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Optimal Boundary Regularity of Proper Harmonic Maps between Asymptotically Hyperbolic Spaces

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Abstract. **This paper studies the optimal boundary regularity of harmonic maps between a class of asymptotically hyperbolic spaces. To be precise, given any smooth boundary map with nowhere vanishing energy density, this paper provides an asymptotic expansion formula for harmonic maps under the assumption of C^1 up to the boundary.**

1 Introduction

Let M denote the interior of a compact smooth manifold with boundary $\overline{M}$, and let $\overline{g}$ be a nondegenerate smooth metric on $\overline{M}$. Let $r \in C^\infty(\overline{M})$ with $r \geq 0$ on M, $r^{-1}(0) = \partial M$. Consider the metric g on M given by $g = r^{-2}\overline{g}$. Then we call g a conformally compact metric and (M, g) a conformally compact manifold. In addition, if (M, g) satisfies $|dr|_{\overline{g}} = 1$ on ∂M, it can be shown in [24] that this condition is equivalent to the sectional curvature uniformly tends to -1 as we approach the boundary ∂M. When this additional condition holds, we say that the conformally compact manifold (M, g) is asymptotically hyperbolic.

Suppose that M and N are Riemannian manifolds of dimensions m and n respectively, and their Riemannian metrics are $\mathrm{d}s_M^2 = \sum\limits_{i,j=1}^{m} g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and $\mathrm{d}s_N^2 = \sum\limits_{\alpha,\beta=1}^{n} h_{pq}\mathrm{d}u^p\mathrm{d}u^q$, respectively. Then the energy density function of a C^1 map $u : M \to N$ is defined by

$$e\left(u\right) = \frac{1}{2}g^{ij}\frac{\partial u^p}{\partial x^i}\frac{\partial u^q}{\partial x^j}h_{pq},$$

and the total energy of u is given by

$$E\left(u\right) = \int_{M^m} e(u)\mathrm{d}x.$$

The harmonic map equation from M into N, which is the Euler Lagrange equation for critical points of the total energy functional, can be written as

$$\tau(u)^s = \Delta_M u^s + g^{ij}\Gamma_{pq}^s\frac{\partial u^p}{\partial x^j}\frac{\partial u^q}{\partial x^i} = 0, s = 1, \cdots, n,$$

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where Γ^s_{pq} are the Christoffel symbols of N.

The model example for asymptotically hyperbolic manifolds is Poincaré model for hyperbolic space $\mathbb{H}^m$ given by the unit ball B^m with metric $\mathrm{d}s^2_{\mathbb{H}^m} = \frac{4}{(1-\rho^2)^2}\mathrm{d}s^2_0$. Here, ρ denotes the Euclidean distance measured from the origin, and $\mathrm{d}s^2_0$ stands for the standard Euclidean metric. Under this equivalence relation, a map from the $(m-1)$-dimensional sphere $\mathbb{S}^{m-1}$ to the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ can be interpreted as a map from the boundary at infinity of $\mathbb{H}^m$ to the boundary at infinity of $\mathbb{H}^n$. Schoen [27] proposed a conjecture: For every quasiconformal mapping $f : \mathbb{S}^1 \to \mathbb{S}^1$, there exists a unique harmonic and quasiconformal extension $\mathcal{H}(f) : \mathbb{H}^2 \to \mathbb{H}^2$. Li and Wang [21] extended this conjecture to the case where $m = n \geq 2$.

The results of uniqueness, existence, and regularity properties for the Dirichlet problem at infinity for proper harmonic maps between hyperbolic spaces was obtained by Li and Tam [18, 19, 20] when the harmonic map is C^1 up to boundary. In their work [20], they investigated the boundary regularity of these harmonic maps and posed some thought-provoking questions. Specifically, they wondered if the singularities of these harmonic maps could be understood by showing that near the boundary, the harmonic map must have an expansion similar to those in the complex Monge - Ampère equation [12] and the Bergman Laplacian [10]. Donnelly [4, 5] generalized a portion of their findings to any rank one symmetric spaces. When the boundary map is a quasiconformal map, Li, Tam and Wang [20, 21] established the uniqueness and Lemm and Markovic [23, 22, 13] proved existence using heat flow theorem in [28]. Benoist and Hulin [2] completed the proof of the Schoen-Li-Wang conjecture when both the domain manifold and the target manifold are symmetric spaces of rank one. The complex case was studied by Donanelly [6] and Li and Ni [15] and Li and Simon [16], etc.. When boundary map is smooth and has nowhere vanishing energy density, Chen, Li and Luo provide an asymptotic expansion formula for the harmonic map between balls in Berman metrics in [3], so they obtained the optimal regularity of harmonic maps in the case of complex spaces.

There also have been several relevant works on harmonic maps between asymptotically hyperbolic spaces. Leung [14] employed the heat flow method to demonstrate the existence of harmonic maps between asymptotically hyperbolic conformally compact manifolds. Regarding boundary regularity, Economakis proved that any $C^{1,1}$ local harmonic map whose boundary map is smooth and has nowhere vanishing energy density admits a polyhomogeneous expansion at the boundary in [8]. Economakis adopted an approach that is purely microlocal, as can be seen in the references [25, 26, 12]. Donnelly [7] proved existence and uniqueness for asymptotic Dirchlet problems for harmonic maps from Hadamard manifolds to complete simply connected manifolds with nonpositive sectional curvature. Kim and Lee [11] proved the existence of nonconstant bounded harmonic maps on a Cartan-Hadamard manifold of pinched negative curvature by solving the asymptotic Dirichlet problem. Akutagawa and Matsumoto [1] generalized the result of Li and Tam [19], they proved an existence theorem for harmonic maps with C^1 boundary conditions between asymptotically hyperbolic manifolds. For other studies on harmonic maps between asymptotically hyperbolic spaces, see [9, 24].

Compared with the above works, this article has two advantages. First, it improves the boundary regularity in Li and Tam [20] and achieves the optimal boundary regularity. Second, it reduces the assumption of [8] for harmonic maps between asymptotically hyperbolic spaces to C^1 up to the boundary, and obtains an expansion of harmonic maps between a class asymptotically hyperbolic spaces. Moreover, in the Poincaré disk model, this expansion is global. In the case where both M and N are hyperbolic spaces, according to the asymptotic expansion obtained in this article, it is easy to see that the coefficient of the first power of the logarithm, which has an impact on the regularity, can be completely determined by the boundary map. Moreover, the specific expression of this coefficient can be calculated.

Let $\mathbb{H}^m$ and $\mathbb{H}^n$ be hyperbolic spaces of dimensions m and n, respectively. The hyperbolic space $\mathbb{H}^m$ is identified with $B^m = \{x \in \mathbb{R}^m | |x| < 1\}$ with the Poincaré metric given by

$$g_{\mathbb{H}^m} = \frac{4(dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m)}{(1-\rho^2)^2},$$

where $\rho^2 = \sum_{i=1}^m (x^i)^2$. Under this identification, the idea boundary of $\mathbb{H}^m$ can be viewed as $\mathbb{S}^{m-1}$. Similarly we identify $\mathbb{H}^n$ with the unit ball B^n in $\mathbb{R}^n$ with Poincaré metric given by

$$g_{\mathbb{H}^n} = \frac{4(du^1 \otimes du^1 + \cdots + du^n \otimes du^n)}{(1-|u|^2)^2}.$$

Also, the idea boundary of $\mathbb{H}^n$ is identified with $\mathbb{S}^{n-1}$. In terms of these coordinates, the tension field of a map $u : \mathbb{H}^m \to \mathbb{H}^n$ is given by

$$\begin{aligned}\tau_0(u)^p =& \frac{(1-\rho^2)^2}{4}\Delta_0 u^p + \sum_{i=1}^m \frac{(m-2)(1-\rho^2)}{2} x^i \frac{\partial u^p}{\partial x^i} \\ &+ \frac{(1-\rho^2)^2}{2(1-|u|^2)}(2\langle u, \nabla_0 u\rangle \nabla_0 u^p - \langle \nabla_0 u, \nabla_0 u\rangle u^p),\ p = 1, \cdots, n,\end{aligned} \tag{1.1}$$

where $\langle u, \nabla_0 u\rangle = \sum_{q=1}^n \langle u^q, \nabla_0 u^q\rangle$ and $\langle \nabla_0 u, \nabla_0 u\rangle = \sum_{q=1}^n \langle \nabla_0 u^q, \nabla_0 u^q\rangle$. The notation Δ_0 denotes the euclidean Laplacian and ∇_0 denotes the euclidean gradient. $\langle \cdot, \cdot\rangle$ represents the inner product of two n-dimensional vectors.

