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ON THE PRESERVATION OF ROOT NUMBERS AND THE BEHAVIOR OF WEIL CHARACTERS UNDER RECIPROCITY EQUIVALENCE

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ABSTRACT. This paper studies how the local root numbers and the Weil additive characters of the Witt ring of a number field behave under reciprocity equivalence. Given a reciprocity equivalence between two fields, at each place we define a local square class which vanishes if and only if the local root numbers are preserved. Thus this local square class serves as a local obstruction to the preservation of local root numbers. We establish a set of necessary and sufficient conditions for a selection of local square classes (one at each place) to represent a global square class. Then, given a reciprocity equivalence that has a finite wild set, we use these conditions to show that the local square classes combine to give a global square class which serves as a global obstruction to the preservation of all root numbers. Lastly, we use these results to study the behavior of Weil characters under reciprocity equivalence.

1. **Introduction.** A *reciprocity equivalence* between two number fields is the formalization of the concept of a *Hilbert-symbol-preserving map*. Reciprocity equivalences were introduced in [6], where it was shown that number fields have isomorphic Witt rings precisely when they are reciprocity equivalent. One consequence of this is a global-local-global principal for the Witt ring (see [6]).

Given a number field K and a selection $\{d_P\}_{P\in\Omega_K}$ of local square classes d_P, one at each place P of K, we give necessary and sufficient conditions for the existence of a unique, totally positive global square class d of K whose localization at every place P is d_P. Next, we give the consequences of this result for the preservation of local root numbers. Lastly, we use this information to study how the Weil additive characters of the Witt ring $W(K)$ of a number field K behave under a reciprocity equivalence. Since a Weil character is related to a Hilbert symbol (via the local root number attached to the real quadratic character defined by the Hilbert symbol) one would expect Weil characters to be preserved by a reciprocity equivalence. We show, however, that Weil characters are *not* always preserved, and describe the extent to which they can vary.

After two brief preliminary sections recalling the relevant material concerning reciprocity equivalence (Section 2) and local root numbers (Section 3), in Section 4 we establish, with the local Lemma, the existence of a local square class which serves as an obstruction to the preservation of the local root numbers under reciprocity equivalence.

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In addition, we give conditions under which *any* selection of local square classes represent a global square class. In Section 5, we use these results to describe a global square class, d, which serves as an obstruction to the preservation of local root numbers by reciprocity equivalence. This is translated in Section 6 into an expression of the failure of reciprocity equivalence to preserve the Weil characters, in terms of the Witt classes of the global square class d and the rational prime 2.

2. **Reciprocity equivalence.** Throughout this paper, K and L will represent number fields.

DEFINITION (PERLIS-SYZMICZEK-CONNER-LITHERLAND). A *reciprocity equivalence* between the number fields K and L is a pair of maps (t, T) where t is a (group) isomorphism between the square class groups of K and L

$$t: K^*/K^{*2} \longrightarrow L^*/L^{*2}$$

and T is a bijection between the sets Ω_K and Ω_L of all places of K and L

$$T: \Omega_K \longrightarrow \Omega_L$$

which together preserve the Hilbert symbols:

$$(a, b)_P = (ta, tb)_{TP} \quad \forall a \in K_P^*/K_P^{*2}, \forall P \in \Omega_K. \tag{2.1}$$

(Note: The archimedean places are included in Ω_K and Ω_L; K_P^*/K_P^{*2} denotes the local square class group of K at P.) As a consequence of this definition, we have well-defined induced local mappings:

$$t_P: K_P^*/K_P^{*2} \longrightarrow L_{TP}^*/L_{TP}^{*2}.$$

K and L are *reciprocity equivalent* if there exists a reciprocity equivalence between them.

We note that in [6] the authors show that given a reciprocity equivalence (t, T) between K and L,

1) the map t takes the square class of -1 in K to the square class of -1 in L; and
2) T maps the infinite places of K to the infinite places of L.

Given a particular reciprocity equivalence (t, T) between K and L and a finite place P of K, [6] defines the equivalence (t, T) to be *tame* at the place P if

$$\operatorname{ord}_P(a) \equiv \operatorname{ord}_{TP}(ta) \pmod 2 \quad \forall a \in K_P^*/K_P^{*2}$$

and *wild* at P otherwise. The set of all (finite) places P of K at which the equivalence (t, T) is wild is called the *wild set* of (t, T). A *tame equivalence* has no wild places. According to [1], we may assume that whenever K and L are Witt equivalent, there exists (at least) one reciprocity equivalence between K and L which has a finite wild set. This fact will be vital in our effort to extend the results from the local level and obtain a global element (see Section 5).

