. $U_n[1]^{-n-1} M \cong \text{Im } e^n$ for all $n = 0, \ldots, t$.
3. For all $n = 0, \ldots, t$, for each $m \in U_n[1]^{-n-1} M$, \[
\left(0 : \left(\frac{m}{(a_1, \ldots, a_n, 1)}\right) \right) \subset \left(0 : \left(\frac{m}{(a_1, \ldots, a_n)}\right)\right) \text{ where } \left(\frac{m}{(a_1, \ldots, a_n, 1)}\right) \in U_n[1]^{-n-1} M.
\]
4. For all $n = 0, \ldots, t$, each element of $U_{n+1}$ forms a poor $M$-sequence.

In particular, let $R$ be a Noetherian local ring and let $M$ be a finitely generated $R$-module of dimension $d$. Assume that the above conditions hold for $t = d - 1$ and $U_d[1]^{-d-1} M \neq 0$. Then $M$ is a Cohen-Macaulay module.

**Proof.** (1) $\Leftrightarrow$ (2) From the short exact sequence ($\ast$) this is clear.

(2) $\Rightarrow$ (3) By Lemma (1.6)(3) this is obvious.

(3) $\Rightarrow$ (4) We proceed by induction on $n$. In the case $n = 0$, assume that $a_i m = 0$ for some $0 \neq m \in M$ and $(a_i) \in U_1$. Then we have $a_1 \in (0 : m) = \left(0 : \frac{m}{(b_1)}\right)$ for some $\frac{m}{(b_1)} \in U_1^{-1} M$ by the hypothesis. This contradicts Lemma (2.1).

Now suppose that each element of $U_n$ is a poor $M$-sequence. Assume that $a_{n+1} m \in (a_1, \ldots, a_n) M$ for some $(a_i, \ldots, a_{n+1}) \in U_{n+1}$ and $m \in M$. Then by Lemma (1.6)(2) we have $\frac{a_{n+1} m}{(a_1, \ldots, a_{n+1})} = 0$. That is, by ([SZ3], 2.1), we have \[
\left(\frac{m}{(a_1, \ldots, a_{n+1})}\right) = 0 \text{ in } U_{n+1}^{-1} M.
\]

Hence by the hypothesis we have \[
\left(\frac{m}{(a_1, \ldots, a_n, 1)}\right) = 0 \text{ in } U_n[1]^{-n-1} M.
\]

Then, by the definition of module of generalized fractions, there are $(b_1, \ldots, b_n, 1) \in U_n[1]$ and $H \in D_{n+1}(R)$ such that $H[a_1 \ldots a_n 1] = [b_1 \ldots b_n 1]$ and $\text{Im } H : m \in (b_1, \ldots, b_n) M$.
On the other hand, since 
\[ h_{n+1,n+1} = 1 - (h_{n+1,1}a_1 + \cdots + h_{n+1,n}a_n), \]
by ([SZ1], 2.2) we have
\[ h_{11} \cdots h_{nn}m = (b_1, \ldots, b_n)M. \]
Note that by the inductive hypothesis \( b_1, \ldots, b_n \) is a poor \( M \)-sequence and \( H'[a_1, \ldots, a_n] = [b_1, \ldots, b_n]^T \) where \( H' \) is the top left \( n \times n \) submatrix of \( H \). Hence by ([O], 3.2) we get
\[ m \in (a_1, \ldots, a_n)M. \]

(4) \( \Rightarrow \) (1) Let \( \frac{m}{(a_1, \ldots, a_n)} \in \ker e^n \) with \( \frac{m}{(a_0)} = m. \) Then \( \frac{m}{(a_1, \ldots, a_n, 1)} = 0 \) in \( U_{n+1}M \). Hence by Lemma (1.6)(2), we have
\[ m \in (a_1, \ldots, a_n)M. \]
Therefore we have \( \frac{m}{(a_1, \ldots, a_n)} \in \im e^{n-1}. \)

For the last assertion, since \( U_d[1]^{-d-1} M \neq 0 \), there is \( (a_1, \ldots, a_d) \in U_d \) such that \( a_1, \ldots, a_d \) is an \( M \)-sequence.

