Dynamical Evolution of Dense Stellar Systems
Proceedings IAU Symposium No. 246, 2007
E. Vesperini, M. Giersz & A. Sills, eds.

doi:10.1017/S1743921308015615

Resonance, Chaos and Stability in the General Three-Body Problem

R. A. Mardling

School of Mathematical Sciences, Monash University, Victoria, 3800, Australia
email: mardling@sci.monash.edu.au

Abstract. Three-body stability is fundamental to astrophysical processes on all length and mass scales from planetary systems to clusters of galaxies, so it is vital we have a deep and thorough understanding of this centuries-old problem. Here we summarize an analytical method for determining the stability of arbitrary three-body hierarchies which makes use of the chaos theory concept of *resonance overlap*. For the first time the dependence on *all* orbital elements and masses can be given explicitly via simple analytical expressions which contain no empirical parameters. For clarity and brevity, analysis in this paper is restricted to coplanar systems including a description of a practical algorithm for use in N-body and other applications. A Fortran routine for arbitrarily inclined systems is available from the author, and animations of stable and unstable systems are available at `www.maths.monash.edu.au/~ro/Capri`.

Keywords. gravitation, instabilities, methods: n-body simulations, planetary systems, globular clusters: general, binaries (including multiple): close

---

1. Introduction

Most stable hierarchical triples are characterized by the following behaviour: (1) no energy exchange between the inner and outer orbits; (2) slow cyclic evolution of the eccentricities associated with angular momentum exchange between the orbits (except for coplanar systems in which the two inner bodies have the same mass); (3) apsidal advance of both orbits; and (4) nutation and precession of the orbital planes for inclined systems. The exceptions to (1) are those systems which are in resonance in which case a slow, cyclic exchange of energy between the orbits pertains.

In contrast, unstable hierarchies, defined here as those in which one body escapes the system (so-called *Lagrange* instability), by necessity involve substantial exchange of energy between the orbits. They therefore must involve orbital resonances, and the difference between a stable and an unstable resonant system is that the latter involves more than one resonance.† This, for example, explains why it is that Neptune and Pluto can exist in a stable resonance (in particular, the 3:2 resonance), while the overlap of two or more resonances explains the *absence* of orbits at some positions in the asteroid belt which are resonant with Jupiter (Murray & Dermott 2000).

It has been known since the 1960s (Walker & Ford 1969) that the overlap of neighbouring resonances is a diagnostic of chaotic behaviour (the so-called *resonance overlap stability criterion*, greatly expanded upon in Chirikov 1979). It is a consequence of the famous KAM theorem (Kolmogorov 1954), itself an outgrowth of Poincare's work on the restricted three-body problem early last century. The first application of this criterion to the restricted problem was performed by Wisdom (1980) who used a Hamiltonian

† More accurately, a stable system may involve more than one resonance if they are, in some sense, linearly superposed (Mardling 2008a, *in preparation*).

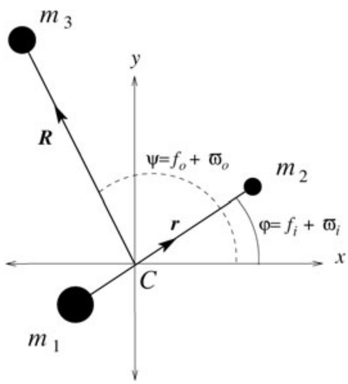

Figure 1. Coordinate system for an aligned coplanar system. The point C (the origin) corresponds to the centre of mass of m_1 and m_2.

formulation, setting the stage for the present application to the general problem (although here we are inspired by the simpler analysis outlined in Murray & Dermott 2000).

Thus the main task in determining the stability or otherwise of a given hierarchical configuration is that of identifying the widths of orbital mean-motion resonances and where they overlap. These resonances are nonlinear, and are defined by the pendulum-like libration (as opposed to circulation) of the so-called *resonance angles*, linear combinations of all the angles appearing in the problem which occur naturally in a Fourier expansion of the disturbing function, that is, the interaction term which couples the inner and outer orbits.

This paper is a summary of Mardling (2008a, *in preparation*).

2. The disturbing function

Using Jacobi coordinates (Fig. 1), the equations of motion for the inner and outer orbits of a hierarchical triple are

$$\mu_i \ddot{\mathbf{r}} + \frac{G m_1 m_2}{r^2}\hat{\mathbf{r}} = \frac{\partial \mathcal{R}}{\partial \mathbf{r}} \tag{2.1}$$

and

$$\mu_o \ddot{\mathbf{R}} + \frac{G m_{12} m_3}{R^2}\hat{\mathbf{R}} = \frac{\partial \mathcal{R}}{\partial \mathbf{R}} \tag{2.2}$$

respectively, where $\mu_i = m_1 m_2/m_{12}$ and $\mu_o = m_{12} m_3/m_{123}$ are the reduced masses associated with the inner and other orbits with $m_{12} = m_1 + m_2$ and $m_{123} = m_{12} + m_3$, and