We assume that M is a class of asymptotically hyperbolic spaces. In the Poincaré disk model $B^m = \{x \in \mathbb{R}^m | |x| < 1\}$, it has a metric

$$g_M = \frac{4(dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m)}{(1-\rho^2)^2} + h_M(x), \tag{1.2}$$

where $\rho = |x|$. The term $h_M(x)$ is a symmetric 2-tensor, which serves as the perturbation term. It has the expansion

$$h_M = \sum_{i,j}^m h_{ij} dx^i \otimes dx^j.$$

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When expressed as a function of the variables (ρ,θ), $h_{ij} \in C^\infty(\overline{B^m})$ adheres to a decay condition. Precisely, for any non-negative integer s, k , the following relation holds:

$$|\nabla_\rho^k \nabla_\theta^s h_{ij}| = O((1-|\rho|^2)^{a-k}), \rho \to 1^-, a > -1.$$

Similarly, we assume that N is an asymptotically hyperbolic space and we identify N with the unit ball B^n in $\mathbb{R}^n$ with Poincaré metric given by

$$g_N = \frac{4(du^1 \otimes du^1 + \cdots + du^n \otimes du^n)}{(1-|u|^2)^2} + h_N(x), \tag{1.3}$$

where $h_N(x)$ is a symmetric 2-tensor (i.e., the perturbation term) and has the expansion

$$h_N = \sum_{i,j}^n \overline{h}_{ij} du^i \otimes du^j.$$

When expressed as a function of the variables (ρ,θ), $\overline{h}_{ij} \in C^\infty(\overline{B^m})$ satisfies the decay condition: for any non-negative integer s, k,

$$|\nabla_\rho^k \nabla_\theta^s \overline{h}_{ij}| = O((1-|u|^2)^{b-k}), \rho \to 1^-, b > -1.$$

In terms of these coordinates, the tension field of a map $u : M \to N$ is given by

$$\begin{aligned}\tau(u)^p = &\sum_{i,j=1}^m \left(\frac{(1-\rho^2)^2}{4}\delta_{ij} + O((1-\rho^2)^{4+a})\right)\frac{\partial^2 u^p}{\partial x^i \partial x^j}\\ &+ \sum_{i=1}^m \frac{(m-2)}{2}((1-\rho^2) + O((1-\rho^2)^{3+a})x^i \frac{\partial u^p}{\partial x^i}\\ &+ \frac{(1-\rho^2)^2 + O((1-\rho^2)^{4+a})}{2}\left(\frac{1}{1-|u|^2} + O((1-|u|^2)^{b+1})\right)\\ &\times (2\langle u, \nabla_0 u\rangle \nabla_0 u^p - \langle \nabla_0 u, \nabla_0 u\rangle u^p), p = 1, \cdots, n.\end{aligned} \tag{1.4}$$

Let $\phi_0 : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$. The Dirichlet boundary value problem for a proper harmonic map is given by

$$\begin{aligned}\tau(u) &= 0 \text{ in } B^m\\ u &= \phi_0 \text{ on } \partial B^m.\end{aligned} \tag{1.5}$$

The main purpose of this paper is to study optimal boundary regularity for proper harmonic maps between asymptotically hyperbolic spaces similar to those in [17]. The main theorem is

***Theorem* 1.1** *Let M and N be asymptotically hyperbolic spaces with metrics* (1.2) *and* (1.3) *respectively in the Poincaré disk model. Let u be a harmonic map from M to N so that $u \in C^1$ as a map from $\overline{B^m}$ to $\overline{B^n}$. Suppose that the boundary map ϕ_0 of u, when restricted to $\mathbb{S}^{m-1}$, is in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$, and has nowhere-vanishing energy density with respect to the stanard metrics. Then $u \in C^{m,\alpha}(\overline{B^m}, \overline{B^n})$ for all*

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$0 < \alpha < 1$. *Moreover, the solution* u *of* (1.5) *has the following asymptotic expansion:*

$$u(x) \sim \phi_0 + \sum_{k=1}^{\infty} \overline{\phi}_k(x) d^k + \sum_{k=m+1}^{\infty} \sum_{l=1}^{[\frac{k-1}{m}]} \overline{\psi}_{k,l}(x) d^k (\log(-d))^l, \tag{1.6}$$

where $\overline{\phi}_k$ *and* $\overline{\psi}_{k,l} \in C^\infty(\overline{B^m}; \overline{B^n})$, $d(x) = |x| - 1$.

Remark 1.2 Here (1.6) is understood in the sense that, for every $m_0 \geq m+1$ and with $w_{m_0} = \phi_0 + \sum_{k=1}^{m_0} \overline{\phi}_k(x) d^k + \sum_{k=1}^{m_0} \sum_{l=m+1}^{[\frac{k-1}{m}]} \overline{\psi}_{k,l}(x) d^k (\log(-d))^l$, we have $|\nabla_\rho^j \nabla_\theta^s (u - w_{m_0})| = O(d^{m_0+1-j-\varepsilon})$, for any $\varepsilon > 0$ and non-negative integers s, j.

Remark 1.3 $|\phi_0|^2 = 1$ on $\mathbb{S}^{m-1}$. Differentiating with respect to $\theta_j \in \mathbb{S}^{m-1}$, we can get $\langle \phi_0, \partial_{\theta_j} \phi_0 \rangle = 0$ on $\mathbb{S}^{m-1}$, $j = 1, \cdots, m-1$.

The main challenge is that since harmonic map is a systems of semi-linear equations, the comparison principle is not applicable in some cases. To address this, we use the method of Fourier series expansion for functions on the Poincaré ball model to prove that taking the derivative in the tangential direction has little effect on the boundary regularity, see Lemma 3.2. For the mixed terms of different components in the system of equations, we regard it as a whole for estimate.

The organization of this paper is as follows. In Sect. 2, we construct a good approximate sollution $\overline{w}$. This approximate solution has the form $\overline{w} = \phi_0 + \sum_{k=1}^{m} \overline{\phi}_k(\rho - 1)^k + \overline{\psi}_{m+1,1}(\rho-1)^{m+1} \log(1-\rho)$, where $\overline{\phi}_k(\theta), \overline{\psi}_{m+1,1}(\theta) \in C^\infty(\mathbb{S}^{m-1}; \mathbb{S}^{n-1})$ and $\tau(\overline{w}) = O((1-\rho)^{m+2} \log(1-\rho))$. In Sect. 3, we extend methods and conclusions of [19] to the harmonic maps between asymptotically hyperbolic spaces . Moreover, by using the method of Fourier series expansion, we prove that taking derivatives in the tangential direction has little effect on the boundary regularity. In Sect. 4, we prove Theorem 1.1 by using the method of solving ordinary differential equations.

2 The coefficient functions $\phi_0, \cdots, \phi_m, \psi_{m+1,1}$

In this section, we mainly prove Theorem 2.1. The coefficients $\phi_0, \cdots, \phi_m, \psi_{m+1,1}$ are completely determined by ϕ_0. Based on the expressions of τ_0 and τ in (1.1) and (1.4), we can make a modification to $\phi_0, \cdots, \phi_m, \psi_{m+1,1}$ in Theorem 2.8. As a result, we obtain $\overline{\phi}_1, \overline{\phi}_2, \cdots, \overline{\phi}_m, \overline{\psi}_{m+1,1}$ in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ and the vector function $\overline{w} = \phi_0 + \sum_{k=1}^{m} \overline{\phi}_k d^k + \overline{\psi}_{m+1,1} d^{m+1} \log(-d)$, $d = \rho - 1$ such that $|\tau(\overline{w})| = O(d^{m+2} \log(-d))$. In fact, according to the proof of Theorem 2.1, for any $m_0 \geq 0$, we can obtain $\overline{w}_{m_0}$ such that $|\tau(\overline{w}_{m_0})| = O(d^{m_0+1} \log(-d))$. However, $\overline{\phi}_{m+1}$ cannot be completely determined by ϕ_0. We temporarily set the value of $\overline{\phi}_{m+1}$ to zero, and Lemma 4.2 will make the correction.