Remark (2.14). In Proposition (2.13), if \( R \) is Noetherian, then we can change the condition (3) for \( \ass(U_d^{-n-1} M) = \ass(U_d[1]^{-n-1} M) \) for all \( n = 0, \ldots, t. \)

Let \((R, m)\) be a Noetherian local ring and let \( M \) be a finitely generated \( R \)-module of dimension \( d \). Let \( \mathcal{U}_f = ((U_f)_i)_{i \geq 1} \) be the chain of the expansions of triangular subsets (Example (1.3)(5)) on \( R \). Then we have the following complex
\[
0 \to M \xrightarrow{f^0} (U_f)_1^{-1} M \xrightarrow{f^1} (U_f)_2^{-2} M \to \cdots \to (U_f)_d^{-d+1} M \xrightarrow{f^{d+1}} (U_f)_d^{-d} M \to 0,
\]
since \( (U_f)_d^{-d+i} M = 0 \) for all \( i \geq 1 \) by ([C], 2.3).

Proposition (2.15). Let \( R, M \) and \( U_f \) be as above. Then the following four conditions are equivalent.

1. \( M \) is an \( f \)-module (see [SZ4], 1.8 (ii)).
2. \( \ker f^n / \im f^{n-1} \cong H_m^n(M) \) for all \( n = 0, \ldots, d. \)
3. \( \ass((U_f)_n[1]^{-n-1} M) \subset \{m\} \cup \ass((U_f)_n^{-n-1} M) \) for all \( n = 0, \ldots, d. \)
4. \( \supp(\ker f^n / \im f^{n-1}) \subset \{m\} \) for all \( n = 0, \ldots, d. \)

In particular, if \( M \) is a Cohen-Macaulay module, then
\[
\{\ass((U_f)_n[1]^{-n-1} M) = \ass((U_f)_n^{-n-1} M) = F_{M_n} \text{ for all } n < d, \}
\{\ass((U_f)_d[1]^{-d-1} M) = \{m\}. \}
\]
Proof. (1) \Rightarrow (2) In the case $n = 0, \ldots, d - 1$, this follows from ([SZ4], 2.4), since $(U_p)_n = (U_d)_n$. In the case $n = d$, we have

\[ \text{Ker} f^d / \text{Im} f^{d - 1} \cong U_d^{-d} M / \text{Im} f^{d - 1} \cong U_d[1]^{d - 1} M \cong (U_d)^{d - 1} M = H_d M \]

by Lemma (1.9) and Lemma (1.7).

(2) \Rightarrow (3) \iff (4) These follow from Corollary (2.12)(2) and Lemma (2.8)(4).

(4) \Rightarrow (1) This follows from ([SZ4], 2.3).

The last assertion follows from (2), Corollary (2.12)(2) and ([C], 2.15).

3. Modules of generalized fractions and complexes of Cousin type

In this section, suppose that $R$ is a Noetherian ring.

Theorem (3.1). Let $R$ be a Noetherian ring and $M$ an $R$-module. Let $\mathcal{U} = (U_i)_{i \geq 1}$ be a chain of triangular subsets on $R$. Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of $\text{Spec}(R)$ which admits $M$. Then

the complex $C(\mathcal{U}, M)$ is of Cousin type for $M$ with respect to $\mathcal{F}$

\[ \text{Ass}(U_n[1]^{-n-1} M) \cap \partial F_n = \text{Ass}(U_{n+1}^{-n-1} M) \text{ for all } n \geq 0 \text{ and } U_{n+1}^{-n-1} M \cong \bigoplus_{v \in \partial F_n} (U_{n+1}^{-n-1} M)_v \text{ for all } n \geq 0. \]

Proof. (↓) We must verify the properties (i)-(iii) of the definition of Cousin type (see (1.4)).

(i) and (ii) By Remark (2.5) these always hold for arbitrary complexes $C(\mathcal{U}, M)$.