$$\mathcal{R} = -\frac{G m_{12} m_3}{R} + \frac{G m_2 m_3}{|\mathbf{R} - \alpha_1 \mathbf{r}|} + \frac{G m_1 m_3}{|\mathbf{R} + \alpha_2 \mathbf{r}|} \tag{2.3}$$

is the disturbing function which here has the dimensions of energy (in the study of the restricted three-body problem it has the dimensions of energy per unit mass). Here $\alpha_i = m_i/m_{12}$, $i = 1, 2$. Since the orbits interact via the disturbing function, it contains all the information about energy and angular momentum exchange and hence stability or otherwise of the system. We therefore focus attention exclusively on $\mathcal{R}$, our aim being to write it in a form which reveals in a simple way the explicit dependence of stability on all the orbital parameters. In particular these are $\{a_i, e_i, I_i, \varpi_i, \Omega_i, M_i\}$, that is, the inner semi major axis, eccentricity, inclination, longitudes of periastron and ascending nodes and mean anomaly respectively, with $\{a_o, e_o, I_o, \varpi_o, \Omega_o, M_o\}$ the corresponding elements

https://doi.org/10.1017/S1743921308015615 Published online by Cambridge University Press

for the outer orbit. This is best done using spherical harmonics (rather than Legendre polynomials, eg., Roy & Haddow 2003) because, together with the use of Euler angles, they allow one to explicitly separate the dependence on the inner and outer elements. Thus (2.3) becomes

$$\mathcal{R} = G\mu_i m_3 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left(\frac{4\pi}{2l+1}\right) \mathcal{M}_l \left(\frac{r^l}{R^{l+1}}\right) Y_{lm}(\theta,\varphi) Y^*_{lm}(\Theta,\psi), \tag{2.4}$$

where r is the distance between bodies 1 and 2, $\alpha_1 r$, θ and φ, and R, Θ and ψ are the spherical polar coordinates of bodies 2 and 3 respectively relative to a fixed (non-inertial) coordinate frame with origin at the centre of mass of bodies 1 and 2 (Fig. 1), Y_{lm} is a spherical harmonic defined as in Jackson (1975), and the mass factor $\mathcal{M}_l$ is given by

$$\mathcal{M}_l = \frac{m_1^{l-1} + (-1)^l m_2^{l-1}}{m_{12}^{l-1}} \tag{2.5}$$

so that $\mathcal{M}_2 = 1$ for any masses while $\mathcal{M}_l = 0$ when l is odd and $m_1 = m_2$. Our intention is to expand $\mathcal{R}$ in a double Fourier series with basis frequencies ν_i and ν_o, the inner and outer orbital frequencies respectively. To clearly demonstrate how the formulation works, we assume that the system is coplanar and choose the coordinate system to be such that all three bodies lie in the $x - y$ plane. Then $\theta = \Theta = \pi/2$, $\varphi = f_i + \varpi_i$ and $\psi = f_o + \varpi_o$, where f_i and f_o are the inner and outer true anomalies, that is, the angular positions of bodies 2 and 3 relative to their periastron directions. The disturbing function then becomes

$$\mathcal{R} = G\mu_i m_3 \sum_{l=2}^{\infty} \sum_{m=-l,2}^{l} c_{lm}^2 \mathcal{M}_l \, e^{im(\varpi_i - \varpi_o)} \left(r^l \, e^{imf_i}\right) \left(\frac{e^{-imf_o}}{R^{l+1}}\right), \tag{2.6}$$

where

$$c_{lm}^2 = \frac{4\pi}{2l+1} \left[Y_{lm}(\pi/2, 0)\right]^2 \tag{2.7}$$

and the sum over m is in steps of two for the coplanar case. This paper will involve quadrupole $l = 2$, $m = -2, 2$ terms only with the relevant value of c_{lm}^2 being $c_{22}^2 = 3/8$. For uncoupled orbits, the quantities in brackets in (2.6) are periodic with period $2\pi/\nu_i$ and $2\pi/\nu_o$ respectively and so can be expanded in individual Fourier series such that

$$(r/a_i)^l e^{imf_i} = \sum_{n'=-\infty}^{\infty} s_{n'}^{(lm)}(e_i) e^{in'M_i}, \tag{2.8}$$

and

$$\frac{e^{-imf_o}}{(R/a_o)^{l+1}} = \sum_{n=-\infty}^{\infty} F_n^{(lm)}(e_o) e^{-inM_o}, \tag{2.9}$$

where the mean anomalies are related to the orbital frequencies by

$$M_i = \int \nu_i(t)\, dt + \epsilon_i - \varpi_i \quad \text{and} \quad M_o = \int \nu_o(t)\, dt + \epsilon_o - \varpi_o. \tag{2.10}$$

Here ϵ_i and ϵ_o are the mean longitudes at epoch of the inner and outer orbits respectively (eg. Murray & Dermott 2000). This definition of the mean anomaly takes into account the fact that the semi major axes and hence the orbital frequencies vary with time when a system is in resonance (equations (3.2) and (3.3); Brouwer & Clements 1961 p. 286).

