J. Aust. Math. Soc. 77 (2004), 149–164

ON $f(n)$ MODULO $\Omega(n)$ AND $\omega(n)$ WHEN f IS A POLYNOMIAL

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(Received 14 August 2002; revised 17 February 2003)

Communicated by W. W. L. Chen

Abstract

In this paper we show that if $f(X) \in \mathbb{Z}[X]$ is a nonzero polynomial, then $\omega(n) | f(n)$ holds only on a set of n of asymptotic density zero, where for a positive integer n the number $\omega(n)$ counts the number of distinct prime factors of n.

2000 *Mathematics subject classification*: primary 11A25, 11N37.
Keywords and phrases: sum-of-divisors function, Euler function, number of prime divisors function.

1. Introduction

Let n be a positive integer and let $\omega(n)$, $\Omega(n)$, $\tau(n)$, $\phi(n)$ and $\sigma(n)$ be the classical arithmetic functions of n, that is, $\omega(n)$, $\Omega(n)$, and $\tau(n)$ count the number of distinct prime divisors of n, the total number of prime divisors of n, and the number of divisors of n, respectively, while $\phi(n)$ and $\sigma(n)$ are the Euler function of n and the sum of divisors function of n, respectively. We also let $f(X) \in \mathbb{Z}[X]$ to be any nonzero polynomial with integer coefficients.

In [2], it was shown that the set of positive integers n for which $\omega(n) | n$ is of density zero, and it was asked whether the same is true for the set of integers n for which $\Omega(n) | n$. This question was answered in a greater generality in [4]. In this paper, we investigate the density of the sets of positive integers n on which one of the given 'small' arithmetic function of n divide either $f(n)$, or the value of f in some other arithmetic function of n.

We have the following result.

THEOREM 1.1. (1) *The set of positive integers*

$$\{n \mid f(n) \equiv 0 \pmod{\omega(n)}\} \tag{1.1}$$

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is of asymptotic density zero. The same is true for the sets obtained if one replaces $\omega(n)$ *in* (1.1) *by either* $\Omega(n)$ *or* $\tau(n)$.

(2) *The set of positive integers*

$$\{n \mid f(\tau(n)) \equiv 0 \pmod{\omega(n)}\} \tag{1.2}$$

is of asymptotic density zero. The same is true for the set obtained if one replaces $\omega(n)$ *in* (1.2) *by* $\Omega(n)$.

(3) *The set of positive integers*

$$\{n \mid \phi(n) \equiv 0 \pmod{\omega(n)}\} \tag{1.3}$$

is of asymptotic density one. The same is true for the other five sets obtained from (1.3) *by independently replacing* $\phi(n)$ *by* $\sigma(n)$, *and* $\omega(n)$ *by either* $\Omega(n)$ *or* $\tau(n)$.

(4) *The set of positive integers*

$$\{n \mid f(\Omega(n)) \equiv 0 \pmod{\omega(n)}\} \tag{1.4}$$

has an asymptotic density for every polynomial $f(X) \in \mathbb{Z}[X]$. *This density is zero unless* f *has nonnegative integer roots, in which case it is positive. Similarly, the set obtained if one interchanges* $\Omega(n)$ *by* $\omega(n)$ *in* (1.4) *has an asymptotic density, which is zero unless* f *has integer roots which are either negative or zero, in which case it is positive.*

The densities of the sets appearing at part (4) of Theorem 1.1 are computable. Namely, the density of the set (1.4) is

$$\sum_{\substack{k \geq 0, k \in \mathbb{Z} \\ f(k)=0}} d_k,$$

where $d_k > 0$ is the Rényi's constant (see [11]) given by

$$d_k := \lim_{x \to \infty} \frac{\#\{1 \leq n \leq x \mid \Omega(n) - \omega(n) = k\}}{x}.$$

Similarly, the density of the set obtained if one interchanges $\Omega(n)$ by $\omega(n)$ in (1.4) is

$$\sum_{\substack{k \leq 0, k \in \mathbb{Z} \\ f(k)=0}} d_{|k|}.$$

Theorem 1.1 gives information about the asymptotic densities of the sets of positive integers $\{n \mid f(\phi(n)) \equiv 0 \pmod{\omega(n)}\}$ and likewise when $\omega(n)$ is replaced by either $\Omega(n)$ or $\tau(n)$, or when $\phi(n)$ is replaced by $\sigma(n)$. Indeed, from part (3) of Theorem 1.1

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we get that the asymptotic density of such sets is zero unless $f(0) = 0$, in which case the asymptotic density is one. We point out that if one replaces the polynomial $f(n)$ by the nth Fibonacci number F_n in (1.1), then the statement asserted at part (1) of Theorem 1.1 still holds. This has been done in [5], and it is likely that a combination of the arguments from the method of proof from there with our present arguments and some results from [7] can yield a similar result when F_n is replaced by any nondegenerate linearly recurrent sequence of integers.

2. Preliminary results

In this section, we point out a 'large' set of integers which is suitable for our purposes, and then in the next section we verify that every positive integer n from our large set satisfies all the congruences or the incongruences asserted by the theorem.