3. **A review of local root numbers.** Let K be a number field, p a rational prime, Q_p the p-adic completion of the field Q of rational numbers and P a place of K lying above p. For each square class a of K_P^*/K_P^{*2}, the Hilbert symbol $(\,,\,)_P$ can be used to define a real quadratic character

$$\chi(a)\colon K_P^* \longrightarrow Z^*$$
$$x \longmapsto (x,a)_P.$$

Following Tate [8, p. 94], there is associated to $\chi(a)$ a local root number $r_P(a)$ which is a complex fourth root of unity defined on the square classes of K_P satisfying Tate's formula: namely, given a, b in K_P^*/K_P^{*2}, they satisfy

$$r_P(ab) = (a,b)_P \cdot r_P(a) \cdot r_P(b). \tag{3.1}$$

Moreover, setting $a = b$ in (3.1) yields

$$[r_P(a)]^2 = (-1,a)_P. \tag{3.2}$$

We also note that for a global element $a \in K^*/K^{*2}$, $r_P(a) = 1$ for almost all places P of K (see [8]). In fact, there is a reciprocity law for local root numbers. Namely, given a global square class a of K,

$$\prod_{P\in\Omega_K} r_P(a) = 1$$

(this is the Fröhlich-Queyrut Theorem - see [8] or [3]).

4. **Local root numbers and reciprocity equivalence—the local Case.**

LEMMA 4.1 (LOCAL LEMMA). *Let (t,T) be a reciprocity equivalence between the number fields K and L. Then for each place P of K, there exists a unique local square class d_P of K_P such that*

$$(a,d_P)_P \cdot r_P(a) = r_{TP}(ta) \quad \text{for each } a \in K_P^*/K_P^{*2}. \tag{4.1}$$

From (4.1) we see that an equivalence preserves local root numbers at the place P when its corresponding local square class d_P equals 1. We emphasize that d_P depends upon the particular reciprocity equivalence used. That is, different equivalences between the same two fields K and L could have different local square classes d_P for the same place P of K. Note also that this Local Lemma holds even for the "worst" equivalences, that is, those which have infinite wild sets.

PROOF. By the non-degeneracy of the Hilbert symbol, we have an isomorphism

$$K_P^*/K_P^{*2} \cong \mathrm{Hom}(K_P^*/K_P^{*2}, \mu_2) \tag{4.2}$$
$$d \longmapsto f$$

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where

$$f(a) := (d, a)_P \text{ for every } a \in K_P^*/K_P^{*2}.$$

Here $\mu_2 = \pm 1$ is the group of square roots of unity.

We would now like to define a special homomorphism from K_P^*/K_P^{*2} to μ_2 and use (4.2) to obtain the desired local square class d_P. Namely, given $a \in K_P^*/K_P^{*2}$, we define

$$g(a) = \frac{r_{TP}(ta)}{r_P(a)}.$$

CLAIM.

$$g \in \operatorname{Hom}(K_P^*/K_P^{*2}, \mu_2).$$

That is, given $a, b \in K_P^*/K_P^{*2}$,

$$\begin{aligned} g(ab) &= \frac{r_{TP}(tab)}{r_P(ab)} = \frac{r_{TP}(tatb)}{r_P(ab)} \\ &= \frac{(ta, tb)_{TP} \cdot r_{TP}(ta) \cdot r_{TP}(tb)}{(a, b)_P \cdot r_P(a) \cdot r_P(b)} \qquad \text{by (3.1)} \\ &= \frac{r_{TP}(ta) \cdot r_{TP}(tb)}{r_P(a) \cdot r_P(b)} \qquad \text{by (2.1)} \\ &= g(a)g(b). \end{aligned}$$

Moreover, by (3.2) we have

$$\begin{aligned} [r_P(a)]^2 &= (-1, a)_P \\ &= (-1, ta)_{TP} \qquad \text{by (2.1) and the fact that } t(-1) = -1 \\ &= [r_{TP}(ta)]^2 \qquad \text{by (3.2)} \end{aligned}$$

which implies

$$r_P(a) = \pm r_{TP}(ta) \qquad \text{since } r_P(a), r_{TP}(ta) \in \mu_4.$$

Hence

$$g(a) = \frac{r_{TP}(ta)}{r_P(a)} = \pm 1 \in \mu_2.$$

So indeed $g \in \operatorname{Hom}(K_P^*/K_P^{*2}, \mu_2)$, as claimed.