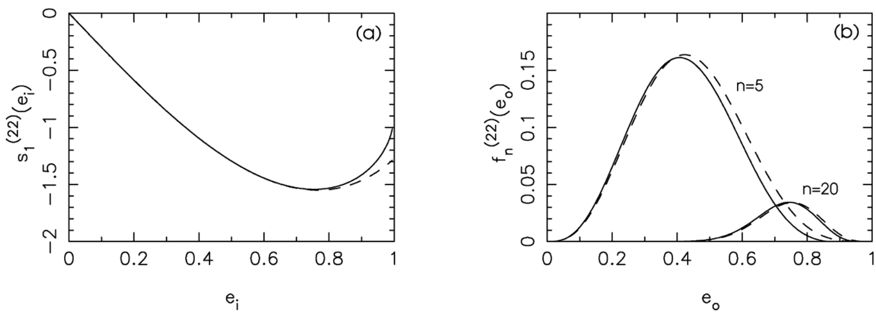

Figure 2. The eccentricity functions (a) $s_1^{(22)}(e_i)$ and (b) $f_n^{(22)}(e_o) \equiv F_n^{(22)}(e_o) \cdot (1-e_o)^{l+1}$, $n = 5, 20$ (solid curves) together with their approximations (dashed curves) (2.13) and (2.14). The values of n were chosen to illustrate the general behaviour of $f_n^{(22)}(e_o)$ as n increases, and also to indicate that the approximation improves with increasing n.

Table 1. Data for scaling function (2.15).

	α_{22}	β_{22}	γ_{22}
$n \leqslant 9$	1.046	0.891	0.097
$n \geqslant 10$	0.448	0.134	2.4×10^{-4}

The Fourier coefficients

$$s_{n'}^{(lm)}(e_i) = \frac{1}{2\pi}\int_0^{2\pi} (r/a_i)^l e^{imf_i} e^{-in'M_i}\, dM_i \tag{2.11}$$

and

$$F_n^{(lm)}(e_o) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-imf_o}}{(R/a_o)^{l+1}} e^{inM_o}\, dM_o \tag{2.12}$$

give the dependence of the disturbing function on the inner and outer eccentricities and may be approximated respectively by Taylor series and by asymptotic expansions (Mardling 2008a, *in preparation*). To eighth-order in e_i, the relevant function of the inner eccentricity for this paper is

$$s_1^{(22)}(e_i) \simeq -3e_i + \tfrac{13}{8}e_i^3 + \tfrac{5}{192}e_i^5 - \tfrac{227}{3072}e_i^7. \tag{2.13}$$

In general, the leading term of such an expansion is $\mathcal{O}(e_i^{|m-n'|})$. This approximation is plotted in Fig. 2a (dashed curve) together with its numerically integrated "exact" solution (solid curve). The approximation diverges slightly for $e_i \gtrsim 0.8$. The asymptotic expression for $F_n^{(lm)}(e_o)$ is (Mardling 2008a, *in preparation*)

$$F_n^{(lm)}(e_o) \simeq s_{lmn} \cdot \frac{2^m/\sqrt{2\pi}}{(l+m-1)!!}\, \frac{(1-e_o^2)^{(3m-l-1)/4}}{e_o^m} n^{(l+m-1)/2} e^{-n\xi(e_o)}, \tag{2.14}$$

where $\xi(e_o) = \mathrm{Cosh}^{-1}(1/e_o) - \sqrt{1-e_o^2}$ and s_{lmn} is an empirical scaling factor designed to match the amplitudes of the exact and asymptotic expressions. It is given by

$$s_{lmn} = 1 - \alpha_{lm} n^{-\beta_{lm}} \exp(\gamma_{lm} n), \tag{2.15}$$

where the relevant fitting constants for this paper are given in Table 1. Note that $\lim_{e_o \to 0} F_n^{(lm)}(e_o) = 1$ when $n = 2$ and is zero otherwise. Note also that $\lim_{e_o \to 1} (1-e_o)^{l+1} F_n^{(lm)}(e_o)$ is finite (Fig. 2) so that $\lim_{e_o \to 1} F_n^{(lm)}(e_o)$ is infinite, ensuring resonance overlap and hence

https://doi.org/10.1017/S1743921308015615 Published online by Cambridge University Press

instability for all configurations as $e_o \to 1$ (see next Section). Fig. 2b plots the numerically integrated "exact" function $f_n^{(lm)}(e_o) \equiv (1-e_o)^{l+1} F_n^{(lm)}(e_o)$ (solid curves) together with the scaled asymptotic version (dashed curves) for the cases $n = 5$ and 20. Since the scale factor is $\mathcal{O}(1)$ ($0.3 \lesssim s_{lmn} \lesssim 0.78$ for $2 \leqslant n \leqslant 1000$), the resonance widths (next Section) are not very sensitive to it and it can be omitted from the asymptotic expression (2.14). Note that (2.14) is closely related to Heggie's analysis of energy exchange during binary flybys (Heggie 1975).