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Theorem 2.1 *Let $\phi_0 \in C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ be a boundary map with nowhere-vanishing energy density. There exist vector functions $\phi_1, \cdots, \phi_m, \psi_{m+1,1}$ in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ such that the vector function*

$$w = \sum_{k=0}^{m} \phi_k d^k + \psi_{m+1,1} d^{m+1} \log(-d), d = \rho - 1.$$

satisfies

$$(1)|\tau_0(w)| = O(d^{m+2} \log(-d)),$$
$$(2)\langle \phi_1, \phi_1 \rangle = \frac{\langle \nabla\phi_0, \nabla\phi_0 \rangle}{m-1}, \langle \phi_0, \phi_1 \rangle = \sqrt{\frac{\langle \nabla\phi_0, \nabla\phi_0 \rangle}{m-1}},$$
$$(3)\phi_1 = \sqrt{\frac{\langle \nabla\phi_0, \nabla\phi_0 \rangle}{m-1}}\phi_0,$$
$$(4)\frac{1}{C} \leq \frac{1-|w|^2}{1-\rho} \leq C \text{ for some constant } C > 0,$$

where $\phi_k(\theta) = \phi_k(s\theta), \psi_{m+1,1}(\theta) = \psi_{m+1,1}(s\theta)$, for any $s \in (0,1), \theta \in \mathbb{S}^{m-1}, k = 1, \cdots, m$.

Since $e(\phi_0) > 0$ on $\mathbb{S}^{n-1}$, we have $\frac{1-|w|^2}{1-\rho} = 2\langle \phi_0, \phi_1 \rangle > 0$ by (3). Let $u \in C^2(B^m, B^n)$. Then $\tau_0(u) = 0$ if and only if

$$\begin{aligned} &\rho(1-|u|^2)(1-\rho^2)\Delta_0 u + \sum_{i=1}^{m} 2(m-2)(1-|u|^2)\rho x^i \frac{\partial u}{\partial x^i} \\ &+ 2(1-\rho^2)\rho(2\langle u, \nabla_0 u \rangle \nabla_0 u - \langle \nabla_0 u, \nabla_0 u \rangle u) = 0. \end{aligned} \tag{2.1}$$

Lemma 2.2 *Let $T = \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i}$. If $\phi \in C^2(\overline{B^m} \backslash \{0\})$ with $\phi(x) = \phi(sx), \forall s \in (0,1)$, then $T\phi(x) = 0, \forall x \in B^m$.*

Proof $\phi(x) = \phi(sx), \forall s \in (0,1)$. Then for any $x \in \mathbb{B}^m$,

$$T\phi(x) = T(\phi(sx)) = \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i} \phi(sx) = \sum_{i=1}^{m} s x^i \frac{\partial \phi}{\partial x^i}(sx) \to 0, \text{ as } s \to 0.$$

■

Next, We will calculate Equation (2.1) in two parts.

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2.1 Computation of $\rho(1-|u|^2)(1-\rho^2)\Delta_0 u + \sum_{i=1}^{m} 2(m-2)(1-|u|^2)\rho x^i \frac{\partial u}{\partial x^i}$

Let $d := \rho - 1$, $T := \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i}$. Let $\phi_k(x) = \phi_k(sx)$ be smooth functions except the origin and

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k d(x)^k. \tag{2.2}$$

For $k \geq 0$ with $\phi_{-1} = \phi_{-2} = 0$, define

$$\begin{aligned} D_k[\phi] =& - [\Delta_0\phi_{k-3} + 3\Delta_0\phi_{k-2} + 2\Delta_0\phi_{k-1} + (k-1)(k+m-3)\phi_{k-1} \\ &+ k(3k+2m-5)\phi_k + 2k(k+1)\phi_{k+1}] + 2(m-2)[(k-1)\phi_{k-1} \\ &+ 2k\phi_k + (k+1)\phi_{k+1}]. \end{aligned} \tag{2.3}$$

Lemma 2.3 *Let $\phi \in C^\infty(\overline{B^m}, \overline{B^n})$ with the asymptotic expansion* (2.2) *near ∂B^m. Then*

$$\rho(1-\rho^2)\Delta_0\phi + \sum_{i=1}^{m} 2(m-2)\rho x^i \frac{\partial\phi}{\partial x^i} = \sum_{k=0}^{\infty} D_k[\phi] d^k. \tag{2.4}$$

Proof Notice that for any non-negative integer k and for each i ranging from 1 to m, it holds that

$$\begin{aligned} \frac{\partial(\phi_k d^k)}{\partial x^i} &= \frac{\partial\phi_k}{\partial x^i} d^k + k\phi_k d^{k-1}\frac{x^i}{\rho}, \\ \frac{\partial^2(\phi_k d^k)}{\partial(x^i)^2} &= \frac{\partial^2\phi_k}{\partial(x^i)^2} d^k + 2k\frac{\partial\phi_k}{\partial x^i} d^{k-1}\frac{x^i}{\rho} \\ &\quad + k(k-1)\phi_k d^{k-2}(\frac{x^i}{\rho})^2 + k\phi_k d^{k-1}(\frac{1}{\rho} - \frac{(x^i)^2}{\rho^3}). \end{aligned} \tag{2.5}$$

Recall the definitions $\rho = d+1$, $1-\rho^2 = -d(d+2)$. Summing the second equation in (2.5) over i from 1 to m and then multiplying the result by $\rho(1-\rho^2)$, we can calculate $\Delta_0(\phi_k d^k)$ multiplied by $(1-\rho^2)\rho$. According to Lemma 2.2, we have

$$\begin{aligned} &(1-\rho^2)\rho\Delta_0(\phi_k d^k) \\ =& - [\Delta_0\phi_k d^{k+1}(d+1)(d+2) + k(k-1)\phi_k d^{k-1}(d+1)(d+2) \\ &+ k(m-1)\phi_k d^k(d+2)] \\ =& - [\Delta_0\phi_k d^{k+3} + 3\Delta_0\phi_k d^{k+2} + (2\Delta_0\phi_k + k(k-1)\phi_k + k(m-1)\phi_k)d^{k+1} \\ &+ (3k(k-1)\phi_k + 2k(m-1)\phi_k)d^k + 2k(k-1)\phi_k d^{k-1}]. \end{aligned}$$

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Next, by summing the above expression over k from 0 to ∞, we get

$$\begin{aligned}
&(1-\rho^2)\rho\Delta_0\phi \\
=&-[\sum_{k=3}^{\infty}\Delta_0\phi_{k-3}d^k+3\sum_{k=2}^{\infty}\Delta_0\phi_{k-2}d^k \\
&+\sum_{k=1}^{\infty}(2\Delta_0\phi_{k-1}+(k-1)(k-2)\phi_{k-1}+(k-1)(m-1)\phi_{k-1})d^k \\
&+\sum_{k=0}^{\infty}(3k(k-1)\phi_k+2k(m-1)\phi_k)d^k+2\sum_{k=0}^{\infty}k(k+1)\phi_{k+1}d^k] \qquad (2.6)\\
=&-\sum_{k=0}^{\infty}[\Delta_0\phi_{k-3}+3\Delta_0\phi_{k-2}+2\Delta_0\phi_{k-1}+(k-1)(k+m-3)\phi_{k-1} \\
&+k(3k+2m-5)\phi_k+2k(k+1)\phi_{k+1}]d^k \\
=:&\sum_{k=0}^{\infty}D_k^1[\phi]d^k.
\end{aligned}$$

We multiply the first equation of (2.5) by $2(m-2)\rho x^i$ and then sum over i from 1 to m. By applying Lemma 2.2, we can derive

$$\begin{aligned}
\sum_{i=1}^{m}2(m-2)\rho x^i\frac{\partial(\phi_k d^k)}{\partial x^i}=&\sum_{i=1}^{m}2(m-2)\rho x^i(\frac{\partial\phi_k}{\partial x^i}d^k+k\phi_k d^{k-1}\frac{x^i}{\rho}) \\
=&2(m-2)k\phi_k d^{k-1}(d+1)^2 \qquad (2.7)\\
=&2(m-2)[k\phi_k d^{k+1}+2k\phi_k d^k+k\phi_k d^{k-1}].
\end{aligned}$$