But then we are done; that is, (4.2) guarantees the existence of a (unique) local square class $d_P \in K_P^*/K_P^{*2}$ such that

$$g(a) = \frac{r_{TP}(ta)}{r_P(a)} = (a, d_P)_P \qquad \text{for every } a \in K_P^*/K_P^{*2}$$

or equivalently,

$$(a, d_P)_P \cdot r_P(a) = r_{TP}(ta) \qquad \text{for every } a \in K_P^*/K_P^{*2}. \qquad \blacksquare$$

These local square classes $\{d_P\}_{P \in \Omega_K}$ satisfy several useful properties which we will need later. In particular,

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LEMMA 4.2. *Let (t,T) be a reciprocity equivalence between two number fields K and L and denote by $\{d_P\}_{P\in\Omega_K}$ the set of local square classes which satisfy the Local Lemma (Lemma 4.1). Then for each global square class $\sigma \in K^*/K^{*2}$, we have*

$$(\sigma, d_P)_P = 1$$

for almost all places P of K and

$$\prod_{P\in\Omega_K} (\sigma, d_P)_P = 1.$$

PROOF. Fix $\sigma \in K^*/K^{*2}$. By (4.1), for each place P of K we have

$$(\sigma, d_P)_P \cdot r_P(\sigma) = r_{TP}(t\sigma).$$

Multiplying both sides by $r_P(\sigma)$ and using (3.2) we obtain

$$(\sigma, d_P)_P \cdot (-1,\sigma)_P = r_P(\sigma)\cdot r_{TP}(t\sigma).$$

Since $[(-1,\sigma)_P]^2 = 1$, multiplying both sides by $(-1,\sigma)_P$ yields

$$(\sigma, d_P)_P = (-1,\sigma)_P \cdot r_P(\sigma)\cdot r_{TP}(t\sigma). \tag{4.3}$$

We now want to compute the right hand side of (4.3). To do so, we define the following sets of places of K:

$$S_1(\sigma) = \{P \in \Omega_K : r_P(\sigma) \neq 1\}$$
$$S_2(\sigma) = \{P \in \Omega_K : r_{TP}(t\sigma) \neq 1\}$$
$$S_3(\sigma) = \{P \in \Omega_K : (-1,\sigma)_P \neq 1\}.$$

Note that each of the sets $S_i(\sigma), i = 1,2,3$, is a finite set of places since each of $r_P(\sigma)$, $r_{TP}(t\sigma)$ and $(-1,\sigma)_P$ is equal to 1 for almost all places P of K. Let $S(\sigma) = S_1(\sigma)\cup S_2(\sigma)\cup S_3(\sigma)$. Then $S(\sigma)$ is also a finite set of places of K. Fix $P \notin S(\sigma)$. Then we have

$$(\sigma, d_P)_P = (-1,\sigma)_P \cdot r_P(\sigma)\cdot r_{TP}(t\sigma) = 1 \qquad \text{since } P \notin S(\sigma). \tag{4.4}$$

That is to say, $(\sigma, d_P)_P = 1$ for almost all places P of K. Moreover,

$$\begin{aligned}
\prod_{P\in\Omega_K} (\sigma, d_P)_P &= \prod_{P\in\Omega_K} (-1,\sigma)_P \cdot r_P(\sigma)\cdot r_{TP}(t\sigma) \\
&= \prod_{P\in S(\sigma)} (-1,\sigma)_P \cdot r_P(\sigma)\cdot r_{TP}(t\sigma) \qquad \text{by (4.4)} \\
&= \prod_{P\in S(\sigma)} (-1,\sigma)_P \cdot \prod_{P\in S(\sigma)} r_P(\sigma)\cdot \prod_{P\in S(\sigma)} r_{TP}(t\sigma) \qquad \text{since } S(\sigma) \text{ is finite} \\
&= \prod_{P\in\Omega_K} (-1,\sigma)_P \cdot \prod_{P\in\Omega_K} r_P(\sigma)\cdot \prod_{P\in\Omega_K} r_{TP}(t\sigma) \qquad \text{again by (4.4)} \\
&= 1 \qquad \text{by reciprocity of Hilbert symbols and root numbers.}
\end{aligned}$$

∎

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Now we pause to prove an independent lemma about local and global square classes. (J. Hsia has mentioned that this result generalizes). A necessary condition for the existence of a *global* square class $d \in K^*/K^{*2}$ which behaves suitably at each place P of K (that is, for which $d = d_P$ in K_P^*/K_P^{*2} for some selection of local square classes d_P, one at each place P of K) is that

$$\mathrm{ord}_P(d_P) \equiv 0 \pmod 2 \qquad \text{for almost all finite places } P \text{ of } K.$$

Moreover, given any global square d of K, d satisfies the Hilbert symbol reciprocity law. Namely, for every global square class a of K,

$$\prod_{P \in \Omega_K} (a, d)_P = 1.$$

As it turns out, these two conditions are in fact necessary and sufficient to guarantee that *any* selection of local square classes, one at each place P of K, represents a global square class d of K.