(iii) We must show that $\text{Supp}(H^n_U(M)) \subseteq F_{n+1}$. Note that $\text{Ass}(U_{n+1}^{-n-1} M) = \text{Ass} \left( \bigoplus_{v \in \partial F_n} (U_{n+1}^{-n-1} M)_v \right) \subseteq \partial F_n$ by Lemma (2.4). By Lemma (2.8)(5) and Lemma (2.4), we have $\text{Supp}(H^n_U(M)) \subseteq \text{Supp}(U_{n+1}[1]^{-n-1} M) \subseteq F_n$. But it follows from the hypothesis and Lemma (2.8)(4)(6) that $\partial F_n \cap \text{Supp}(H^n_U(M)) = \emptyset$.

(↑) It is enough to show that the first condition of Theorem holds. By the third and the fourth conditions of the definition of Cousin type, we have $\partial F_n \cap \text{Supp}(H^n_U(M)) = \emptyset$ and $\text{Ass}(U_{n+1}^{-n-1} M) \subseteq \partial F_n$. Hence Lemma (2.8)(6) completes the proof of Theorem.
Corollary (3.2). With the same notation and assumption as in Theorem (3.1), we have the following.

(1) Suppose that \( \partial F_m \cap \partial F_n = \text{Ass}(U_{n+1}^{-n-1} M) \) for all \( n \geq 0 \) and

\[ U_{n+1}^{-n-1} M \cong \bigoplus_{p \in \partial F_n} (U_{n+1}^{-n-1} M)_p \] for all \( n \geq 0 \).

Then the complex \( C(\mathcal{U}, M) \) is of Cousin type for \( M \) with respect to \( \mathcal{F} \).

(2) In particular, assume that \( \partial F_m \cap \partial F_n \subseteq \text{Supp}(U_n[1]^{-n-1} M) \) for all \( n \geq 0 \). Then the converse of (1) is true.

Proof. (1) This follows from Theorem (3.1), since \( \text{Ass}(U_{n+1}^{-n-1} M) \subseteq \text{Ass}(U_n[1]^{-n-1} M) \cap \partial F_n \subseteq \partial F_m \cap \partial F_n \) by Corollary (2.12)(1) and Lemma (2.6)(2).

(2) It is sufficient to show that \( \partial F_m \cap \partial F_n = \text{Ass}(U_{n+1}^{-n-1} M) \), since the second isomorphisms hold by the definition of Cousin type.

\( \implies \) Since \( \text{Ass}(U_{n+1}^{-n-1} M) \subseteq \partial F_n \), it follows from Lemma (2.6)(2) that \( \text{Ass}(U_{n+1}^{-n-1} M) \subseteq \partial F_m \cap \partial F_n \).

\( \subseteq \) We proceed by induction on \( n \). In the case \( n = 0 \), let \( p \in \partial F_m \cap \partial F_n \).

Consider the following complex

\[ 0 \rightarrow M \rightarrow U_1^{-1} M \rightarrow U_2^{-2} M \rightarrow \cdots. \]

Then by the definition of Cousin type, we have the following exact sequence

\[ 0 \rightarrow M_p \rightarrow (U_1^{-1} M)_p \rightarrow 0. \]

Since \( p \in \text{Ass}(M) \), we have \( p \in \text{Ass}(U_1^{-1} M) \) by ([M], p. 38 Corollary).

Suppose that \( n \geq 1 \). Let \( p \in \partial F_m \cap \partial F_n \). Consider the following complex

\[ \cdots \rightarrow U_{n-1}^{-n+1} \rightarrow U_n^{-n} M \rightarrow U_{n+1}^{-n-1} M \rightarrow \cdots. \]

It follows from the definition of Cousin type that we have the following exact sequence

\[ 0 \rightarrow \text{Im}(e)^{-n-1})_p \rightarrow (U_n^{-n} M)_p \rightarrow (U_{n+1}^{-n-1} M)_p \rightarrow 0, \]

since \( \text{Ker}(e)^{-n-1})_p \cong (\text{Im}(e)^{-n-1})_p \). Hence by the inductive hypothesis and Lemma (2.6) (3), we have \( (U_n^{-n} M)_p \neq 0 \). On the other hand, by Proposition (2.10)(2) and the assumption \( \partial F_m \cap \partial F_n \subseteq \text{Supp}(U_n[1]^{-n-1} M) \), we get

\( (\text{Im}(e)^{-n-1})_p \neq (U_n^{-n} M)_p \).
That is \((U_{n+1}^{-n-1}M)_p \neq 0\). Hence we conclude that \(p \in \text{Ass}(U_{n+1}^{-n-1}M)\) by Lemma (2.4).