We sum the equation (2.7) over k from 0 to ∞. This summation yields

$$\begin{aligned}
&\sum_{i=1}^{m}2(m-2)\rho x^i\frac{\partial\phi}{\partial x^i} \\
=&\sum_{k=0}^{\infty}2(m-2)[k\phi_k d^{k+1}+2k\phi_k d^k+k\phi_k d^{k-1}] \\
=&\sum_{k=0}^{\infty}2(m-2)[(k-1)\phi_{k-1}d^k+2k\phi_k d^k+(k+1)\phi_{k+1}d^k] \qquad (2.8)\\
=:&\sum_{k=0}^{\infty}D_k^2[\phi]d^k.
\end{aligned}$$

By adding the equation (2.6) and (2.8), we obtain

$$\begin{aligned}
&(1-\rho^2)\rho\Delta_0\phi+\sum_{i=1}^{m}2(m-2)\rho x_i\frac{\partial\phi}{\partial x_i} \\
=&\sum_{k=0}^{\infty}(D_k^1[\phi]+D_k^2[\phi])d^k=\sum_{k=0}^{\infty}D_k[\phi]d^k.
\end{aligned}$$

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Next, we will use the following notations:

$$\psi_l(x) = \sum_{k=1}^{\infty} \psi_{k,l} d^k \text{ with } \psi_{k,l} = 0 \text{ when } k < lm + 1$$

and

$$u(x) = \phi(x) + \sum_{l=1}^{\infty} \psi_l(x)(\log(-d))^l.$$

First, we have the operator $D_k[u]$ given by

$$\begin{aligned} D_k[u] =& D_k[\phi] - 2\xi_{k,1}[\psi_1] + \eta_{k,1}[\psi_1] - (m-1)\zeta_{k,1}[\psi_1] - 2\eta_{k,2}[\psi_2] \\ &+ 2(m-2)(\psi_{k-1,1} + 2\psi_{k,1} + \psi_{k+1,1}) \end{aligned} \tag{2.9}$$

and the operator $D_{k,l}[u]$ defined as

$$\begin{aligned} D_{k,l}[u] =& D_k[\psi_l] - 2(l+1)\xi_{k,l+1}[\psi_{l+1}] + (l+1)\eta_{k,l+1}[\psi_{l+1}] \\ &- (l+1)(m-1)\zeta_{k,l+1}[\psi_{l+1}] - (l+1)(l+2)\eta_{k,l+2}[\psi_{l+2}] \\ &+ 2(m-2)(l+1)(\psi_{k-1,l+1} + 2\psi_{k,l+1} + \psi_{k+1,l+1}), \end{aligned} \tag{2.10}$$

where $D_k[\phi]$ is defined by (2.3) and we have the following definitions for the terms involving $\xi_{k,l}[\psi_l]$, $\eta_{k,l}[\psi_l]$, and $\zeta_{k,l}[\psi_l]$,

$$\xi_{k,l}[\psi_l] = (k-1)\psi_{k-1,l} + 3k\psi_{k,l} + 2(k+1)\psi_{k+1,l}, \tag{2.11}$$

$$\eta_{k,l}[\psi_l] = \psi_{k-1,l} + 3\psi_{k,l} + 2\psi_{k+1,l}, \tag{2.12}$$

$$\zeta_{k,l}[\psi_l] = \psi_{k-1,l} + 2\psi_{k,l}. \tag{2.13}$$

For the function $(\log(-d))^l$, notice the following partial derivative relationships

$$\begin{aligned} \frac{\partial}{\partial x_i}(\log(-d))^l =& l(\log(-d))^{l-1}\frac{x^i}{d\rho}, \\ \frac{\partial^2(\log(-d))^l}{\partial(x^i)^2} =& l(l-1)(\log(-d))^{l-2}\frac{1}{d^2}\frac{(x^i)^2}{\rho^2} \\ &- l(\log(-d))^{l-1}\frac{1}{d^2}\frac{(x^i)^2}{\rho^2} + l(\log(-d))^{l-1}\frac{1}{d}(\frac{1}{\rho} - \frac{(x^i)^2}{\rho^3}). \end{aligned} \tag{2.14}$$

Additionally, we have the following summation and Laplacian relationships

$$\begin{aligned} &\sum_{i=1}^{m} x^i \frac{\partial(\log(-d))^l}{\partial x^i} = l\frac{d+1}{d}(\log(-d))^{l-1}, \\ &\Delta_0(\log(-d))^l \\ =& l(l-1)d^{-2}(\log(-d))^{l-2} - ld^{-2}(\log(-d))^{l-1} + l(m-1)d^{-1}\rho^{-1}(\log(-d))^{l-1}. \end{aligned}$$

Let $v_l = \psi_l(\log(-d))^l$. Recall the definitions $\rho = d+1, 1-\rho^2 = -d(d+2)$. According to the second equation in (2.14), we can calculate $(1-\rho^2)\rho\Delta_0 v_l$,

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$$
\begin{aligned}
&(1-\rho^2)\rho\Delta_0 v_l \\
=&(1-\rho^2)\rho[\Delta_0\psi_l(\log(-d))^l + \sum_{i=1}^{m} 2\frac{\partial\psi_l}{\partial x^i}\frac{\partial(\log(-d))^l}{\partial x^i} + \psi_l\Delta_0(\log(-d))^l] \\
=&-[\Delta_0\psi_l d(d+1)(d+2)(\log(-d))^l + \sum_{i=1}^{m} 2(d+2)\frac{\partial\psi_l}{\partial x^i}x^i l(\log(-d))^{l-1} \\
&+ l(l-1)d^{-1}(d+1)(d+2)\psi_l(\log(-d))^{l-2} - ld^{-1}(d+1)(d+2)\psi_l(\log(-d))^{l-1} \\
&+ l(m-1)(d+2)\psi_l(\log(-d))^{l-1}] \\
=&-[\Delta_0\psi_l d(d+1)(d+2)(\log(-d))^l \\
&+ (2(d+2)lT\psi_l - ld^{-1}(d+1)(d+2)\psi_l + l(m-1)(d+2)\psi_l)(\log(-d))^{l-1} \\
&+ l(l-1)d^{-1}(d+1)(d+2)\psi_l(\log(-d))^{l-2}],
\end{aligned}
$$

where $T = \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i}$. Finally, for the infinite sum $\sum_{l=1}^{\infty} v_l$, we have

$$
\begin{aligned}
&(1-\rho^2)\rho\Delta_0(\sum_{l=1}^{\infty} v_l) \\
=&-[\sum_{l=1}^{\infty}\Delta_0\psi_l d(d+1)(d+2)(\log(-d))^l \\
&+ \sum_{l=1}^{\infty}(2(d+2)(l+1)T\psi_{l+1} - (l+1)d^{-1}(d+1)(d+2)\psi_{l+1} \\
&+ (l+1)(m-1)(d+2)\psi_{l+1})(\log(-d))^l \\
&+ \sum_{l=1}^{\infty}(l+2)(l+1)d^{-1}(d+1)(d+2)\psi_{l+2}(\log(-d))^l + 2(d+2)T\psi_1 \\
&- d^{-1}(d+1)(d+2)\psi_1 + (m-1)(d+2)\psi_1 + 2d^{-1}(d+1)(d+2)\psi_2].
\end{aligned}
\tag{2.15}
$$

Similar to the computation of (2.6), we have

$$
-\sum_{l=1}^{\infty}\Delta_0\psi_l d(d+1)(d+2) = \rho(1-\rho^2)\Delta_0\psi_l = \sum_{k=1}^{\infty} D_k^1[\psi_l]d^k. \tag{2.16}
$$

By Lemma 2.2, $T\psi_{k,l} = 0$. Notice that $\psi_{k,l} = 0$ when $k < ml + 1$. We can computer the term $2(d+2)(l+1)T\psi_{l+1}$,

$$
\begin{aligned}
2(d+2)(l+1)T\psi_{l+1} =& 2(d+2)(l+1)\sum_{k=1}^{\infty}\psi_{k,l+1}Td^k \\
=& 2(d+2)(l+1)\sum_{k=1}^{\infty}\psi_{k,l+1}k(d^k + d^{k-1})
\end{aligned}
$$