We denote by $C_1, C_2, \ldots$ positive computable constants which are either absolute or depend only on the polynomial f. For a positive integer k and positive real number x, we denote by $\log_k x$ the recursively defined function given by $\log_1 x := \max\{\log x, 1\}$ and $\log_k x := \max\{\log(\log_{k-1} x), 1\}$, where log denotes the natural logarithm function. When $k = 1$, we simply write $\log_1 x$ as $\log x$ and we thus understand that it is always ≥ 1. We also use the Landau symbols O and o and the Vinogradov symbols $\gg$ and $\ll$ with their usual meanings. We write $\pi(x)$ for the number of prime numbers $p \leq x$. For a positive integer n, we use $P(n)$ and $p(n)$ to denote the largest prime factor of n and the smallest prime factor of n, respectively.

We set $\delta(x) := \log_5 x$, and we use the notations p, q, and r to denote prime numbers. For a positive integer n, we write $\omega_1(n)$ and $\omega_3(n)$ for the number of distinct prime factors of n which are congruent to 1 and 3 modulo 4, respectively. Thus, we always have $\omega(n) = \omega_1(n) + \omega_3(n) + \varepsilon$, where $\varepsilon = 0$ or 1 according to whether n is odd or even.

We begin with the following claim.

LEMMA 2.1. *Let x be a large positive real number and let $A(x)$ be the set of all positive integers n in the range $\sqrt{x} < n < x$ and which satisfy the following conditions*:

(1) $\max\{|\omega(n) - \log_2 x|, |\Omega(n) - \log_2 x|\} < \delta(x)\log_2^{1/2} x$.
(2) $\min\{\omega_1(n), \omega_3(n)\} > \log_2(x)/4$.
(3) *Write*

$$n := \prod_{p^{a_p} \| n} p^{a_p}. \tag{2.1}$$

Then, $\max_{p|n}\{a_p\} < \log_7 x$ *and* $a_p = 1$ *when* $p \geq \log_7 x$.

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Then the set $A(x)$ contains all positive integers $n < x$ except for $o(x)$ of them.

REMARK. Note that if $n \in A(x)$ then, by condition (3) of Lemma 2.1, we have that the inequality

$$\Omega(n) - \omega(n) = \sum_{a_p > 1}(a_p - 1) < \pi(\log_7 x)(\log_7 x - 1) < \log_7^2 x < \log_6 x \tag{2.2}$$

holds for large values of x.

PROOF OF LEMMA 2.1. It is obvious that there are $\lfloor \sqrt{x} \rfloor = o(x)$ numbers $n < x$ which are not in the range $\sqrt{x} < n < x$.

(1) Since both estimates

$$\sum_{1 \le n < x} (\omega(n) - \log_2 x)^2 = O(x \log_2 x) \tag{2.3}$$

and

$$\sum_{1 \le n < x} (\Omega(n) - \log_2 x)^2 = O(x \log_2 x) \tag{2.4}$$

hold (see [14]), it follows that there are at most $O(x/\delta^2(x)) = o(x)$ positive integers $n < x$ which fail to satisfy the inequality asserted at part (1) of the lemma.

(2) Let E be any set of prime numbers and for $x > 0$ write

$$E(x) := \sum_{\substack{p < x \\ p \in E}} \frac{1}{p}. \tag{2.5}$$

For any positive integer n, write $\omega(E, n)$ for the number of primes dividing n which belong to E, and let $0 < \alpha < 1$ be any fixed positive real number. Then, a result of Norton (see [9, 10]), says that if $E(x) > 0$, then the number of positive integers $n < x$ such that $|\omega(E, n) - E(x)| > \alpha E(x)$ is at most $C(\alpha)x/E(x)^{1/2}$, where $C(\alpha)$ is some computable number depending on α and E. Take $E = E_i := \{p \mid p \equiv i \pmod 4\}$ with $i = 1$ or 3, take $\alpha := 1/3$, and assume that $x > 5$. Then $E_i(x) \neq 0$ and the estimate

$$E_i(x) = \sum_{p \equiv i \pmod 4} \frac{1}{p} = \frac{\log_2 x}{2} + O(1) \tag{2.6}$$

holds for both $i = 1$ and 3. And so, if we assume that $n < x$ fails condition (2) of the lemma for some $i = 1$ or 3, then $\omega_i(n) \le \log_2(x)/4$ holds for such n, and with estimate (2.6) we conclude that $\omega_i(n) < 2E(x)/3$ holds for such $n < x$ and for large values of x. Thus, the inequality $|\omega(E_i, n) - E(x)| > E(x)/3$ holds for such n, and by the above result from [9, 10], we know that the number of such positive integers $n < x$ is

$$\ll \frac{x}{E(x)^{1/2}} \ll \frac{x}{\log_2^{1/2} x} = o(x).$$

(3) Suppose first that there exists a prime number $p \geq \log_7 x$ such that $p^2 | n$. The totality of such $n < x$ is at most

$$\sum_{p \geq \log_7 x} \frac{x}{p} = O\left(\frac{x}{\log_7 x \log_8 x}\right) = o(x).$$