Substituting (2.8) and (2.9) into (2.6) gives

$$\mathcal{R} = G\mu_i m_3 \sum_{l=2}^{\infty} \sum_{m=-l,2}^{l} \sum_{n'=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{lm}^2 \mathcal{M}_l \left(\frac{a_i^l}{a_o^{l+1}}\right) s_{n'}^{(lm)}(e_i) F_n^{(lm)}(e_o) \exp[i\phi_{mnn'}] \tag{2.16}$$

$$= 2\, G\mu_i m_3 \sum_{L0} \zeta_m c_{lm}^2 \mathcal{M}_l \left(\frac{a_i^l}{a_o^{l+1}}\right) s_{n'}^{(lm)}(e_i) F_n^{(lm)}(e_o) \, \cos\phi_{mnn'} \tag{2.17}$$

where

$$\phi_{mnn'} = n' M_i - n M_o + m(\varpi_i - \varpi_o) \tag{2.18}$$

is a *resonance angle*,

$$\sum_{L0} \equiv \sum_{l=2}^{\infty} \sum_{m=m_{min},2}^{l} \sum_{n'=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}, \tag{2.19}$$

$$\zeta_m = \begin{cases} 1/2, & m = 0 \\ 1, & \text{otherwise} \end{cases} \qquad \text{and} \qquad m_{min} = \begin{cases} 0, & l \text{ even} \\ 1, & l \text{ odd.} \end{cases} \tag{2.20}$$

In going from (2.16) to (2.17) we have used the fact that $s_{n'}^{(lm)}$ and $F_n^{(lm)}$ are real so that $s_{n'}^{(lm)^*} = s_{n'}^{(lm)}$ and $F_n^{(lm)^*} = F_n^{(lm)}$ and consequently, $s_{-n'}^{(l-m)} = s_{n'}^{(lm)}$ and $F_{-n}^{(l-m)} = F_n^{(lm)}$, and have grouped together terms with the same value of $|m|$ (thus the factor 1/2 in the definition of ζ_m).

3. Pendulum-like behaviour of the resonance angle

We now have the disturbing function expressed in such a way that the dependence on all the orbital elements is evident. In particular, all angles (orbital phases and longitudes of periasta) appear in various linear combinations in the resonance angles. For most configurations (in particular, non-resonant stable configurations), all resonance angles cycle rapidly through all angles with a frequency dominated by the inner orbital frequency. The main consequence of this is that no energy is exchanged *on average* between the inner and outer orbits. Energy is exchanged during outer periastron passage, but this is "returned" as the system moves away from periastron. This behaviour is demonstrated in an animation available at `www.maths.monash.edu.au/~ro/Capri`. However, for some configurations, one or more resonance angles *librate* between two fixed values, and this results in substantial permanent energy exchange between the orbits, except when a system is *exactly* in resonance which occurs when $\dot\phi_{mnn'} = 0$ for some $\{mnn'\}$. Permanent energy exchange in an unstable system is demonstrated at the above website.

A system is defined to be in resonance if a resonance angles librates, and is unstable if it resides in two overlapping resonances. Hence the task now is to identify the resonances and determine which configurations reside in two or more. This can be done by examining the behaviour of the resonance angles which, not surprisingly, satisfy the equation of

https://doi.org/10.1017/S1743921308015615 Published online by Cambridge University Press

motion of a pendulum. Referring to the definition of a resonance angle, (2.18), we label a resonance with the notation $[n:n'](m)$. In particular, resonances with $m=0$ or $m=2$ are referred to as *quadrupole* resonances while those with $m=1$ or $m=3$ are *octopole* resonances.

Starting with the definition (2.18) as well as (2.10) for the mean anomalies, we have

$$\begin{aligned}\ddot{\phi}_{mnn'} &= n'\dot{\nu}_i - n\dot{\nu}_o + n'\ddot{\epsilon}_i - n\ddot{\epsilon}_o + (m-n')\ddot{\varpi}_i - (m-n)\ddot{\varpi}_o \\ &\simeq n'\dot{\nu}_i - n\dot{\nu}_o. \end{aligned} \tag{3.1}$$