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After arranging according to the powers of d, we can obtain

$$\begin{aligned}
&2(d+2)(l+1)T\psi_{l+1} \\
=&2(l+1)\sum_{k=1}^{\infty} k\psi_{k,l+1}(d^{k+1}+3d^k+2d^{k-1}) \\
=&2(l+1)\sum_{k=1}^{\infty}[(k-1)\psi_{k-1,l+1}+3k\psi_{k,l+1}+2(k+1)\psi_{k+1,l+1}]d^k \\
=&2(l+1)\sum_{k=1}^{\infty}\xi_{k,l+1}[\psi_{l+1}]d^k,
\end{aligned} \tag{2.17}$$

where $\xi_{k,l+1}[\psi_{l+1}]$ is defined in (2.11). Next, we can perform a calculation on the terms $(l+1)d^{-1}(d+1)(d+2)\psi_{l+1}$, $(l+1)(m-1)(d+2)\psi_{l+1}$ similar to that on $2(d+2)(l+1)T\psi_{l+1}$ to obtain

$$\begin{aligned}
&(l+1)d^{-1}(d+1)(d+2)\psi_{l+1} \\
=&(l+1)(d^2+3d+2)\sum_{k=1}^{\infty}\psi_{k,l+1}d^{k-1} \\
=&(l+1)\sum_{k=1}^{\infty}(\psi_{k-1,l+1}+3\psi_{k,l+1}+2\psi_{k+1,l+1})d^k \\
=&(l+1)\sum_{k=1}^{\infty}\eta_{k,l+1}[\psi_{l+1}]d^k,
\end{aligned} \tag{2.18}$$

where $\eta_{k,l+1}[\psi_{l+1}]$ is defined in (2.12) and

$$\begin{aligned}
(l+1)(m-1)(d+2)\psi_{l+1} =&(l+1)(m-1)(d+2)\sum_{k=1}^{\infty}\psi_{k,l+1}d^k \\
=&(l+1)(m-1)\sum_{k=1}^{\infty}(\psi_{k-1,l+1}+2\psi_{k,l+1})d^k \\
=&(l+1)(m-1)\sum_{k=1}^{\infty}\zeta_{k,l+1}[\psi_{l+1}]d^k,
\end{aligned} \tag{2.19}$$

where $\zeta_{k,l+1}[\psi_{l+1}]$ is defined in (2.13). And finally

$$\begin{aligned}
&(l+2)(l+1)d^{-1}(d+1)(d+2)\psi_{l+2} \\
=&(l+1)(l+2)\sum_{k=1}^{\infty}(\psi_{k-1,l+2}+3\psi_{k,l+2}+2\psi_{k+1,l+2})d^k \\
=&(l+1)(l+2)\sum_{k=1}^{\infty}\eta_{k,l+2}[\psi_{l+2}]d^k.
\end{aligned} \tag{2.20}$$

Therefore substituting (2.16)-(2.20) into (2.15), we have

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$$
\begin{aligned}
&(1-\rho^2)\rho\Delta_0(\sum_{l=1}^{\infty} v_l)\\
&=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}(D_k^1[\psi_l]-2(l+1)\xi_{k,l+1}[\psi_{l+1}]+(l+1)\eta_{k,l+1}[\psi_{l+1}]\\
&\quad-(l+1)(m-1)\zeta_{k,l+1}[\psi_{l+1}]-(l+1)(l+2)\eta_{k,l+2}[\psi_{l+2}])d^k(\log(-d))^l\\
&\quad+\sum_{k=1}^{\infty}(-2\xi_{k,1}[\psi_1]+\eta_{k,1}[\psi_1]-(m-1)\zeta_{k,1}[\psi_1]-2\eta_{k,2}[\psi_2])d^k.
\end{aligned}
\tag{2.21}
$$

For the second part

$$
\begin{aligned}
&2(m-2)\rho\sum_{i=1}^{m} x_i\frac{\partial v_l}{\partial x_i}=2(m-2)\rho[\sum_{i=1}^{m} x_i\frac{\partial \psi_l}{\partial x_i}(\log(-d))^l+\psi_l x^i\frac{\partial(\log(-d))^l}{\partial x^i}]\\
&=\sum_{k=1}^{\infty} D_k^2[\psi_l]d^k(\log(-d))^l+2(m-2)l\psi_l(d+2+\frac{1}{d})(\log(-d))^{l-1}.
\end{aligned}
$$

Summing l from 1 to ∞, we get

$$
\begin{aligned}
&2(m-2)\rho\sum_{i=1}^{m} x_i\frac{\partial}{\partial x_i}(\sum_{l=1}^{\infty} v_l)\\
&=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} D_k^2[\psi_l]d^k(\log(-d))^l+\sum_{l=1}^{\infty}2(m-2)l\psi_l(d+2+\frac{1}{d})(\log(-d))^{l-1}\\
&=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} D_k^2[\psi_l]d^k(\log(-d))^l\\
&\quad+\sum_{l=1}^{\infty}2(m-2)(l+1)\psi_{l+1}(d+2+\frac{1}{d})(\log(-d))^l+2(m-2)\psi_1(d+2+\frac{1}{d})\\
&=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}(D_k^2[\psi_l]+2(m-2)(l+1)(\psi_{k-1,l+1}+2\psi_{k,l+1}+\psi_{k+1,l+1}))\\
&\quad\times d^k(\log(-d))^l+2(m-2)\sum_{k=1}^{\infty}(\psi_{k-1,1}+2\psi_{k,1}+\psi_{k+1,1})d^k.
\end{aligned}
\tag{2.22}
$$

Therefore the summation of equations (2.21), (2.22) and (2.4) yields

$$
\begin{aligned}
&((1-\rho^2)\rho\Delta_0+2(m-2)\rho T)(\phi+\sum_{l=1}^{\infty}\psi_l(x)(\log(-d))^l)\\
&=\sum_{k=0}^{\infty} D_k[u]d^k+\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} D_{k,l}[u]d^k(\log(-d))^l,
\end{aligned}
$$

where $D_k[u]$ and $D_{k,l}[u]$ are defined in (2.9) and (2.10).

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As a summary, we have proved the following lemma.

***Lemma* 2.4** *Let $\psi_{k,l} = 0$ when $k < ml + 1$ and let*

$$u(x) = \sum_{k=0}^{\infty} \phi_k(x) d^k + \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \psi_{k,l}(x) d^k (\log(-d))^l. \tag{2.23}$$

Then

$$\begin{aligned} &((1-\rho^2)\rho\Delta_0 + 2(m-2)\rho T)u(x) \\ =& \sum_{k=0}^{\infty} D_k[u] d^k + \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} D_{k,l}[u] d^k (\log(-d))^l, \end{aligned}$$

where $D_k[u]$ and $D_{k,l}[u]$ are defined in (2.9) *and* (2.10)*.*

Define

$$A_{p,q}[\phi] := \langle \phi_p, \phi_q \rangle, \; A_k[\phi] := \sum_{p+q=k} A_{p,q}[\phi],$$

$$A_{k,l}[u] := \sum_{\alpha+\beta=k} (\langle \phi_\alpha, \psi_{\beta,l} \rangle + \langle \phi_\beta, \psi_{\alpha,l} \rangle) + \sum_{s+t=l} \sum_{\alpha+\beta=k} \langle \psi_{\alpha,s}, \psi_{\beta,t} \rangle,$$

$$B_k[u] := - \sum_{\alpha+\beta=k} A_\alpha[\phi] D_\beta[u], \tag{2.24}$$

and

$$\begin{aligned} B_{k,l}[u] := &- \sum_{\alpha+\beta=k} (A_\alpha[\phi] D_{\beta,l}[u] + A_{\alpha,l} D_\beta[u]) \\ &- \sum_{s+t=l} \sum_{\alpha+\beta=k} A_{\alpha,s}[u] D_{\beta,t}[u]. \end{aligned} \tag{2.25}$$

***Theorem* 2.5** *Let u be defined in* (2.23) *with $|\phi_0| = 1$, then*

$$\begin{aligned} &(1-|u|^2)((1-\rho^2)\rho\Delta_0 + 2(m-2)\rho T)u(x) \\ =& \sum_{k=0}^{\infty} B_k[u] d^k + \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} B_{k,l}[u] d^k (\log(-d))^l. \end{aligned}$$