Assume now that $a_p \geq \log_7 x$ holds for some p. Since we may assume that $p < \log_7 x$, it follows that the number of such positive integers $n < x$ is at most

$$\sum_{2 \leq p < \log_7 x} \frac{x}{p^{\log_7 x}} < \frac{x\pi(\log_7 x)}{2^{\log_7 x}} < \frac{x \log_7 x}{(\log_6 x)^{C_1}} = o(x),$$

where we put $C_1 := \log 2$. □

LEMMA 2.2. *Let x be a large positive real number and let $A(x)$ be the subset appearing in Lemma 2.1. Let $B(x)$ be the subset of $n \in A(x)$ with the following property.*

Write

$$\omega(n) := w_1(n) w_2(n), \quad \textit{where} \quad w_1(n) := \prod_{\substack{q^{b_q} \| n \\ q \leq \log_3 x \log_5 x}} q^{b_q}.$$

Then, $w_2(n) > \log_2^{1/3} x$ is squarefree, coprime to n, and has $p(w_2(n)) < 2\log_2^{1/2} x$.

Similarly, if one writes

$$\Omega(n) := W_1(n) W_2(n), \quad \textit{where} \quad W_1(n) := \prod_{\substack{r^{c_r} \| \Omega(n) \\ r \leq \log_3 x \log_5 x}} r^{c_r},$$

then $W_2(n) > \log_2^{1/3} x$ is squarefree, coprime to n, and has $p(W_2(n)) < 2\log_2^{1/2} x$.

Then the set $B(x)$ contains all positive integers $n < x$ except for $o(x)$ of them.

PROOF OF LEMMA 2.2. We shall deal only with the statement concerning the function $\omega(n)$ because the statement about $\Omega(n)$ can be dealt with in an entirely similar way.

Assume that $n \in A(x)$ but that $w_2(n) \leq \log_2^{1/3} x$. Then $\omega(n) = w_1(n) w_2(n)$, where $w_1(n) < \log_2 x + \delta(x) \log_2^{1/2} x < 2\log_2 x$, and $P(w_1(n)) \leq \log_3 x \log_5 x$. We estimate the number of values $w_1(n)$ can take. Suppose that $y > z > 0$ and put $\Psi(y, z) := \#\{n \le y \mid P(n) \le z\}$. We shall show that if $z := 2\log y \log_3 y$, then $\Psi(y, z) = y^{o(1)}$. To see this, we put

$$Z := \frac{\log y}{\log z}\log\left(1 + \frac{z}{\log y}\right) + \frac{z}{\log z}\log\left(1 + \frac{\log y}{z}\right)$$

and then, by [13, Theorem 2 on page 359], we know that the estimate

$$\log \psi(y, z) = Z\left(1 + O\left(1 + \frac{1}{\log z} + \frac{1}{\log_2(2y)}\right)\right) \tag{2.7}$$

holds uniformly in $y > z > 0$. It is clear that the factor that multiplies Z appearing on the right-hand side of (2.7) is $O(1)$, and with our choice for z we have

$$Z \ll \frac{z}{\log z} + \frac{\log y}{\log z} \ll \frac{\log y \log_3 y}{\log_2 y}.$$

And thus, we have

$$\psi(y, z) = \exp(O(Z)) = \exp\left(O\left(\frac{\log y \log_3 y}{\log_2 y}\right)\right) = y^{o(1)}.$$

Setting $y := 2\log_2 x$, and noting that $\log_3 x \log_5 x < 2\log y \log_3 y = z$, we get

$$\psi(2\log_2 x, \log_3 x \log_5 x) = (2\log_2 x)^{o(1)} = (\log_2 x)^{o(1)}. \tag{2.8}$$

In particular, the inequality

$$\psi(2\log_2 x, \log_3 x \log_5 x) < \log_2^{1/12} x \tag{2.9}$$

holds for large values of x. Inequality (2.9) tells us that $w_1(n)$ can take no more than $\log_2^{1/12} x$ values. Thus, the total number of values of $\omega(n) = w_1(n)w_2(n)$ which are smaller than $2\log_2 x$ and for which $w_2(n) \le \log_2^{1/3} x$ holds is at most $(\log_2 x)^{1/3+1/12} = \log_2^{5/12} x$. However, from [3, page 303], we know that if j is any fixed positive integer, then the number of positive integers $n < x$ having $\omega(n) = j$ is

$$\ll \frac{x}{\log^{1/2} x}. \tag{2.10}$$

Since our j can take only $\log_2^{5/12} x$ values, we conclude that the number of positive integers $n < x$ for which $w_2(n) < \log^{1/3} x$ is

$$\ll \frac{x}{\log^{1/2} x}\log_2^{5/12} x = \frac{x}{\log_2^{1/12} x} = o(x),$$

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which takes care of the first condition from the lemma.