PROPOSITION 4.3. *Let K be a number field and $\{d_P\}_{P \in \Omega_K}$ denote any selection of local square classes d_P in K_P^*/K_P^{*2}, one at each place P of K. Then there exists a (necessarily unique) global square class $d \in K^*/K^{*2}$ such that $d = d_P$ in K_P^*/K_P^{*2} for each place P of K if and only if*

1) $\mathrm{ord}_P(d_P) \equiv 0 \pmod 2$ *for almost all finite places P of K, and*

2) for all global square classes $a \in K^*/K^{*2}$,

$$\prod_{P \in \Omega_K} (a, d_P)_P = 1.$$

PROOF. The necessity is clear. We prove the sufficiency. First, we observe that the assumption in statement 2) above can be made, since for almost all P in Ω_K, P is finite, nondyadic and 1) holds. That is, $(a, d_P)_P = 1$ for almost all $P \in \Omega_K$.

Following [6] and [1], we choose a finite set S of places of K satisfying each of the following conditions:

(i) S contains all infinite places of K;

(ii) S contains all finite dyadic places of K;

(iii) the S-class number $h^S(K)$ of K is odd;

(iv) $\mathrm{ord}_P(d_P) \equiv 0 \pmod 2$ for all $P \notin S$;

(to accomplish (iii), add generators of the 2-Sylow subgroup of the ideal class group to the set S). Let $U_K(S)$ represent the group of S-units of K and define the direct product of the spaces $\big(K_P^*/K_P^{*2}, (\,,\,)_P\big)$ over all $P \in S$

$$G(S) = \prod_{P \in S} K_P^*/K_P^{*2} \text{ and } \langle\,,\,\rangle_S = \prod_{P \in S} (\,,\,)_P$$

where $(\,,\,)_P$ represents the Hilbert symbol at the place P. Note that $\big(G(S), \langle\,,\,\rangle_S\big)$ forms a non-degenerate F_2-inner product space. As in [6, Lemma 5.6], we can assemble the localizations

$$i_P \colon U_K(S)/U_K(S)^2 \longrightarrow K_P^*/K_P^{*2}, \quad P \in S$$

into a single monomorphism which diagonally embeds the group of S-units modulo squares into $G(S)$:

$$i_S\colon U_K(S)/U_K(S)^2 \longrightarrow G(S)$$
$$y \longmapsto \mathbf{y}$$

where $y \in U_K(S)/U_K(S)^2$ denotes the square class of the element $y \in K^*$ and $\mathbf{y} \in G(S)$ is the tuple $(y, \ldots, y)$ whose P-th coordinate represents the image of the global square class y in K_P^*/K_P^{*2} (note that this map is an embedding because of conditions (i)–(iii) above—see [6]). To simplify notation, we will identify $U_K(S)/U_K(S)^2$ with its image in $G(S)$. Now the subspace $U_K(S)/U_K(S)^2$ of $G(S)$ is a metabolizer in $\big(G(S), \langle\,,\rangle_S\big)$. That is,

$$U_K(S)/U_K(S)^2 = \big(U_K(S)/U_K(S)^2\big)^{\perp}$$

where $\perp$ denotes the orthogonal complement. Namely,

$$U_K(S)/U_K(S)^2 \subseteq \big(U_K(S)/U_K(S)^2\big)^{\perp}$$

by Hilbert reciprocity; $\#U_K(S)/U_K(S)^2 = 2^s$ by the unit theorem; $\#\prod_{P\in S} K_P^*/K_P^{*2} = 4^s = 2^{2s}$ as in [5, page 178]. Thus, by the non-degeneracy of $\langle\,,\rangle$,

$$\#\big(U_K(S)/U_K(S)^2\big)^{\perp} = 2^{2s}/2^s = 2^s = \#U_K(S)/U_K(S)^2.$$

Moreover, in [6] the authors show that, given $a \in K^*/K^{*2}$,

$$(4.5)\qquad a \in U_K(S)/U_K(S)^2 \iff \mathrm{ord}_P(a) \equiv 0 \pmod 2 \qquad \text{for all } P \notin S$$

(this uses the fact that $h^S(K)$ is odd).

Let $\mathbf{d} = \{d_P\}_{P\in S} \in G(S)$.

CLAIM. $\mathbf{d} \in \big(U_K(S)/U_K(S)^2\big)^{\perp}$.