Neglecting the second time derivatives of the longitudes is valid as long as the eccentricities are not vanishingly small. For systems near the stability boundary, this is only ever possible in the case of extreme mass ratios since eccentricity is always induced otherwise (Mardling 2008a, *in preparation*). To proceed we note that $\dot{\nu}_i/\nu_i = -\frac{3}{2}\dot{a}_i/a_i$ and similarly for $\dot{\nu}_o$, and use Lagrange's planetary equation for the rate of change of the semi major axis (Brouwer & Clements 1961):

$$\begin{aligned}\frac{1}{a_i}\frac{da_i}{dt} &= \frac{2}{\mu_i\nu_i a_i^2}\frac{\partial\mathcal{R}}{\partial\lambda_i} = \frac{2}{\mu_i\nu_i a_i^2}\frac{\partial\mathcal{R}}{\partial M_i} \\ &= -4\,\nu_i\left(\frac{m_3}{m_{12}}\right)\sum_{L0} n'\,\zeta_m\,c_{lm}^2\,\mathcal{M}_l\left(\frac{a_i}{a_o}\right)^{l+1} s_{n'}^{(lm)}(e_i)F_n^{(lm)}(e_o)\,\sin\left(\phi_{mnn'}\right)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\frac{1}{a_o}\frac{da_o}{dt} &= \frac{2}{\mu_o\nu_o a_o^2}\frac{\partial\mathcal{R}}{\partial\lambda_o} = \frac{2}{\mu_o\nu_o a_o^2}\frac{\partial\mathcal{R}}{\partial M_o} \\ &= 4\,\nu_o\left(\frac{m_1 m_2}{m_{12}^2}\right)\sum_{L0} n\,\zeta_m\,c_{lm}^2\,\mathcal{M}_l\left(\frac{a_i}{a_o}\right)^{l} s_{n'}^{(lm)}(e_i)F_n^{(lm)}(e_o)\,\sin\left(\phi_{mnn'}\right),\end{aligned} \tag{3.3}$$

where $\lambda_{i,o} = M_{i,o} + \varpi_{i,o}$. The ratio of orbital frequencies is the most fundamental quantity in this problem. Putting

$$\sigma = \frac{\nu_i}{\nu_o} = \left[\left(\frac{m_{12}}{m_{123}}\right)\left(\frac{a_o}{a_i}\right)^3\right]^{1/2} \tag{3.4}$$

and using this to write a_i/a_o in terms of σ, then combining (3.1), (3.2) and (3.3), gives a pendulum equation for the evolution of the $[n:n'](m)$ resonance

$$\ddot{\phi}_{mnn'} = -{n'}^2\nu_o^2\mathcal{A}_{mnn'}\sin\left(\phi_{mnn'}\right), \tag{3.5}$$

where

$$\begin{aligned}\mathcal{A}_{mnn'} &\equiv -6\,\zeta_m\sum_{l=l_{min},2}^{\infty} c_{lm}^2 s_{n'}^{(lm)}(e_i)F_n^{(lm)}(e_o)\left[M_i^{(l)}\,\sigma^{-(2l-4)/3} + M_o^{(l)}(n/n')^2\sigma^{-2l/3}\right] \\ &\simeq -6\,\zeta_m\,c_{lm}^2\,s_{n'}^{(lm)}(e_i)F_n^{(lm)}(e_o)\,\sigma^{-(2l-4)/3}\left[M_i^{(l)} + M_o^{(l)}\sigma^{2/3}\right],\end{aligned} \tag{3.6}$$

and we have put $n/n' \simeq \sigma$ in the last step. Here we have assumed that the only term contributing to the variation of $\phi_{mnn'}$ is the one depending on $\phi_{mnn'}$. Except for low-order resonances with relatively low values of σ, it is also adequate to include only the lowest value of l in (3.6). Note that $l_{min}=2$ if m is even and $l_{min}=3$ if m is odd. The dependence on the masses is through the functions

$$M_i^{(l)} = \mathcal{M}_l\left(\frac{m_3}{m_{12}}\right)\left(\frac{m_{12}}{m_{123}}\right)^{(l+1)/3} \quad \text{and} \quad M_o^{(l)} = \mathcal{M}_l\left(\frac{m_1 m_2}{m_{12}^2}\right)\left(\frac{m_{12}}{m_{123}}\right)^{l/3}. \tag{3.7}$$

Taking $\mathcal{A}_{mnn'}$ to be approximately constant over one libration period, (3.5) can be integrated once to give

$$\tfrac{1}{2}\dot{\phi}^2_{mnn'} - n'^2\nu_o^2\mathcal{A}_{mnn'}\cos\phi_{mnn'} = \text{constant}. \tag{3.8}$$