Proof According to the expression of u, we have

$$\begin{aligned} 1-|u|^2 =& 1 - \sum_{k=0}^{\infty} \sum_{\alpha+\beta=k} \langle \phi_\alpha, \phi_\beta \rangle d^k \\ &- \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \sum_{\alpha+\beta=k} (\langle \phi_\alpha, \psi_{\beta,l} \rangle + \langle \phi_\beta, \psi_{\alpha,l} \rangle) d^k (\log(-d))^l \\ &- \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \sum_{\alpha+\beta=k} \sum_{s+t=l} \langle \psi_{\alpha,s}, \psi_{\beta,t} \rangle d^k (\log(-d))^l \end{aligned}$$

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Since $\langle \phi_0, \phi_0 \rangle = 1$, we can get

$$1 - |u|^2 = -\sum_{k=1}^{\infty} A_k[\phi] d^k - \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} A_{l,k}[u] d^k (\log(-d))^l.$$

By Lemma 2.4, substituting the expression for $1 - |u|^2$ obtained above, we get

$$\begin{aligned}
&-(1-|u|^2)((1-\rho^2)\rho\Delta_0 + 2(m-2)\rho T)u(x)\\
=&\sum_{k=1}^{\infty}\sum_{\alpha+\beta=k} A_\alpha[\phi] D_\beta[u] d^k\\
&+\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\alpha+\beta=k}(A_\alpha[\phi]D_{\beta,l}[u] + A_{\alpha,l}D_\beta[u])d^k(\log(-d))^l\\
&+\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\alpha+\beta=k}\sum_{s+t=l} A_{\alpha,s}[u]D_{\beta,t}[u]d^k(\log(-d))^l\\
=&-\sum_{k=1}^{\infty} B_k[u]d^k - \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} B_{k,l}[u]d^k(\log(-d))^l.
\end{aligned}$$

The proof of the proposition is complete. ■

2.2 Computation of $2(1-\rho^2)\rho(2\langle u, \nabla_0 u\rangle \nabla_0 u - \langle \nabla_0 u, \nabla_0 u\rangle u)$

Given that $u(x) = \sum_{k=0}^{\infty} \phi_k(x) d^k + \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} \psi_{k,l}(x) d^k (\log(-d))^l$, we can proceed to calculate its partial derivative with respect to x_i,

$$\begin{aligned}
\frac{\partial u}{\partial x_i} =& \sum_{k=0}^{\infty}\left(\frac{\partial \phi_k}{\partial x_i} d^k + k\phi_k d^{k-1}\frac{x_i}{\rho}\right) + \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\Big[\frac{\partial \psi_{k,l}}{\partial x_i} d^k (\log(-d))^l\\
&+ k\psi_{k,l} d^{k-1}\frac{x_i}{\rho}(\log(-d))^l + l\psi_{k,l} d^{k-1}\frac{x_i}{\rho}(\log(-d))^{l-1}\Big]\\
=& \sum_{k=0}^{\infty}\left(\frac{\partial \phi_k}{\partial x_i} d^k + [(k+1)\phi_{k+1} + \psi_{k+1,1}] d^k \frac{x_i}{\rho}\right.\\
&+ \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\frac{\partial \psi_{k,l}}{\partial x_i} d^k (\log(-d))^l\\
&+ [(k+1)\psi_{k+1,l} + (l+1)\psi_{k+1,l+1}] d^k \frac{x_i}{\rho}(\log(-d))^l
\end{aligned} \tag{2.26}$$

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Recall that the definition of $\langle\cdot,\cdot\rangle$, we can compute

$$\begin{aligned}\langle u, \frac{\partial u}{\partial x_i}\rangle \frac{\partial u}{\partial x_i} &= \sum_{k=0}^{\infty} \Omega_k d^k + \sum_{l=1}^{\infty}\sum_{k=m}^{\infty}\sum_{p+q+j=k} \Omega_{pqj} d^k (\log(-d))^l \\ &+ \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p+q+j=k}\sum_{\alpha+\beta=l} \Omega_{pqj\alpha\beta} d^k (\log(-d))^l \\ &+ \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p+q+j=k}\sum_{\alpha+\beta+\gamma=l} \Omega_{pqj\alpha\beta\gamma} d^k (\log(-d))^l.\end{aligned} \tag{2.27}$$

Here, the coefficients Ω_k, Ω_{pqj}, $\Omega_{pqj\alpha\beta}$ and $\Omega_{pqj\alpha\beta\gamma}$ are defined as follows,

$$\begin{aligned}\Omega_k = &\sum_{p+q+j=k} \langle \frac{\partial\phi_p}{\partial x_i}, \phi_q\rangle \frac{\partial\phi_j}{\partial x_i} \\ &+ \left(\langle \frac{\partial\phi_p}{\partial x_i}, \phi_q\rangle(((j+1)\phi_{j+1} + \psi_{j+1,1})) + \langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \phi_q\rangle \frac{\partial\phi_j}{\partial x_i}\right) \frac{x_i}{\rho} \\ &+ \left(\langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \phi_q\rangle [(j+1)\phi_{j+1} + \psi_{j+1,1}]\right) (\frac{x_i}{\rho})^2,\end{aligned}$$

and

$$\begin{aligned}\Omega_{pqj} = &(\langle \frac{\partial\psi_{p,l}}{\partial x_i}, \phi_q\rangle \frac{\partial\phi_j}{\partial x_i} + \langle \frac{\partial\phi_p}{\partial x_i}, \phi_q\rangle \frac{\partial\psi_{j,l}}{\partial x_i} + \langle \frac{\partial\phi_p}{\partial x_i}, \psi_{q,l}\rangle \frac{\partial\phi_j}{\partial x_i}) \\ &+ \Bigg(\langle \frac{\partial\psi_{p,l}}{\partial x_i}, \phi_q\rangle [(j+1)\phi_{j+1} + \psi_{j+1,1}] + \langle (p+1)\psi_{p+1,l} + (l+1)\psi_{p+1,l+1}, \phi_q\rangle \frac{\partial\phi_j}{\partial x_i} \\ &+ \langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \phi_q\rangle \frac{\partial\psi_{j,l}}{\partial x_i} + \langle \frac{\partial\phi_p}{\partial x_i}, \phi_q\rangle [(j+1)\psi_{j+1,l} + (l+1)\psi_{j+1,l+1}] \\ &+ \langle \frac{\partial\phi_p}{\partial x_i}, \psi_{q,l}\rangle [(j+1)\phi_{j+1} + \psi_{j+1,1}] + \langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \psi_{q,l}\rangle \frac{\partial\phi_j}{\partial x_i}\Bigg) \frac{x^i}{\rho} \\ &+ \big(\langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \phi_q\rangle [(j+1)\psi_{j+1,l} + (l+1)\psi_{j+1,l+1}] \\ &+ \langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \psi_{q,l}\rangle [(j+1)\phi_{j+1} + \psi_{j+1,1}] \\ &+ \langle (p+1)\psi_{p+1,l} + (l+1)\psi_{p+1,l+1}, \phi_q\rangle [(j+1)\phi_{j+1} + \psi_{j+1,1}]\big) (\frac{x_i}{\rho})^2,\end{aligned}$$

and the expression of $\Omega_{pqj\alpha\beta}$ and $\Omega_{pqj\alpha\beta\gamma}$ are omitted for the sake of brevity.