**Remark (3.3).** Using Lemma (2.6)(2), Lemma (2.8)(6), the third and the fourth conditions of the definition of Cousin type, we have another proof of Corollary (3.2)(2) as follows:

\[
\partial F_{M^n} \cap \partial F_n = \partial F_n \cap \text{Supp}(U_n[1]^{-n-1}M) = \partial F_n \cap \text{Ass}(U_n[1]^{-n-1}M) = (\partial F_n \cap (H^n_U(M))) \cup (\partial F_n \cap \text{Ass}(U_{n+1}^{-n-1}M)) = \partial F_n \cap \text{Ass}(U_{n+1}^{-n-1}M) = \text{Ass}(U_{n+1}^{-n-1}M).
\]

**Remark (3.4).** If \(M\) is a finitely generated \(R\)-module and a complex \(C(\mathcal{U}, M)\) is of Cousin type for \(M\) with respect to \(\mathcal{U}\), then \(\text{Ass}(U_{n+1}^{-n-1}M) = \{p \in \text{Supp}(M) : \text{ht}_M p = n\}\) by ([RSZ], 3.3), ([C], 2.11) and the following Corollary (3.5) (1).

**Corollary (3.5).** Let \(M\) be a finitely generated \(R\)-module of dimension \(d\). Let \(\mathcal{F} = (F_i)_{i \geq 0}\) be a filtration of \(\text{Spec}(R)\) which admits \(M\). Let \(\mathcal{F}_M = (F_{M,i})_{i \geq 0}\) be the \(M\)-height filtration.

1. (cf. [SY], 3.9) \(C(\mathcal{U}_h, M)\) is of Cousin type for \(M\) w. r. t. \(\mathcal{F}_M\), where \(\mathcal{U}_h = ((U_h)_i)_{i \geq 0}\).
2. ([RSZ], 3.4) \(C(\mathcal{U}_k, M)\) is of Cousin type for \(M\) w. r. t. \(\mathcal{F}\), where \(\mathcal{U}_k = ((U_k)_i)_{i \geq 0}\).
3. ([GO], 3.6) Let \(\mathcal{U} = (U_i)_{i \geq 0}\) be a chain of saturated triangular subsets on \(R\). Put \(G_0 = \text{Supp}(M)\) and for \(i \in \mathbb{N}\), define \(G_i = \{p \in \text{Supp}(M) : \text{there exists } (a_1, \ldots, a_i) \in U_i \text{ with } (a_1, \ldots, a_i)R \subset p\}\). Assume that \(G = (G_i)_{i \geq 0}\), induced by \(\mathcal{U}\) and \(M\), is a filtration of \(\text{Spec}(R)\) which admits \(M\). Then \(C(\mathcal{U}, M)\) is of Cousin type for \(M\) w. r. t. \(G\).
4. If \(\dim M = \text{ht}_M q + \dim M/qM\) for all \(q \in \text{Supp}(M)\), then \(C(\mathcal{U}_r, M)\) is of Cousin type for \(M\) w. r. t. \(\mathcal{F}_M\), where \(\mathcal{U}_r = ((U_r)_i)_{i \geq 0}\).
5. Let \(\mathcal{U}_r = ((U_r)_i)_{i \geq 0}\). Then we have the following equivalent conditions.

\(M\) is a Cohen-Macaulay module

\(\iff C(\mathcal{U}_r, M)\) is of Cousin type for \(M\) w. r. t. \(\mathcal{F}_M\)

\(\iff (U_r^{-n-1}M)_{n+1} \cong \bigoplus_{\text{ht}_M p = n} ((U_r^{-n-1}M)_p \text{ for all } n \geq 0.
\)

6. Let \(R\) be a Noetherian local ring. Then

\(M\) is a Gorenstein module

\(\iff C(\mathcal{U}_r, M)\) is of Cousin type for \(M\) w. r. t. \(\mathcal{F}_M\) and

\((U_r)^{-d-1}M\) is an injective \(R\)-module.
\( M = \Theta (U^\mu)^{n-1} M \) for all \( n \geq 0 \), and \( (U_\mu)^{n-1} M \) is an injective \( R \)-module.