We next show that $w_2(n)$ is squarefree for almost all $n < x$. Assume that $n \in A(x)$ but $w_2(n)$ is not squarefree. Notice that $p(w_2(n)) > \log_3 x \log_5 x$. Pick a prime number $p > \log_3 x \log_5 x$ and assume that $j := \omega(n)$ is a number which is divisible by p^2. Since $\omega(n) < 2\log_2 x$, it follows that $p \le C_2 \log_2^{1/2} x$, where $C_2 := \sqrt{2}$. But j is also a number in the interval

$$\mathscr{I} := (\log_2 x - \delta(x)\log_2^{1/2} x, \log_2 x + \delta(x)\log_2^{1/2} x) \tag{2.11}$$

whose length is $2\delta(x)\log_2^{1/2} x$, and so the number of such numbers j which can be multiples of p^2 is

$$\le \frac{2\delta(x)\log_2^{1/2} x}{p^2} + 1. \tag{2.12}$$

For every one of these numbers j, the number of positive integers $n < x$ with $\omega(n) = j$ is, by (2.10), $\ll x/\log_2^{1/2} x$. Thus, for a fixed prime number p, the number of positive integers $n \in A(x)$ and for which $p^2|\omega(n)$ is

$$\ll \frac{x\delta(x)}{p^2} + \frac{x}{\log_2^{1/2} x}.$$

Summing up the above inequalities over all the prime numbers p in the range $\log_3 x \log_5 x < p \le C_2 \log_2^{1/2} x$, we get that the totality of the positive integers $n \in A(x)$ and for which $w_2(n)$ is not squarefree is

$$\ll x\delta(x) \sum_{p > \log_3 x \log_5 x} \frac{1}{p^2} + \frac{x\pi(C_2 \log_2^{1/2} x)}{\log_2^{1/2} x}. \tag{2.13}$$

Since

$$\sum_{p > \log_3 x \log_5 x} \frac{1}{p^2} = O\left(\frac{1}{\log_3 x \log_4 x \log_5 x}\right), \tag{2.14}$$

and

$$\frac{\pi(C_2 \log_2^{1/2} x)}{\log_2^{1/2} x} = O\left(\frac{1}{\log_3 x}\right),$$

it follows that (2.13) is bounded above by

$$\ll \frac{x\delta(x)}{\log_3 x \log_4 x \log_5 x} + \frac{x}{\log_3 x} = o(x).$$

We now show that n and $w_2(n)$ are coprime for almost all $n < x$. Let $n \in A(x)$ and let p be a prime number dividing both n and $w_2(n)$. We now have $\log_3 x \log_5 x < p < 2\log_2 x$. Fix such a prime p. By condition (3) from Lemma 2.1, we know that $p \parallel n$, therefore $n = pm$ and $\omega(m) = \omega(n) - 1$. Fix also j such that $\omega(n) = pj$. Then $m < x/p$ and $\omega(m) = pj - 1$ is fixed. The number of such numbers m is, by (2.9),

$$\ll \frac{x}{p} \frac{1}{\log_2^{1/2}(x/p)} \ll \frac{x}{p \log^{1/2} x},$$

where the last inequality above follows from the fact that the inequalities $p < 2\log_2 x < x^{1/2}$ hold for large x. Moreover, since pj is a number in the interval $\mathscr{I}$ shown at (2.11), it follows that j can take at most

$$\frac{2\delta(x)\log_2^{1/2} x}{p} + 1$$

values. Thus, the number of numbers $n \in A(x)$ for which $p \mid \gcd(n, w_2(n))$ with a fixed value of p is

$$\ll \frac{x\delta(x)}{p^2} + \frac{x}{p \log_2^{1/2} x}.$$

Summing up the above inequalities over all the possible values of p, it follows that the number of positive integers $n \in A(x)$ for which $w_2(n)$ and n are not coprime is

$$\ll x\delta(x) \sum_{p > \log_3 x \log_5 x} \frac{1}{p^2} + \frac{x}{\log^{1/2} x} \sum_{p < 2\log_2 x} \frac{1}{p}. \tag{2.15}$$

Since

$$\sum_{p < 2\log_2 x} \frac{1}{p} \ll \log_4 x, \tag{2.16}$$

we get, with (2.14) and (2.16), that (2.15) is bounded above by

$$\ll \frac{x\delta(x)}{\log_3 x \log_4 x \log_5 x} + \frac{x \log_4 x}{\log_2^{1/2} x} = o(x).$$

Finally, we show that for almost all $n < x$ we have $p(w_2(n)) < 2\log_2^{1/2} x$. Assume that this is not so for some $n \in A(x)$. In this case, since $\omega(n) < 2\log_2 x$, it follows that $w_2(n)$ is a prime number $p \geq 2\log_2^{1/2} x$, and $\omega(n) = pj$, where $j < \log_2^{1/2} x$ has $P(j) \leq \log_3 x \log_5 x$. We now fix the number j and notice that since pj belongs to the interval $\mathscr{I}$ shown at (2.11), then p must be a prime number in the interval

$$\mathscr{I}_j := \left(\frac{\log_2 x}{j} - \frac{2\delta(x)\log_2^{1/2} x}{j}, \frac{\log_2 x}{j} + \frac{2\delta(x)\log_2^{1/2} x}{j} \right). \tag{2.17}$$