Take $\mathbf{u} \in U_K(S)/U_K(S)^2$. Then $\forall P \notin S$, we have

$$\mathrm{ord}_P(u) \equiv \mathrm{ord}_P(d_P) \equiv 0 \pmod 2$$

or $(u, d_P)_P = 1 \quad \forall P \notin S$. Thus,

$$\begin{aligned}
1 &= \prod_{P\in\Omega_K} (u, d_P)_P && \text{by assumption}\\
&= \prod_{P\in S}(u, d_P)_P \cdot \prod_{P\notin S}(u, d_P)_P\\
&= \prod_{P\in S}(u, d_P)_P && \text{since } (u, d_P)_P = 1 \text{ for all } P \notin S\\
&= \langle \mathbf{u}, \mathbf{d}\rangle_S.
\end{aligned}$$

Hence we have

$$\mathbf{d} \in \big(U_K(S)/U_K(S)^2\big)^{\perp} = U_K(S)/U_K(S)^2,$$

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as claimed. That is, there exists a global square class $d \in K^*/K^{*2}$ with $d \in U_K(S)/U_K(S)^2$ and $i_P(d) = d_P \in K_P^*/K_P^{*2}$ for all $P \in S$.

It remains to show $i_P(d) = d_P$ for all $P \notin S$ and the uniqueness of d.

Fix $P_0 \notin S$. Then $\mathrm{ord}_{P_0}(d_{P_0}) \equiv 0 \pmod 2$ by choice of S. Let $S_1 = S \cup \{P_0\}$. Then the set S_1 satisfies conditions (i)–(iv) above. Hence, by the above argument, there exists a global square class $d_1 \in U_K(S_1)/U_K(S_1)^2$ such that

$$i_P(d_1) = d_P \qquad \text{for all } P \in S_1,$$

in particular

$$i_{P_0}(d_1) = d_{P_0} \text{ and } \mathrm{ord}_{P_0}(d_1) \equiv \mathrm{ord}_{P_0}(d_{P_0}) \equiv 0 \pmod 2.$$

But, by (4.5), $d_1 \in U_K(S_1)/U_K(S_1)^2$ and $\mathrm{ord}_{P_0}(d_1) \equiv 0 \pmod 2$ implies

$$d_1 \in U_K(S)/U_K(S)^2 \subseteq U_K(S_1)/U_K(S_1)^2.$$

Thus we have $d, d_1 \in U_K(S)/U_K(S)^2$ and

$$i_P(d) = i_P(d_1) \text{ in } K_P^*/K_P^{*2} \qquad \text{for all } P \in S.$$

Hence $i_S(d) = i_S(d_1)$. Since i_S is a monomorphism, we must have $d = d_1$ in $U(S_1)/U(S_1)^2$. But then $i_{P_0}(d) = d_{P_0}$ or, more generally,

$$i_P(d) = d_P \in K_P^*/K_P^{*2} \qquad \text{for all } P \notin S,$$

as desired. Clearly d must be unique, for if $d, d' \in K^*/K^{*2}$ satisfy

$$i_P(d) = i_P(d') \qquad \text{for all places } P \text{ of } K,$$

then $d = d'$ in K^*/K^{*2} by the global squares theorem (see [5, page 182]). ∎

5. **Local root numbers and reciprocity equivalence—the global case.** Reciprocity equivalences between number fields do not, in general, preserve the local root numbers of the equivalent fields. Our goal here, then, is to describe when reciprocity equivalence preserves or *fails* to preserve the local root numbers of equivalent fields. We want to use the Local Lemma (Lemma 4.1) and Proposition 4.3 to obtain a global object which we can view as an obstruction to the preservation of the local root numbers.

We are now in a position to establish the existence of a global square class d which satisfies the Local Lemma (Lemma 4.1) at every place P of K; that is, for which $d = d_P$ in K_P^*/K_P^{*2} for all places P of K. Note that, as in the local case, the global square class d will depend upon the particular reciprocity equivalence used.

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THEOREM 5.1 GLOBAL THEOREM. *Given K and L number fields and (t, T) a reciprocity equivalence between K and L which has a finite wild set, then there exists a unique totally positive global square class $d \in K^*/K^{*2}$ such that*

$$(a, d)_P \cdot r_P(a) = r_{TP}(ta) \tag{5.1}$$

for each $a \in K_P^*/K_P^{*2}$ and for each place P of K.

As in the local case, a reciprocity equivalence preserves local root numbers precisely when its unique global square class d is equal to 1. We remind the reader that requiring the equivalence (t, T) to have a finite wild set does not create a problem in the sense that whenever K and L are reciprocity equivalent, there exists (at least) one reciprocity equivalence between them which has a finite wild set (see [1]). It does mean, however, that only equivalences which are tame outside a finite set can globally preserve all local root numbers.