**Proof.** (1) This follows from ([C], 2.11 and 3.3(2)) and Corollary (3.2).

(2) By ([RSZ], 2.6 or [C], 3.3(1)), we have for all \( n \in \mathbb{N} \cup \{0\} \)

\[
(U_\mu)^{n-1} M \cong \bigoplus_{p \in \partial F_n} ((U_\mu)^{n-1} M)_p.
\]

Hence by Lemma (2.4) we get

\[
\text{Ass}((U_\mu)^{n-1} M) = \text{Ass}\left( \bigoplus_{p \in \partial F_n} ((U_\mu)^{n-1} M)_p \right) = \bigcup_{p \in \partial F_n} \text{Ass}((U_\mu)^{n-1} M)_p \subseteq \partial F_n.
\]

By Lemma (2.7) and the definition of \((U_\mu)^n\), we have, for all \( p \in \partial F_n \cap \text{Supp}(M), \)

\[
(H^n_\mu(M))_p = 0.
\]

Therefore we have \( \partial F_n \cap \text{Ass}(H^n_\mu(M)) = \emptyset \), since \( \text{Ass}(H^n_\mu(M)) \subseteq \text{Supp}(M) \).

Hence we obtain

\[
\partial F_n \cap \text{Ass}((U_\mu)^{n-1} M) = \partial F_n \cap \text{Ass}((U_\mu)^{n-1} M) = \text{Ass}((U_\mu)^{n-1} M),
\]

by Lemma (2.8)(6). Then Theorem (3.1) completes the proof.

(3) By ([GO], 3.6), we have for all \( n \in \mathbb{N} \cup \{0\} \)

\[
U_{n-1}^{n-1} M \cong \bigoplus_{p \in \partial G_n} (U_{n-1}^{n-1} M)_p.
\]

Hence we get \( \text{Ass}(U_{n-1}^{n-1} M) \subseteq \partial G_n \).

Next for all \( p \in \partial G_n \) we have

\[
(H^n_\mu(M))_p = 0.
\]

In fact, if \((H^n_\mu(M))_p \neq 0\), then there is \( x \in H^n_\mu(M) \) such that \( (0 : x) \subseteq p \). But by Lemma (2.7), we have \((a_1, \ldots, a_{n+1}) R \subseteq (0 : x) \subseteq p \) for some \((a_1, \ldots, a_{n+1}) \in U_{n+1}\). Hence from the definition of \( G_{n+1} \) we have \( p \in G_{n+1} \). This contradicts \( p \in \partial G_n \).

Therefore we have \( \partial G_n \cap \text{Ass}(H^n_\mu(M)) = \emptyset \).

Then by Lemma (2.8)(6) we get

\[
\partial G_n \cap \text{Ass}(U_n[1]^{n-1} M) = (\partial G_n \cap \text{Ass}(H^n_\mu(M))) \cup (\partial G_n \cap \text{Ass}(U_{n+1}^{n-1} M)) = \text{Ass}(U_{n+1}^{n-1} M).
\]
The result follows from Theorem (3.1).

(4) This follows from ([C], 2.12 and 3.3(3)) and Corollary (3.2).

(5) Since \( C(\mathcal{U}, M) \) is an exact sequence by Proposition (2.13), the first equivalence follows from ([S2], 2.4). From Proposition (2.13)(3) and Theorem (3.1), we have the second equivalence.

(6) This follows from (5) and ([S2], 3.11).