Let $\pi_j(x)$ be the number of prime numbers in the interval $\mathscr{I}_j$ shown at (2.17). Then, for j fixed, the number of values of $\omega(n) = pj$ is at most $\pi_j(x)$, and for each one of these values, by (2.10), the number of positive integers $n < x$ with $\omega(n) = pj$ is $\ll x/\log_2^{1/2} x$. So, the number of positive integers $n < x$ for which $\omega(n) = pj$ with j fixed and p prime is $\ll x\pi_j(x)/\log_2^{1/2} x$, and so the totality of the positive integers $n \in A(x)$ for which $p(w_2(n)) \geq 2\log_2^{1/2} x$ is

$$\ll \frac{x}{\log_2^{1/2} x} \sum_{\substack{j < \log_2^{1/2} x \\ P(j) \leq \log_3 x \log_5 x}} \pi_j(x). \tag{2.18}$$

Let us now notice that the interval $\mathscr{I}_j$ is an interval of length $2\delta(x)\log_2^{1/2} x/j$, and, by a result of Montgomery (see [8, page 34]), any interval of length y can contain no more than $2y/\log y$ prime numbers. Thus, since $\delta(x) > 1$ and $j < \log_2^{1/2} x = o(\delta(x)\log_2^{1/2} x)$, we get the inequality

$$\begin{aligned} \pi_j(x) &\ll \frac{\delta(x)\log_2^{1/2} x}{j\log(2\delta(x)\log_2^{1/2} x/j)} \\ &\ll \begin{cases} \dfrac{\delta(x)\log_2^{1/2} x}{j\log_3 x}, & \text{if } j < \log_2^{1/4} x; \\ \dfrac{\delta(x)\log_2^{1/2} x}{j}, & \text{if } \log_2^{1/4} x \leq j < \log_2^{1/2} x. \end{cases} \end{aligned} \tag{2.19}$$

In particular, (2.18) can be bounded from above by

$$\ll \frac{x\delta(x)}{\log_3 x} \sum_{P(j) < \log_3^2 x} \frac{1}{j} + \frac{x\delta(x)}{\log_2^{1/4} x} \psi(\log_2^{1/2} x, \log_3 x \log_5 x). \tag{2.20}$$

Clearly, $\psi(\log_2^{1/2} x, \log_3 x \log_5 x) < \psi(2\log_2 x, \log_3 x \log_5 x) = (\log_2 x)^{o(1)}$ (see (2.8)), therefore

$$\frac{x\delta(x)}{\log_2^{1/4} x} \psi(\log_2^{1/2} x, \log_3 x \log_5 x) = \frac{x\delta(x)}{(\log_2 x)^{1/4+o(1)}} = o(x). \tag{2.21}$$

Finally, note that

$$\begin{aligned} \sum_{P(j) < \log_3^2 x} \frac{1}{j} &= \prod_{p < \log_3^2 x} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \exp\left(O(1) + \sum_{p < \log_3^2 x} \frac{1}{p}\right) = \exp(\log_5 x + O(1)) \ll \log_4 x, \end{aligned}$$

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therefore

$$\text{(2.22)} \qquad \frac{x\delta(x)}{\log_3 x} \sum_{P(j)<\log_3^2 x} \frac{1}{j} \ll \frac{x\delta(x)\log_4 x}{\log_3 x} = o(x),$$

and now (2.21) and (2.22) imply that the right-hand side of (2.20) is $o(x)$.

Finally, the statement about $\Omega(n)$ follows in an entirely similar way due to the fact that a similar upper bound as (2.11) holds for the number of numbers $n < x$ having a fixed value of $\Omega(n)$. □

3. The proof of Theorem 1.1

PROOF. We let x be a large positive real number and we shall assume that $n \in B(x)$, where $B(x)$ is the set described in Lemma 2.2. Part (1) of the theorem is the toughest cookie in the jar and so we shall prove it last.

(2) Let $n \in B(x)$, and write n as in (2.1). Then $\tau := \tau(n) = 2^{\omega(n)-k}m$, where $k := \#\{p|n \mid a_p > 1\}$ and $m := \prod_{a_p>1}(a_p + 1)$. Condition 3 of Lemma 2.1 insures that $k \le \pi(\log_7 x) < \log_7 x$, and that the inequality

$$m < \exp\bigl(\pi(\log_7 x)\log(\log_7 x + 1)\bigr) < \exp(2\log_7 x) = \log_6^2 x$$

holds when x is large. In particular, there are a number $< \log_7 x \log_6^2 x < \log_5 x$ such pairs (k, m), and they all have $P(m) < \log_7 x + 1$. Let p be a prime number in the interval $\mathcal{J} := (\log_3 x \log_5 x, 2\log_2^{1/2} x)$. By Lemma 2.2, we know that $\omega(n)$ has such a prime factor for all $n \in B(x)$. Assume now that n is a number in $B(x)$ such that $p|\omega(n)$, write $\omega(n) := pj$, and assume further that n satisfies congruence (1.2). With the fixed value of p, the congruence $f(\tau) \equiv 0 \pmod p$ puts τ into at most d residue classes modulo p, where $d := \deg(f)$. Let α be one of these residue classes. Since $\tau = 2^{\omega(n)-k}m \equiv 2^{j-k}m \pmod p$, we get that $2^{j-k}m \equiv \alpha \pmod p$. Note that both 2 and m are invertible modulo p. Put $t(p)$ for the multiplicative order of 2 modulo p. For fixed values of α, k, m, the congruence $2^{j-k}m \equiv \alpha \pmod p$ puts j into a fixed congruence class modulo $t(p)$. In particular, with p fixed, the number $\omega(n) = pj$ belongs to at most $d\log_5 x$ congruence classes modulo $pt(p)$. Since this number is also in the interval $\mathcal{J}$ shown in (2.11), we get that the number of values that pj can assume for a fixed value of p is

$$\ll \left(\frac{\delta(x)\log_2^{1/2} x}{pt(p)} + 1\right)\log_5 x = \frac{\delta(x)\log_2^{1/2} x \log_5 x}{pt(p)} + \log_5 x.$$