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Sum for i from 1 to m and multiply ρ in (2.27) to find

$$\begin{aligned}
\rho\langle u, \nabla u\rangle \nabla u &= \sum_{k=0}^{\infty} \sum_{p+q+j=k} (\langle \phi_p, \nabla \phi_q\rangle \nabla \phi_j \\
&+ \langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \phi_q\rangle [(j+1)\phi_{j+1} + \psi_{j+1,1}]) d^k (d+1) \\
&+ \sum_{l=1}^{\infty} \sum_{k=m}^{\infty} \sum_{p+q+j=k} (\langle \nabla \psi_{p,l}, \phi_q\rangle \nabla \phi_j + \langle \nabla \phi_p, \phi_q\rangle \nabla \phi_{j,l} + \langle \nabla \phi_p, \psi_{q,l}\rangle \nabla \phi_j \\
&+ \langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \phi_q\rangle [(j+1)\psi_{j+1,l} + (l+1)\psi_{j+1,l+1}] \\
&+ \langle (p+1)\phi_{p+1} + \psi_{p+1,1}, \psi_{q,l}\rangle [(j+1)\phi_{j+1} + \psi_{j+1,1}] \\
&+ \langle (p+1)\psi_{p+1,l} + (l+1)\psi_{p+1,l+1}, \phi_q\rangle [(j+1)\phi_{j+1} + \psi_{j+1,1}]) d^k (d+1)(\log(-d))^l \\
&+ \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \sum_{p+q+j=k} (\sum_{\alpha+\beta=l} \Lambda_{pqj\alpha\beta} + \sum_{\alpha+\beta+\gamma=l} \Lambda_{pqj\alpha\beta\gamma}) d^k (d+1)(\log(-d))^l,
\end{aligned}$$

where $\Lambda_{pqj\alpha\beta}$ and $\Lambda_{pqj\alpha\beta\gamma}$ are omitted for the sake of brevity.

Define $H_k[u]$ and $H_{k,l}[u]$ such that

$$\rho\langle u, \nabla u\rangle \nabla u =: \sum_{k=0}^{\infty} H_k[u] d^k + \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} H_{k,l}[u] d^k (\log(-d))^l.$$

Consider $1-\rho^2 = -d(d+2)$. By multiplying both sides of the equation $\rho\langle u, \nabla u\rangle \nabla u$ by $4(1-\rho^2)$, we get

$$\begin{aligned}
4(1-\rho^2)\rho\langle u, \nabla u\rangle \nabla u &= -4d(d+2)\rho\langle u, \nabla u\rangle \nabla u \\
= -4(\sum_{k=2}^{\infty} H_{k-2}[u] d^k &+ \sum_{l=1}^{\infty} \sum_{k=3}^{\infty} H_{k-2,l}[u] d^k (\log(-d))^l) \\
-8(\sum_{k=1}^{\infty} H_{k-1}[u] d^k &+ \sum_{l=1}^{\infty} \sum_{k=2}^{\infty} H_{k-1,l}[u] d^k (\log(-d))^l) \\
=: \sum_{k=1}^{\infty} F_k[u] d^k &+ \sum_{l=1}^{\infty} \sum_{k=2}^{\infty} F_{k,l}[u] d^k (\log(-d))^l,
\end{aligned}$$

where

$$F_k[u] = -4H_{k-2}[u] - 8H_{k-1}[u] \tag{2.28}$$

and

$$F_{k,l}[u] = -4H_{k-2,l}[u] - 8H_{k-1,l}[u]. \tag{2.29}$$

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According to the expression of (2.23) and (2.26), we deduce

$$\begin{aligned}\langle\frac{\partial u}{\partial x^i},\frac{\partial u}{\partial x^i}\rangle u=&\sum_{k=0}^{\infty}X_k d^k+\sum_{l=1}^{\infty}\sum_{k=m}^{\infty}\sum_{p+q+j=k}X_{pqj}d^k(\log(-d))^l\\&+\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p+q+j=k}\sum_{\alpha+\beta=l}X_{pqj\alpha\beta}d^k(\log(-d))^l\\&+\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p+q+j=k}\sum_{\alpha+\beta+\gamma=l}X_{pqj\alpha\beta\gamma}d^k(\log(-d))^l,\end{aligned}\tag{2.30}$$

where

$$\begin{aligned}X_k=&\sum_{p+q+j=k}\langle\frac{\partial\phi_p}{\partial x^i},\frac{\partial\phi_q}{\partial x^i}\rangle\phi_j+\langle\frac{\partial\phi_p}{\partial x^i},(q+1)\phi_{q+1}+\psi_{q+1,1}\rangle\phi_j\frac{x^i}{\rho}\\&+\langle(p+1)\phi_{p+1}+\psi_{p+1,1},(q+1)\phi_{q+1}+\psi_{q+1,1}\rangle\phi_j(\frac{x^i}{\rho})^2,\end{aligned}$$

and

$$\begin{aligned}X_{pqj}=&\left(\langle\frac{\partial\phi_p}{\partial x^i},\frac{\partial\phi_q}{\partial x^i}\rangle\psi_{j,l}+\langle\frac{\partial\phi_p}{\partial x^i},\frac{\partial\psi_{q,l}}{\partial x^i}\rangle\phi_j\right)+\left(\langle\frac{\partial\phi_p}{\partial x^i},(q+1)\phi_{q+1}+\psi_{q+1,1}\rangle\psi_{j,l}\right.\\&\left.+\langle\frac{\partial\phi_p}{\partial x^i},(q+1)\psi_{q+1,l}+(l+1)\psi_{q+1,l+1}\rangle\phi_j+\langle(p+1)\phi_{p+1}+\psi_{p+1,1},\frac{\partial\psi_{q,l}}{\partial x^i}\rangle\phi_j\right)\frac{x^i}{\rho}\\&+(\langle(p+1)\phi_{p+1}+\psi_{p+1,1},(q+1)\phi_{q+1}+\psi_{q+1,1}\rangle\psi_{j,l}\\&+\langle(p+1)\phi_{p+1}+\psi_{p+1,1},(q+1)\psi_{q+1,l}+(l+1)\psi_{q+1,l+1}\rangle\phi_j)(\frac{x^i}{\rho})^2,\end{aligned}$$

and $X_{pqj\alpha\beta}$ and $X_{pqj\alpha\beta\gamma}$ are omitted for the sake of brevity.

Summing i from 1 to m and multiplying ρ in (2.30), we find that

$$\begin{aligned}&\rho\langle\nabla u,\nabla u\rangle u\\=&\sum_{k=0}^{\infty}\sum_{p+q+j=k}(\langle\nabla\phi_p,\nabla\phi_q\rangle\phi_j+\langle(p+1)\phi_{p+1}+\psi_{p+1,1},(q+1)\phi_{q+1}+\psi_{q+1,1}\rangle\phi_j)\\&\times d^k(d+1)+\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p+q+j=k}(\langle\nabla\phi_p,\nabla\phi_q\rangle\psi_{j,l}+\langle\nabla\phi_p,\nabla\psi_{q,l}\rangle\phi_j\\&+\langle(p+1)\phi_{p+1}+\psi_{p+1,1},(q+1)\phi_{q+1}+\psi_{q+1,1}\rangle\psi_{j,l}\\&+\langle(p+1)\phi_{p+1}+\psi_{p+1,1},(q+1)\psi_{q+1,l}+(l+1)\psi_{q+1,l+1}\rangle\phi_j)d^k(d+1)(\log(-d))^l\\&+\sum_{l=2}^{\infty}\sum_{k=1}^{\infty}\sum_{p+q+j=k}(\sum_{\alpha+\beta=l}\Xi_{pqj\alpha\beta}+\sum_{\alpha+\beta+\gamma=l}\Xi_{pqj\alpha\beta\gamma})d^k(d+1)(\log(-d))^l,\end{aligned}$$

where $\Xi_{pqj\alpha\beta}$ and $\Xi_{pqj\alpha\beta\gamma}$ are omitted for the sake of brevity.