PROOF. By the Local Lemma (Lemma 4.1), there exists a collection of local square classes $d_P \in K_P^*/K_P^{*2}$, one for each place P of K, for which (4.1) holds. By Proposition 4.3, there exists a unique global square class $d \in K^*/K^{*2}$ such that $d = d_P$ in K_P^*/K_P^{*2} for all places P of K if and only if

1) $\operatorname{ord}_P(d_P) \equiv 0 \pmod 2$ for almost all finite places P of K, and
2) for all global square classes $a \in K^*/K^{*2}$,

$$\prod_{P \in \Omega_K} (a, d_P)_P = 1.$$

Condition 2) holds by Lemma 4.2. Hence, it remains to show

CLAIM. $\operatorname{ord}_P(d_P) \equiv 0 \pmod 2$ for almost all finite places P of K.

First we recall that, given a finite nondyadic place P of K and the local nonsquare unit u_P at P, we have

$$\operatorname{ord}_P(d_P) \equiv 0 \pmod 2 \iff (u_P, d_P)_P = 1.$$

Thus we will show $(u_P, d_P)_P = 1$ for almost all finite places P of K. To accomplish this, we will evaluate $r_P(u_P), r_{TP}(tu_P)$ and use the Local Lemma (Lemma 4.1).

Choose a finite set S of places of K such that S contains

(1) all infinite places of K;
(2) all dyadic places of K;
(3) all places P of K at which the equivalence (t, T) is wild,
(4) all places P of K for which either P or TP divides the local absolute different ideal D_{K_P} or $D_{L_{TP}}$ of K_P, L_{TP} respectively

(note that $P \notin S$ implies P is a finite, nondyadic place of K).

Fix $P \notin S$ and choose a generator π of P. Let u_P denote the local nonsquare unit at P. We want to consider the local characters

$$\chi: K_P^* \longmapsto Z^*.$$

Recall that these are defined by the Hilbert symbols; namely, given a local character χ, there exists a local square class $b \in K_P^*/K_P^{*2}$ such that $\chi = \chi_b$ is defined by

$$\chi_b(c) = (b, c)_P \qquad \text{for all } c \in K_P^*.$$

In particular, we have a character χ_{u_P} associated with the local nonsquare unit u_P. Since $P \notin S$, P is nondyadic and χ_{u_P} is unramified (that is, $K_P\left(\sqrt{(u_P)}\right)$ is the unique unramified quadratic extension of K_P). Thus $\chi_{u_P}(w) = 1$ for all units w of K_P^*.

Tate ([8]) evaluates $r_P(u_P)$ as follows:

$$r_P(u_P) = \chi_{u_P}(g)$$

where g is a generator of the local absolute different ideal D_{K_P} (note $\chi_{u_P} \equiv 1$ on units implies $r_P(u_P)$ is independent of the choice of the generator g). Thus, let us take $g = \pi^j$ for some $j \in Z$. Then

$$r_P(u_P) = \chi_{u_P}(g) = \chi_{u_P}(\pi^j) = (u_P, \pi^j)_P = (-1)^{\operatorname{ord}_P D_{K_P}}.$$

That is,

$$(u_P, \pi^j) = \begin{cases} -1 & j \equiv 1 \pmod 2 \\ \ \ 1 & j \equiv 0 \pmod 2 \end{cases}$$

or we have

$$(u_P, \pi^j) = (-1)^j.$$

Since $j = \operatorname{ord}_P D_{K_P}$, we obtain

$$r_P(u_P) = (-1)^{\operatorname{ord}_P D_{K_P}} = 1$$

as $P \notin S$ does not divide D_{K_P}. We claim this also gives us $r_{TP}(tu_P) = 1$. Namely, the equivalence (t, T) is tame at P since $P \notin S$. Hence by definition,

$$\operatorname{ord}_P(u_P) = \operatorname{ord}_{TP}(tu_P) \equiv 0 \pmod 2.$$

Since $P \notin S$ is nondyadic, $tu_P = v_{TP}$, where v_{TP} is the nonsquare unit in L_{TP} (note TP is finite, nondyadic by tameness of P). An argument analogous to that above shows $r_{TP}(tu_P) = 1$.

Thus $r_P(u_P) = 1$, $r_{TP}(u_P) = 1$ for almost all finite places P of K. By the Local Lemma (Lemma 4.1),

$$(u_P, d_P)_P \cdot r_P(u_P) = r_{TP}(tu_P)$$

yields $(u_P, d_P)_P = 1$ for almost all finite places P of K or $\operatorname{ord}_P(d_P) \equiv 0 \pmod 2$ for almost all finite places P of K, as claimed.

To see that d is totally positive, let P be a real infinite place of K (if there are none, then the claim is vacuously true). Then, as we noted earlier, TP must be an infinite place of L. Taking $a = -1$ in (5.1), we have

$$(-1, d)_P \cdot r_P(-1) = r_{TP}(-1),$$

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since, as we observed, t always takes -1 to -1. Tate shows in [8, p. 94] that the local root number of -1 at a real infinite place is always equal to $-i$, hence

$$r_P(-1) = r_{TP}(-1) = -i.$$

By (5.1), that forces

$$(-1, d)_P = 1$$

at every real infinite place of K. Thus, d is indeed a totally positive global square class of K. ∎

6. **Weil characters and reciprocity equivalence.** We are now in a position to examine the behavior, under reciprocity equivalence, of the Weil characters. Recall that these are additive characters from the Witt ring of a number field K to the multiplicative group of complex numbers.