**Remark (3.6).** Let \((R, m)\) be a Noetherian ring and let \(M\) be a finitely generated \(R\)-module of dimension \(d\). Then \(C(\mathcal{U}, M)\) is of Cousin type for \(M\) with respect to \(\mathcal{F}_M\) (Corollary (3.5)(4)) but \(C(\mathcal{U}, M)\) is not, even though \((U_j)_{n+1}^{d-1} M \cong \bigoplus_{ht p=n}(U_j)_{n+1}^{d-1} M\) for all \(n \geq 0\) ([C], 3.3(5)). For, by ([C], 2.15), we have \(\text{Ass}((U_j)_{d+1}^{d-1} M) = \emptyset\) but \(\text{Ass}((U_j)_{d+1}^{d-1} M) \cap \partial F_M = \text{Ass}((U_j)_{d+1}^{d-1} M) \cap \partial F_M = \{m\}.\) Hence we have

\[
\text{Ass}((U_j)_{d+1}^{d-1} M) \cap \partial F_M \neq \text{Ass}((U_j)_{d+1}^{d-1} M).
\]

Therefore the result follows from Theorem (3.1).

**Example (3.7).** Let \(R = \mathbb{k}[x, y, z]\). Let \(U_1 = \{(tx^\alpha) \in R^1 : 0 \neq t \in k\}\) and \(\alpha \in N \cup \{0\}\). Let \(U_i = U_{i-1}[1]\) for \(i = 2, 3, \ldots\). Then \(\mathcal{U} = (U_i)_{i \geq 1}\) is a chain of saturated triangular subsets on \(R\). Put \(G_0 = \text{Spec}(R), G_1 = (p \in \text{Spec}(R) : x \in p)\) and \(G_i = \emptyset\) for \(i \geq 2\). Then \(\mathcal{G} = (G_i)_{i \geq 0}\) is induced by \(\mathcal{U}\) and \(M\) as in (3) of Corollary (3.5), but is not a filtration of \(\text{Spec}(R)\). For, \(\partial G_0 = G_0 \setminus G_1 \supset \{(y), (y, z)\}\).

**Example (3.8).** Let \(R = \mathbb{k}[X, Y, Z]/(X) \cap (Y, Z) = \mathbb{k}[x, y, z]\). Then \(R\) is not equidimensional and \(\{(x), (y, z)\} = \partial F_{R0} \cap \text{Spec}((U_j)_{d+1}^{d-1} R) \subset \text{Ass}((U_j)_{d+1}^{d-1} R) = \{(x)\}\). Hence \(C(\mathcal{U}, R)\) is not of Cousin type for \(R\) w. r. t. \(\mathcal{F}_R\). In fact, \(k\langle y, z \rangle \times k\langle x \rangle \cong (U_j)_{d+1}^{d-1} R \not\cong (U_j)_{d+1}^{d-1} R \cong k\langle y, z \rangle\) (cf. Corollary (3.5)(1)(4)).

**Example (3.9).** Let \(R = \mathbb{k}[x, y]\). Let \(U_1 = \{(x^\alpha) \in R^1 : \alpha \in N \cup \{0\}\}\) and \(U_n = \{(x^\alpha, 1, \ldots, 1) \in \mathbb{R}^n : \alpha \in N \cup \{0\}\}\) for \(n \geq 2\). Then we have \(\text{Ass}(U_1^{-1} R) = \{(0)\} = \partial F_{R0} \cap \text{Supp}(U_0[1]^{-1} R), \text{Ass}(U_2^{-2} R) = \{(x)\} = \partial F_{R1} \cap \text{Supp}(U_2^{-2} R) = \partial F_{R1} \cap \text{Ass}(U_1[1]^{-2} R)\) and \(U_i^{-1} R = 0\) for all \(i \geq 3\). But \(U_i^{-1} R \not\cong (U_2^{-2} R)_x\).

**Example (3.10).** Let \(R = \mathbb{k}[X, Y, Z]\) and \(M = k[[X, Y, Z]]/(X) \cap (X^2, Y) = k[[x, y, z]]\). Let \(U_1 = \{(y^\alpha) \in R^1 : \alpha \in N \cup \{0\}\}\). Let \(F_i = \{p \in \text{Spec}(R) : \text{ht } p \geq\)
$i + 1$) for $i \geq 0$. Then $\text{Ass}(U_i^{-1}M) = (\mathcal{X}) = \partial F_0 \cap \text{Ass}(M) = \partial F_0 \cap \text{Ass}(U_0[1]^{-1}M)$ but $M_T \cong U_i^{-1}M \not\cong (U_i^{-1}M)_{(X)} \cong M_{(X)}$.

REFERENCES