From (3.5) we see that if $\mathcal{A}_{mnn'} > 0$ libration is about $\phi_{mnn'} = 0$, while for $\mathcal{A}_{mnn'} < 0$ it is about $\phi_{mnn'} = \pi$. The separatrix is the solution curve which contains the hyperbolic fixed point $(\phi_{mnn'}, \dot{\phi}_{mnn'}) = (\pi, 0)$ for $\mathcal{A}_{mnn'} > 0$ and $(0, 0)$ for $\mathcal{A}_{mnn'} < 0$. For most systems of interest in this paper, $\mathcal{A}_{mnn'} > 0$. Therefore we define the quantity

$$\mathcal{E}_{mnn'} = \tfrac{1}{2}\dot{\phi}^2_{mnn'} - n'^2\nu_o^2\mathcal{A}_{mnn'}(1+\cos\phi_{mnn'}), \tag{3.9}$$

so that the separatrix corresponds to $\mathcal{E}_{mnn'} = 0$ and is given by

$$\dot{\phi}_{mnn'} = \pm n'\nu_o\sqrt{2\mathcal{A}_{mnn'}(1+\cos\phi_{mnn'})} = \pm 2\,n'\nu_o\sqrt{\mathcal{A}_{mnn'}}\cos\left(\frac{\phi_{mnn'}}{2}\right). \tag{3.10}$$

Solutions are libratory when $\mathcal{E}_{mnn'} < 0$ and circulatory when $\mathcal{E}_{mnn'} > 0$. From (2.18) we have that

$$\dot{\phi}_{mnn'} \simeq n'\nu_i - n\nu_o = n'\nu_o(\sigma - n/n') \equiv n'\nu_o\,\delta\sigma_{nn'}, \tag{3.11}$$

where we have neglected the contributions from the rates of change of the longitudes (since generally $\dot{\varpi}_{i,o} \ll \nu_o$). Thus we define the auxiliary quantity

$$\overline{\mathcal{E}}_{mnn'} = \tfrac{1}{2}(\delta\sigma_{nn'})^2 - \mathcal{A}_{mnn'}(1+\cos\phi_{mnn'}), \tag{3.12}$$

which again indicates the libratory or circulatory nature of a system according to whether $\overline{\mathcal{E}}_{mnn'} < 0$ or $\overline{\mathcal{E}}_{mnn'} > 0$ respectively. Exact resonance corresponds to $(\phi_{mnn'}, \delta\sigma_{nn'}) = (0, 0)$.

Since a system is defined to be in resonance when a resonance angle librates, from (3.10) and (3.11) the resonance half-width is defined to be

$$\Delta\sigma_{mnn'} = 2\sqrt{\mathcal{A}_{mnn'}}. \tag{3.13}$$

Since the expression for $\mathcal{A}_{mnn'}$, (3.6), depends on the orbital parameters through simple functions, it becomes easy to determine the stability or otherwise of any given system. However, formally a system potentially resides in infinitely many resonances so our next task is to determine which resonances govern the stability of a system.

4. The $[n:1](2)$ resonances

The first thing one must do to determine whether or not a system is in resonance is to check which exact resonances it is near. Since the rational numbers are dense on the number line, a system is always inside some resonance. However, as we have shown, resonance widths are proportional to $e^{-n\xi(e_o)}$ through the functions $F_n^{(lm)}(e_o)$, where $n \simeq n'\sigma$ so that for a given σ, $n \propto n'$. Thus the $[n:1](m)$ resonances always dominate. Moreover, the dependence of the resonance widths on n and $\sigma \simeq n$ together with their dependence on the mass ratios ensures that, for coplanar systems, the quadrupole $m = 2$ resonances are widest. Thus the $[n:1](2)$ resonances determine the stability of most coplanar configurations except planetary-like systems for which *both* m_2 and m_3 are less than around $0.01m_1$. In the latter case resonances with $n' > 1$, that is, resonances which sit between the $[n:1](2)$ resonances (see Fig. 3) are important (Mardling 2008a, *in preparation*) because the $[n:1](2)$ widths are limited by the mass ratio dependence in (3.6).

https://doi.org/10.1017/S1743921308015615 Published online by Cambridge University Press

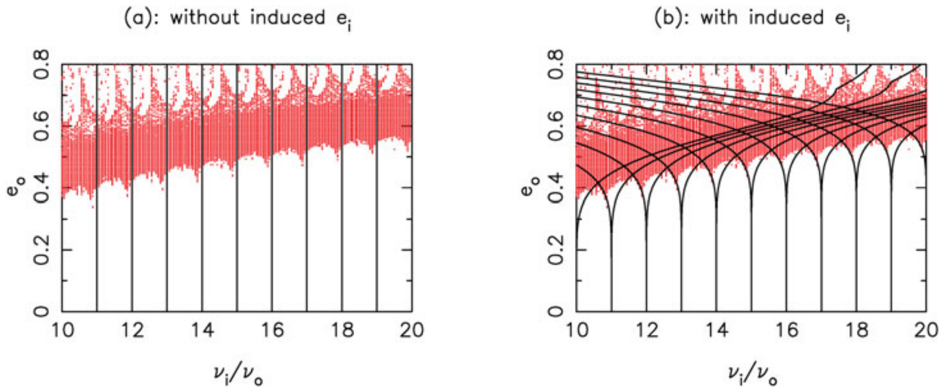

Figure 3. An illustration of the importance of including the induced inner eccentricity. Here the masses are equal and $e_i(0) = 0$. The curves are the resonance boundaries given by (3.6): the resonance widths are zero in (a) for $e_i = 0$. The resonance overlap stability criterion correctly predicts the stability boundary in (b) when $e_i = e_i^{(ind)}$ (eqn (5.1)).