First we examine the behavior of a particular shifted Weil character (defined in [3] and referred to as a Tate character), then state the desired result concerning the behavior of the Weil characters themselves.

Fix a number field K. We will define the shifted Weil characters in terms of scaled Fourier transforms for the local root numbers of K. Thus, fix a place P of K. Given the local root number function

$$r_P\colon K_P^*/K_P^{*2} \longrightarrow \mu_4$$

we define the scaled Fourier transform for r_P, following [2, page 3]: for $a \in K_P^*/K_P^{*2}$ we set:

$$\hat{r}_P(a) = \frac{1}{\sqrt{\#K_P^*/K_P^{*2}}} \sum_{b \in K_P^*/K_P^{*2}} (a, b)_P \cdot r_P(b).$$

It is not difficult to prove that these scaled Fourier transforms satisfy the following identities (see [2, pages 3 and 4]): for $a \in K_P^*/K_P^{*2}$,

$$\hat{r}_P(a) = (-1, a)_P \cdot r_P(a) \cdot \hat{r}_P(1)$$

and

$$\hat{r}_P(1)^2 = r_P(-1).$$

We note that this implies that $\hat{r}_P(a)$ is always an eighth root of unity, since $r_P(-1)$ is a fourth root of unity.

LEMMA 6.1. *Let K and L be number fields and (t, T) a reciprocity equivalence between K and L. Fix a place P of K. Given $a \in K_P^*/K_P^{*2}$ we have*

$$\hat{r}_{TP}(ta) = \hat{r}_P(d_P a)$$

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where $d_P \in K_P^*/K_P^{*2}$ satisfies the Local Lemma (Lemma 4.1).

PROOF. Since $K_P^*/K_P^{*2} \cong L_{TP}^*/L_{TP}^{*2}$ (by definition of the reciprocity equivalence (t, T)), we can write

$$\begin{aligned}\hat{r}_{TP}(ta) &= \frac{1}{\sqrt{\#L_{TP}^*/L_{TP}^{*2}}} \sum_{b \in K_P^*/K_P^{*2}} (ta, tb)_{TP} \cdot r_{TP}(tb)\\ &= \frac{1}{\sqrt{\#K_P^*/K_P^{*2}}} \sum_{b \in K_P^*/K_P^{*2}} (ta, tb)_{TP} \cdot r_{TP}(tb).\end{aligned}$$

Since the reciprocity equivalence (t, T) preserves Hilbert symbols between K and L, we have

$$\hat{r}_{TP}(ta) = \frac{1}{\sqrt{\#K_P^*/K_P^{*2}}} \sum_{b \in K_P^*/K_P^{*2}} (a, b)_P \cdot r_{TP}(tb).$$

By the Local Lemma (Lemma 4.1), we can rewrite $r_{TP}(tb)$ to obtain

$$\begin{aligned}\hat{r}_{TP}(ta) &= \frac{1}{\sqrt{\#K_P^*/K_P^{*2}}} \sum_{b \in K_P^*/K_P^{*2}} (a, b)_P \cdot (d_P, b)_P \cdot r_P(b)\\ &= \frac{1}{\sqrt{\#K_P^*/K_P^{*2}}} \sum_{b \in K_P^*/K_P^{*2}} (d_P a, b)_P \cdot r_P(b)\\ &= \hat{r}_P(d_P a),\end{aligned}$$

as desired. ∎

Following [2, page 5], we define the shifted Weil character

$$\tau_P \colon W(K) \longrightarrow \mu_8$$

as follows: take a Witt class $X \in W(K)$ and let (V, b) be a representative of the class X; fix an orthogonal diagonalization $\langle a_1, \ldots, a_m \rangle$ of (V, b). Then, again following [2, page 5], we define

$$\tau_P(X) = \prod_{j=1}^{m} \hat{r}_P(a_j).$$

Clearly $\tau_P(X) \in \mu_8$. In [2, pages 5-6], the author verifies that this is in fact a well-defined additive character on W(K).

These shifted Weil characters behave rather nicely under reciprocity equivalence; namely, they are preserved up to the local square class d_P from the Local Lemma (Lemma 4.1).