For everyone of these values of pj, by inequality (2.10), there are $\ll x/\log_2^{1/2} x$ numbers numbers n for which $\omega(n) = pj$. Thus, for fixed p, the number of numbers

$n \in B(x)$ satisfying congruence (1.2) and for which $p|\omega(n)$ is

$$\ll \frac{x\delta(x)\log_5 x}{pt(p)} + \frac{x\log_5 x}{\log_2^{1/2} x}.$$

Summing up the above inequalities over all the primes p in the interval $\mathcal{J}$, and using the obvious fact that $t(p) \gg \log p$, we get that the number of numbers $n \in B(x)$ satisfying congruence (1.2) is

$$\ll x\delta(x)\log_5 x \sum_{p>\log_3 x} \frac{1}{p\log p} + \frac{x\log_5 x}{\log_2^{1/2} x}\pi(2\log_2^{1/2} x). \tag{3.1}$$

Since the estimate $\sum_{p>y} 1/(p\log p) \ll 1/\log y$ holds for all $y > 1$, we get that (3.1) is bounded from above by

$$\ll \frac{x\delta(x)\log_5 x}{\log_4 x} + \frac{x\log_5 x}{\log_3 x} = o(x).$$

The same argument applies when $\omega(n)$ is replaced by $\Omega(n)$.

(3) In [6, Lemma 2], it is shown that there exists an absolute constant C_3 such that if x is large and if we set $g(x) := C_3\log_2 x/\log_3 x$, then both $\phi(n)$ and $\sigma(n)$ are divisible by the least common multiple of all the prime powers $p^a < g(x)$ for all $n < x$ with $o(x)$ exceptions (in [6, Lemma 2] this is only shown for the function ϕ but the argument from there can be adapted in a straightforward way to yield the corresponding result for the function σ). Let $M(x)$ denote the least common multiple of all the prime powers up to $g(x)$. To get statement (3) of Theorem 1.1 for $\omega(n)$ and $\Omega(n)$, we show that both $\omega(n)$ and $\Omega(n)$ divide $M(x)$. To see this, assume that $p^a \parallel \omega(n)$. If $p \le \log_3 x\log_5 x$, then, by Lemma 2.2, we have $p^a \le w_1(n) = \omega(n)/w_2(n) \ll \log_2^{2/3} x = o(g(x))$. Assume now that $p^a \parallel w_2(n)$. In this case, by Lemma 2.2, we have that $a = 1$. If $w_2(n)$ is not prime, then there exists another prime number q (necessarily larger than $\log_3 x\log_5 x$) such that $pq|w_2(n)$. Thus, $p \le w_2(n)/q \ll \log_2 x/(\log_3 x\log_5 x) = o(g(x))$. Finally, if $w_2(n)$ is prime, then $w_2(n) = p(w_2(n)) < 2\log_2^{1/2} x = o(g(x))$. And thus, we have shown that $\omega(n)$ divides $M(x)$, and therefore both $\phi(n)$ and $\sigma(n)$, and a similar argument applies to $\Omega(n)$.

To see that $\tau(n)$ divides both $\phi(n)$ and $\sigma(n)$, write

$$\tau(n) := \prod_{p^{a_p}\parallel n}(a_p+1) = \prod_{q^{c_q}\parallel\tau(n)} q^{c_q}.$$

We first assume that q is an odd prime divisor of $\tau(n)$. Then, by condition (3) of Lemma 2.1,

$$q^{c_q} \Bigm| \prod_{\substack{p<\log_7 x\\ p^{a_p}\parallel n}} (a_p+1),$$

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and $a_p < \log_7 x$, therefore we deduce that

$$\log(q^{c_q}) \leq \sum_{\substack{p<\log_7 x \\ p^{a_p}\|n}} \log(a_p+1) \ll \pi(\log_7 x)\log(\log_7 x+1) \ll \log_7 x < \log_6 x$$

holds for large x, and so $q^{c_q} < \exp(\log_6 x) = \log_5 x = o(g(x))$. Hence, q^{c_q} divides $M(x)$. Assume now that $q = 2$ and for every $p|n$ write d_p for the exponent at which 2 divides $a_p + 1$. Then, $c_2 = \sum_{p|n} d_p$ and, by condition (3) of Lemma 2.1, we have $d_p = 1$ whenever $p \geq \log_7 x$ and $d_p \leq C_4 \log(\log_7 x + 1) < \log_7 x$ with $C_4 := 1/\log 2$, whenever $p < \log_7 x$. Thus, with conditions (1) and (3) of Lemma 2.1, the inequality