5. Induced eccentricity and secular effects

Consider an equal mass coplanar configuration for which the initial inner eccentricity is zero. According to (3.13), (3.6) and (2.13) the resonance widths should be zero. Fig. 3a is a stability plot for equal mass configurations with initially circular inner binaries, for various initial period ratios and outer eccentricities. A dot corresponding to the initial conditions is plotted if a direct numerical integration of the three-body equations of motion results in an unstable system. Rather than integrating the system until one of the bodies escapes, two almost identical systems (the given system and its "ghost") are integrated in parallel and the difference in the inner semi major axes at outer apastron is monitored (because this variable is approximately constant for non-resonant systems). Taking advantage of the sensitivity of a chaotic system to initial conditions, this difference will grow in proportion to the initial difference between two systems (10^{-7} in the inner eccentricity) for a stable system, but will grow exponentially for an unstable system (Mardling 2001). The stability boundary should correspond to points where neighbouring resonances overlap. Clearly zero resonance width is incorrect! However, if one uses the inner eccentricity *induced* after the first outer periastron passage, resonance overlap correctly predicts the stability boundary (Fig. 3b).

If the initial inner eccentricity is $e_i(0)$, the inner eccentricity following outer periastron passage, $e_i^{(ind)}$, is given approximately by (Mardling 2008a, *in preparation*)

$$e_i^{(ind)} = \left[e_i(0)^2 - 2\beta_n e_i(0)\sin(\phi_{2n1}) + \beta_n^2\right]^{1/2}, \tag{5.1}$$

where

$$\beta_n = \tfrac{9}{2}\pi(m_3/m_{123})f_n^{(22)}(e_o)/n. \tag{5.2}$$

5.1. *Octopole variations for coplanar systems*

For systems with $m_1 \neq m_2$, secular octopole contributions to the disturbing function (ie., terms with $n = n' = 0$) can cause the inner eccentricity to vary considerably on timescales of thousands of inner orbits (Murray & Dermott 2000; Mardling 2008b, *in preparation*). This is especially important for close planetary systems. While the outer eccentricity also varies, the main effect on the resonance widths comes from the variation of $s_1^{(22)}(e_i)$ which is a maximum at the maximum of the octopole cycle in e_i. Referring

https://doi.org/10.1017/S1743921308015615 Published online by Cambridge University Press

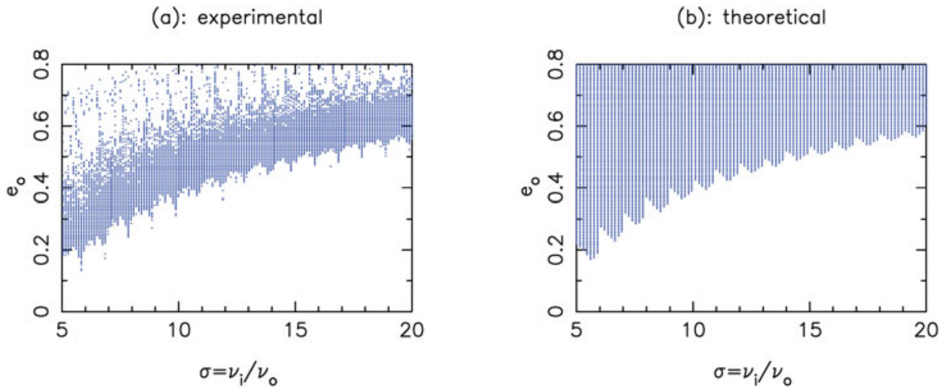

Figure 4. Comparison of (a) experimental and (b) theoretical data for equal mass coplanar systems with $e_i(0) = 0$. The structure at the top of (a) is a result of exchanges being deemed stable (the "ghost" orbit behaves almost identically before rapid escape).

to this maximum as $e_i^{(oct)}$, it is given approximately by (Mardling 2007)

$$e_i^{(oct)} = \begin{cases} (1+A)e_i^{(eq)}, & A \leqslant 1, \\ e_i(0) + 2e_i^{(eq)}, & A > 1, \end{cases} \tag{5.3}$$

where $A = |1 - e_i(0)/e_i^{(eq)}|$ and $e_i^{(eq)}$ is the "equilibrium" or "fixed point" eccentricity

$$e_i^{(eq)} = \frac{(5/4)(a_i/a_o)\, e_o/(1-e_o^2)}{\left|1 - \sqrt{a_i/a_o}(m_2/m_3)/\sqrt{1-e_o^2}\right|}. \tag{5.4}$$