LEMMA 6.2. *Let K and L be reciprocity equivalent number fields and (t, T) a reciprocity equivalence between them. Fix a place P of K. Let the local square class $d_P \in K_P^*/K_P^{*2}$ be as in the Local Lemma (Lemma 4.1). Given $X \in W(K)$, we have*

$$\tau_{TP}(t_* X) = \tau_P(\langle d_P \rangle X)$$

where $t_*\colon W(K) \cong W(L)$ *is the ring isomorphism induced by* $t\colon K^*/K^{*2} \cong L^*/L^{*2}$ *and* $\langle d_P\rangle \in W(K)$ *represents the image of the square class of* $d_P \in K_P^*/K_P^{*2}$.

Note that the use of $\langle d_P\rangle$ here is well-defined here. Namely consider the sequence

$$K^*/K^{*2} \longrightarrow K_P^*/K_P^{*2} \longrightarrow 1.$$

Given any pullback δ_P of d_P, it differs from d_P by at most a local square class $\alpha \in K_P^*/K_P^{*2}$. But

$$\begin{aligned}\hat{r}_P(d_P) &= \hat{r}_P(\delta_P\alpha)\\ &= \frac{1}{\sqrt{\#K_P^*/K_P^{*2}}} \sum_{b\in K_P^*/K_P^{*2}} (\delta_P\alpha, b)_P \cdot r_P(b)\\ &= \frac{1}{\sqrt{\#K_P^*/K_P^{*2}}} \sum_{b\in K_P^*/K_P^{*2}} (\delta_P, b)_P \cdot (\alpha, b)_P \cdot r_P(b)\\ &= \frac{1}{\sqrt{\#K_P^*/K_P^{*2}}} \sum_{b\in K_P^*/K_P^{*2}} (\delta_P, b)_P \cdot r_P(b)\end{aligned}$$

since α is a local square at P

$$= \hat{r}_P(\delta_P),$$

hence $\tau_P(\langle d_P\rangle) = \tau_P(\langle\delta_P\alpha\rangle)$.

Thus d_P serves as an obstruction to the preservation of the shifted Weil characters at P by reciprocity equivalence. In the case that the given equivalence is tame outside a finite set, the global square class d from the Global Theorem (Theorem 5.1) measures how close the equivalence comes to preserving all of the shifted Weil characters (*i.e.*, when $d = 1$, they are preserved at every place P of K).

PROOF. Choose a representative (V, b) for X and let $\langle a_1, \ldots, a_m\rangle$ be a diagonalization of (V, b), $a_i \in K_P^*$ for $i = 1, \ldots, m$. Then $\langle ta_1, \ldots, ta_m\rangle$ serves as a diagonalization of a representative of t_*X, so we can write

$$\begin{aligned}\tau_{TP}(t_*X) &= \prod_{j=1}^{m} \hat{r}_{TP}(ta_j)\\ &= \prod_{j=1}^{m} \hat{r}_P(d_Pa_j) \qquad \text{by Lemma 6.1}\\ &= \tau_P(\langle d_P\rangle X)\end{aligned}$$

as desired, since $\langle d_Pa_1, \ldots, d_Pa_m\rangle$ also serves as a diagonalization of (V, b). ■

We will define the Weil characters of K as follows (for a formal definition, see [7]). Fix a place P of K and let $X \in W(K)$ be a Witt class of K. The Weil character γ_P can be defined on $W(K)$ by

$$\gamma_P(\langle 2\rangle X) = \tau_P(X)$$

where $\langle\cdot\rangle$ denotes the corresponding 1-dimensional form in the Witt ring of K (see [3]). We now see how equivalence relates to the Weil characters themselves.

THEOREM 6.3. *Let K and L be reciprocity equivalent number fields and (t, T) an equivalence between them. Fix a place P of K. Let d_P be as in the Local Lemma (Lemma 4.1). Then for $X \in W(K)$ a Witt class of K, we have*

$$\gamma_P(\langle 2d_P\rangle X) = \gamma_{TP}(\langle 2\rangle t_* X)$$

where $\langle\cdot\rangle$ denotes the class of the corresponding 1-dimensional form in the appropriate Witt ring.

Again, we note that when the equivalence is tame outside a finite set, the corresponding global square class d can be used at each place P. Unfortunately, however, the local square classes $\{d_P\}_{P\in\Omega_K}$ alone are not enough to completely determine when an equivalence preserves the Weil characters. This also depends on the square class of 2, which need not be the same in both fields.

PROOF. By definition we have

$$\begin{aligned}\gamma_P(\langle 2d_P\rangle X) &= \gamma_P(\langle 2\rangle\langle d_P\rangle X)\\ &= \tau_P(\langle d_P\rangle X)\\ &= \tau_{TP}(t_* X) \qquad \text{by Lemma 6.2}\\ &= \gamma_{TP}(\langle 2\rangle t_* X),\end{aligned}$$

as desired. ∎

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