$$\begin{aligned} c_2 &\leq \omega(n) + \sum_{\substack{p<\log_7 x \\ p^{a_p}\|n}} d_p \\ &< \log_2 x + \delta(x)\log_2^{1/2} x + \pi(\log_7 x)\log_7 x < \log_2 x + 2\delta(x)\log_2^{1/2} x \end{aligned} \tag{3.2}$$

holds for large values of x. However, the power at which 2 which divides $\phi(n)$ is, by Lemma 2.1, at least

$$\omega(n) + \omega_1(n) - 1 \geq \frac{5\log_2 x}{4} - \delta(x)\log_2^{1/2} x - 1, \tag{3.3}$$

and it is clear that the right-hand side of (3.3) is larger than the right-hand side of (3.2) for large x. Thus, $2^{c_2}|\phi(n)$. To see the statement for σ, notice that by Lemma 2.1, we have that the power at which 2 divides $\sigma(n)$ is at least

$$\omega(n) + \omega_3(n) - 1 - 2\pi(\log_7 x) > \frac{5\log_2 x}{4} - 1 - \delta(x)\log_2^{1/2} x - \log_7 x, \tag{3.4}$$

and it is clear that the right-hand side of (3.4) is also larger than the right-hand side of (3.2) for large x. This shows that 2^{c_2} divides $\sigma(n)$ as well. We point out that the fact that the set shown at (1.3) with $\phi(n)$ replaced by $\sigma(n)$ and $\omega(n)$ replaced by $\tau(n)$ is of asymptotic density zero has also been proved in [1].

(4) Write $\Delta(n) := \Omega(n) - \omega(n)$. If n satisfies congruence (1.4), then $f(\Delta(n)) \equiv 0 \pmod{\omega(n)}$. By the remark following Lemma 2.1, we know that $\Delta(n) < \log_6 x$, therefore $|f(\Delta(n))| \ll \log_6^d x$, where we use again d for the degree of the polynomial f. However, by condition (1) of Lemma 2.1, we know that $\omega(n) \gg \log_2 x$, and therefore $|f(\Delta(n))| < \omega(n)$ holds when $n \in B(x)$ and x is large. Thus, except for a set of asymptotic density zero of positive integers n, the congruence (1.4) forces $f(\Delta(n)) = 0$. Since $\Delta(n)$ is a nonnegative integer, this will happen only if f has nonnegative integer roots k, and $\Delta(n) = k$ for such n and with k one of these nonnegative integer roots. Conversely, if f has nonnegative integer roots k, then any number

https://doi.org/10.1017/S1446788700013537 Published online by Cambridge University Press

n with $\Delta(n) = k$ will satisfy congruence (1.4). The corresponding statement about $f(\omega(n))$ being a multiple of $\Omega(n)$ can be dealt with in a similar way.

(1) Let w_2 be a squarefree number belonging to the interval $\mathscr{K} := (\log_2^{1/3} x, 2\log_2 x)$ having $p(w_2) > \log_3 x \log_5 x$, and assume that n satisfies congruence (1.1) and that $w_2(n) = w_2$. By Lemma 2.2, we know that if $n \in B(x)$, then n has such a factor w_2 which is moreover coprime to n. Since w_2 is squarefree and its smallest prime factor is large, it follows that for large x the congruence $f(n) \equiv 0 \pmod{w_2}$ implies that we may replace f by the product of all its primitive nonassociated factors in $\mathbb{Z}[X]$, that is, we may assume that $f(X)$ is primitive and squarefree as an element of $\mathbb{Z}[X]$. Write again d for the degree of f. The congruence $f(n) \equiv 0 \pmod{w_2}$ puts n into at most $d^{\omega(w_2)}$ congruence classes modulo w_2. Let $\alpha \pmod{w_2}$ be such a congruence class. Then $n = \alpha + w_2 m$ holds with some nonnegative integer m, and since by Lemma 2.2 we know that w_2 and n are coprime, it follows that α and w_2 and coprime. We also write $\omega(n) = w_2 j$. Fixing j and α, we conclude that $n < x$ is in the arithmetic progression $\alpha \pmod{w_2}$, and has a fixed value of $\omega(n) = w_2 j$. It is known that for every fixed positive integer l, the number of positive integers $n < x$ with $n \equiv \alpha \pmod{w_2}$ having $\omega(n) = l$ is

$$\ll \frac{x}{\phi(w_2)\log_2^{1/2} x} \tag{3.5}$$

and that estimate (3.5) above is uniform in our range $l < 2\log_2 x$ and $w_2 < 2\log_2 x$. Indeed, this can be obtained from the main result in [15] together with inequality (2.10). Now $\omega(n) = w_2 j$ is a number in the interval $\mathscr{I}$ shown at (2.11), and so the number of numbers j when w_2 is fixed is

$$\le \frac{2\delta(x)\log_2^{1/2} x}{w_2} + 1.$$

Summing up the above inequalities over all the allowable values of j, we conclude that if w_2 and α are fixed, then the number of such numbers $n \in B(x)$ which are congruent to α modulo w_2 is

$$\ll \frac{x\delta(x)}{w_2\phi(w_2)} + \frac{x}{\phi(w_2)\log_2^{1/2} x}.$$

Summing up the above inequalities over all the possible values of α, we conclude that the number of numbers $n \in B(x)$ satisfying congruence (1.1) and for which $w_2(n) = w_2$ is fixed is

$$\ll \frac{x\delta(x)d^{\omega(w_2)}}{w_2\phi(w_2)} + \frac{x d^{\omega(w_2)}}{\phi(w_2)\log_2^{1/2} x}. \tag{3.6}$$