6. A stability algorithm

Here we summarize the steps one can follow to implement the resonance overlap stability criterion for coplanar systems for which the $[n:1](2)$ resonances alone determine the stability. This scheme is valid for coplanar systems with *both* $m_2/m_1 > 0.01$ and $m_3/m_1 > 0.01$, <u>or</u>, for systems with *at least one of* $m_2/m_1 > 0.05$ or $m_3/m_1 > 0.05$.
(1) Identify which $[n:1](2)$ resonance the system is near and calculate the distance $\delta\sigma_n$ from that resonance: $\delta\sigma_n = \sigma - n$, where $n = \lfloor\sigma\rfloor$ (the nearest integer for which $n \leqslant \sigma$);
(2) Take the associated resonance angle to be zero rather than the definition (2.18) (see discussion below): $\phi_{2n1} = 0$;
(3) Calculate the induced eccentricity from (5.1) and (if $m_1 \neq m_2$) the maximum octopole eccentricity from (5.3). Determine $e_i = \max[e_i^{(ind)}, e_i^{(oct)}]$ for use in $s_1^{(22)}(e_i)$;
(4) Calculate $\mathcal{A}_{2n1}$ from (3.6);
(5) Calculate $\overline{\mathcal{E}}_{2n1}$ and $\overline{\mathcal{E}}_{2\,n+1\,1}$: deem the system unstable if $\overline{\mathcal{E}}_{2n1} < 0$ *and* $\overline{\mathcal{E}}_{2\,n+1\,1} < 0$.
Fig. 4 compares the experimental data shown in Fig. 3 with data generated using the algorithm above. A dot is plotted if a system is deemed to be unstable. The boundary structure is reproduced reasonably well, although the boundary itself should be slightly lower, a result of the fact that the resonance overlap criterion does not recognize the unstable nature of points near to but outside the separatrix.

A Fortran routine for arbitrarily inclined systems is available from the author.

7. Summary and Discussion

A stability criterion for the coplanar general three-body problem has been presented which involves *no empirical parameters*. While previous studies have concentrated on the coplanar circular restricted three-body problem, we can now see how stability works for all mass ratios and eccentricities through a simple transparent expression for the resonance widths. The $[n:1](2)$ resonances determine stability for most configurations, while $[n:n'](2)$, $n'>2$ resonances are important for two-planet-type configurations. The latter is a result of the dependence of the resonance widths on the mass functions (3.7).

We have also shown that it is vital to include the induced inner eccentricity as well as the maximum inner eccentricity achieved in an octopole cycle when $m_1 \neq m_2$. We note here that inclined systems also require a knowledge of the maximum eccentricity achieved in a *Kozai* cycle (Mardling 2008a, *in preparation*).

Given the simple functional form of (3.6), it is possible to determine which systems have similar stability properties. For example, given $e_i(0)$, e_o and σ, a system with $m_2/m_1 = 0.1$ and $m_3/m_1 = 0.35$ will have similar stability properties to a system with $m_2/m_1 = 20$ and $m_3/m_1 = 7.45$ (ie, $M_i^{(l)} + M_o^{(l)}\sigma^{2/3}$ is the same for both; note that m_3/m_{123} in (5.1) is similar for both systems).

An analysis of the success or otherwise of this formulation of the stability problem is given in Mardling (2008a, *in preparation*). Here we note that its main drawback is that the resonance overlap stability criterion doesn't recognize that systems outside but near the separatrix are often unstable (thus we take $\phi_{2n1} = 0$ in the algorithm). Nonetheless, it is possible to invent remedies for individual applications, many of which do not require such high resolution anyway. For example, the study of the evolution of triple systems formed through binary-binary collisions in N-body simulations requires that one knows unequivocally when such a system is stable, since a mistake has the potential to grind the whole simulation to a halt. The fact that the algorithm presented here will sometimes make the opposite mistake (deem an unstable triple stable) if the configuration is close to the stability boundary should not have much effect on the overall evolution of the cluster.

References

Brouwer, D. & Clements, G. M. 1961, *Methods of Celestial Mechanics*, New York and London: Academic Press

Chirikov, B. V. 1979, *Physics Reports*, 52, 263

Heggie, D. C. 1975, *MNRAS*, 173, 729

Jackson, J. D. 1975, New York: Wiley, 1975, 2nd ed.

Kolmogorov, A. N. 1954, *Dokl. Akad. Nauk. SSSR*— 98, 527

Mardling, R. A. 2001, in *Evolution of Binary and Multiple Star Systems*, eds Podsiadlowski *et al.* ASP Conference Series, 229, 101

Mardling, R. A. 2007, *MNRAS*, in press

Murray, C. D. & Dermott, S. F. 2000, *Solar System Dynamics*, Cambridge, UK: Cambridge University Press

Roy, A. & Haddow, M. 2003, *Cel. Mec. Dyn. Ast.*, 87, 411

Walker, G. H. & Ford, J. 1969, *Physical Review*, 188, 416

Wisdom, J. 1980, *AJ*, 85, 1122