Finally, summing up inequalities (3.6) over all the allowable values of w_2, we get that the number of numbers $n \in B(x)$ satisfying congruence (1.1) is

$$(3.7)\qquad \ll x\delta(x)\sum_{w_2\in\mathcal{K}}\frac{d^{\omega(w_2)}}{w_2\phi(w_2)}+\frac{x}{\log_2^{1/2}x}\sum_{w_2\in\mathcal{K}}\frac{d^{\omega(w_2)}}{\phi(w_2)}.$$

Notice that that $d^{\omega(w_2)} = (2^{\omega(w_2)})^{C_5} < (\tau(w_2))^{C_5}$, with $C_5 := \log d/\log 2$. Since for every $\varepsilon > 0$ the inequality $\tau(t) < t^{\varepsilon}$ holds for all positive integers $t > t_{\varepsilon}$, it follows that if we set $\varepsilon := (6C_5)^{-1}$, then the inequality $d^{\omega(w_2)} < w_2^{1/6}$ holds for all $w_2 \in \mathcal{K}$ and for sufficiently large values of x. Since we also know that $\phi(m) \gg m/\log_2 m$ holds for all positive integers m, it follows that the inequality $\phi(w_2) > w_2^{5/6}$ holds for all $w_2 \in \mathcal{K}$ and when x is sufficiently large. Thus, if x is large the expression appearing at (3.7) can be bounded above by

$$\begin{aligned}&\ll x\delta(x)\sum_{w_2\in\mathcal{K}}\frac{1}{w_2^{5/3}}+\frac{x}{\log_2^{1/2}x}\sum_{w_2\in\mathcal{K}}\frac{1}{w_2^{2/3}}\\&\ll x\delta(x)\int_{\log_2^{1/3}x}^{2\log_2 x}\frac{ds}{s^{5/3}}+\frac{x}{\log_2^{1/2}x}\int_{\log_2^{1/3}x}^{2\log_2 x}\frac{ds}{s^{2/3}}\\&\ll\frac{x\delta(x)}{\log_2^{2/9}x}+\frac{x}{\log_2^{1/6}x}=o(x).\end{aligned}$$

The same argument applies when $\omega(n)$ is replaced by $\Omega(n)$.

Finally, to see the statement with $\omega(n)$ replaced by $\tau(n)$ in (1.1), write $h(x) := \lfloor(\log_2 x)/2d\rfloor$ and let $n \in B(x)$ be a number satisfying congruence (1.1) with $\tau(n)$ instead of $\omega(n)$. By condition (3) of Lemma 2.1, if we set again c_2 to be the exponent at which 2 divides $\tau(n)$, we have that the inequality

$$(3.8)\qquad c_2 \geq \omega(n)-\pi(\log_7 x)\geq \log_2 x-\delta(x)\log_2^{1/2}x-\log_7 x>\frac{\log_2 x}{2}$$

holds for large x. We write K for the number of irreducible factors of f, and we write $f_1, \dots, f_K$ for these irreducible factors. The congruence $f(n) \equiv 0 \pmod{2^{c_2}}$ together with the lower bound (3.8) on c_2, imply that there exists $i = 1, \dots, K$ such that $f_i(n) \equiv 0 \pmod{2^{h(x)}}$ holds. Since all the f_i's are irreducible, with i fixed the above congruence puts n into at most C_6 arithmetic progressions of ratio $2^{h(x)}$, where one can choose C_6 to be an upper bound for the absolute values of all the discriminants of the polynomials f_i for $i = 1, \dots, K$. Thus, if $n \in B(x)$ satisfies (1.1) with $\omega(n)$ replaced by $\tau(n)$, then n is in at most $C_7 := K\,C_6$ arithmetic progressions of ratio $2^{h(x)}$. Each one of these arithmetic progressions will contain at most

$$\frac{x}{2^{h(x)}}+1$$

positive integers $n < x$, thus the total number of numbers $n \in B(x)$ satisfying $f(n) \equiv 0 \pmod{\tau(n)}$ is at most

$$\frac{C_7 x}{2^{h(x)}} + C_7 = o(x),$$

which completes the proof of our theorem. □

We point out that when $f(n) := n$, the problem of estimating the number of positive integers $n < x$ for which $\tau(n) | n$ was treated by Spiro in [12].

Acknowledgements

I would like to express my gratitude to Professors P. D. T. A. Elliott, C. Pomerance, and A. Sankaranarayanan for their helpful advice. I thank the referee for comments and suggestions which considerably improved the quality of this paper. Finally, I am grateful to CONACyT and SEP for their support via Grant SEP-CONACyT 37259-E.

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