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Define $W_k[u]$ and $W_{k,l}[u]$ such that

$$\rho\langle\nabla u,\nabla u\rangle u =: \sum_{k=0}^{\infty} W_k[u]d^k + \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} W_{k,l}[u]d^k(\log(-d))^l.$$

Then by multiplying both sides of the equation $\rho\langle\nabla u,\nabla u\rangle u$ by $-2(1-\rho^2)$, we conclude that

$$\begin{aligned}
&-2(1-\rho^2)\rho\langle\nabla u,\nabla u\rangle u = 2d(d+2)\rho\langle\nabla u,\nabla u\rangle u\\
=&2\Big(\sum_{k=2}^{\infty} W_{k-2}[u]d^k + \sum_{l=1}^{\infty}\sum_{k=3}^{\infty} W_{k-2,l}[u]d^k(\log(-d))^l\Big)\\
&+4\Big(\sum_{k=1}^{\infty} W_{k-1}[u]d^k + \sum_{l=1}^{\infty}\sum_{k=2}^{\infty} W_{k-1,l}[u]d^k(\log(-d))^l\Big)\\
=:&\sum_{k=1}^{\infty} E_k[u]d^k + \sum_{l=1}^{\infty}\sum_{k=2}^{\infty} E_{k,l}[u]d^k(\log(-d))^l.
\end{aligned}$$

where

$$E_k[u] = 2W_{k-2}[u] + 4W_{k-1}[u] \tag{2.31}$$

and

$$E_{k,l}[u] = 2W_{k-2,l}[u] + 4W_{k-1,l}[u]. \tag{2.32}$$

***Lemma* 2.6** *With notations above, one has*

$$\begin{aligned}
&2(1-\rho^2)\rho(2\langle u,\nabla_0 u\rangle\nabla_0 u - \langle\nabla_0 u,\nabla_0 u\rangle u)\\
=&\sum_{k=1}^{\infty}(F_k[u]+E_k[u])d^k + \sum_{l=1}^{\infty}\sum_{k=2}^{\infty}(F_{k,l}[u]+E_{k,l}[u])d^k(\log(-d))^l.
\end{aligned}$$

2.3 Proof of Theorem 2.1

***Theorem* 2.7** *With notations above, one has*

$$\begin{aligned}
&\rho(1-|u|^2)[(1-\rho^2)\Delta_0 + 2(m-2)T]u\\
&+2(1-\rho^2)\rho(2\langle u,\nabla_0 u\rangle\nabla_0 u - \langle\nabla_0 u,\nabla_0 u\rangle u)\\
=&\sum_{k=1}^{\infty}(B_k[u]+F_k[u]+E_k[u])d^k\\
&+\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}(B_{k,l}[u]+F_{k,l}[u]+E_{k,l}[u])d^k(\log(-d))^l,
\end{aligned}$$

where B_k and $B_{k,l}$ are given by (2.24) *and* (2.25)*, F_k and $F_{k,l}$ are given by* (2.28) *and* (2.29)*, E_k and $E_{k,l}$ are given by* (2.31) *and* (2.32)*.*

From the definition of $B_{k,l}$, $F_{k,l}$, and $E_{k,l}$, one has

$$B_{k,l} = F_{k,l} = E_{k,l} = 0 \text{ for } k < ml + 1.$$

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Theorem 2.7 implies that if $\tau_0(u) = 0$, then

$$\begin{cases} B_k[u] + E_k[u] + F_k[u] = 0, k \geq 1, \\ B_{k,l}[u] + E_{k,l}[u] + F_{k,l}[u] = 0, l \geq 1, k \geq ml + 1. \end{cases}$$

We will solve ϕ_k and $\psi_{k,l}$ through the above system of equations.

Let's analyze the coefficients B_k for different values of k. When $k = 1$, the coefficient $B_1[u]$ is given by

$$B_1[u] = -A_1[\phi]D_0[u], \tag{2.33}$$

where $A_1[\phi] = 2\langle\phi_0, \phi_1\rangle, D_0[u] = D_0[\phi] = 2(m-2)\phi_1$. When $2 \leq k \leq m$, the coefficient $B_k[u]$ can be expressed as

$$B_k[u] = -\sum_{q=1}^{k} A_q[\phi]D_{k-q}[u] = -A_k D_0 - \sum_{q=1}^{k-1} A_q[\phi]D_{k-q}[u]. \tag{2.34}$$

Since $1 \leq k - q \leq m - 1$, we have $D_{k-q}[u] = D_{k-q}[\phi]$. Recall that $A_{p,q}[\phi] = \langle\phi_p, \phi_q\rangle$ and $A_k[\phi] = \sum\limits_{p+q=k} A_{p,q}[\phi]$. When $1 \leq k \leq m - 1$, according to (2.3), we have

$$\begin{aligned} D_k[\phi] =& - [\Delta_0\phi_{k-3} + 3\Delta_0\phi_{k-2} + 2\Delta_0\phi_{k-1}] + (k-1)(m-k-1)\phi_{k-1} \\ &+ k(2m-3k-3)\phi_k + 2(k+1)(m-2-k)\phi_{k+1} \\ =&:2(k+1)(m-2-k)\phi_{k+1} + \widetilde{D}[\phi_k, \phi_{k-1}, \phi_{k-2}, \phi_{k-3}], \end{aligned} \tag{2.35}$$

where $\widetilde{D}[\phi_k, \phi_{k-1}, \phi_{k-2}, \phi_{k-3}]$ is determined by $\phi_k, \phi_{k-1}, \phi_{k-2}, \phi_{k-3}$.

Let us examine the coefficients F_k for different values of k. When $k = 1$, the coefficient $F_1[u]$ is given by

$$F_1[u] = -8H_0[u], H_0[u] = \langle\phi_0, \nabla\phi_0\rangle\nabla\phi_0 + \langle\phi_0, \phi_1\rangle\phi_1 = \langle\phi_0, \phi_1\rangle\phi_1 \tag{2.36}$$

and for $2 \leq k \leq m$, the coefficient $F_k[u]$ is defined by

$$\begin{aligned} F_k[u] =& - 4H_{k-2}[u] - 8H_{k-1}[u] \\ =& - 8k\langle\phi_0, \phi_k\rangle\phi_1 - 8k\langle\phi_0, \phi_1\rangle\phi_k + \widetilde{F}[\phi_0, \cdots, \phi_{k-1}], \end{aligned} \tag{2.37}$$

where $\widetilde{F}[\phi_0, \cdots, \phi_{k-1}]$ depends on $\phi_0, \cdots, \phi_{k-1}$.

Let us focus on the coefficients E_k and analyze their expressions for different values of k. When $k = 1$, the coefficient $E_1[u]$ is given by

$$E_1[u] = 4W_0[u], W_0[u] = \langle\nabla\phi_0, \nabla\phi_0\rangle\phi_0 + \langle\phi_1, \phi_1\rangle\phi_0 \tag{2.38}$$

and for $2 \leq k \leq m$, the coefficient $E_k[u]$ is defined through a combination of $W_{k-2}[u]$ and $W_{k-1}[u]$,

$$E_k[u] = 2W_{k-2}[u] + 4W_{k-1}[u] = 8k\langle\phi_k, \phi_1\rangle\phi_0 + \widetilde{E}[\phi_0, \cdots, \phi_{k-1}], \tag{2.39}$$

where $\widetilde{E}[\phi_0, \cdots, \phi_{k-1}]$ depends on $\phi_0, \cdots, \phi_{k-1}$.

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2.3.1 $B_1[u] + E_1[u] + F_1[u] = 0$

According to (2.33), (2.36) and (2.38), $B_1 + E_1 + F_1 = 0$ yields

$$-m\langle\phi_0, \phi_1\rangle\phi_1 + \langle\nabla\phi_0, \nabla\phi_0\rangle\phi_0 + \langle\phi_1, \phi_1\rangle\phi_0 = 0, \tag{2.40}$$

where we have used $|\phi_0|^2 = 1$ and $\langle\phi_0, \partial_{x^i}\phi_0\rangle = 0$. Then multiplying (2.40) by ϕ_0 and ϕ_1 respectively, we get

$$\begin{cases} -mA_{1,0}^2 + A_{1,1} + \langle\nabla\phi_0, \nabla\phi_0\rangle = 0, \\ (1-m)A_{1,0}A_{1,1} + \langle\nabla\phi_0, \nabla\phi_0\rangle A_{1,0} = 0. \end{cases}$$

Choosing $A_1 > 0$, we can get

$$A_{1,1} = A_{1,0}^2 = \frac{\langle\nabla\phi_0, \nabla\phi_0\rangle}{m-1}. \tag{2.41}$$

So we can solve ϕ_1 from (2.40) that

$$\phi_1 = \frac{\langle\nabla\phi_0, \nabla\phi_0\rangle\phi_0 + \langle\phi_1, \phi_1\rangle\phi_0}{m\langle\phi_0, \phi_1\rangle} = \sqrt{\frac{\langle\nabla\phi_0, \nabla\phi_0\rangle}{m-1}}\phi_0. \tag{2.42}$$

In particularly, ϕ_0 and ϕ_1 are linearly dependent.

2.3.2 $B_k[u] + E_k[u] + F_k[u] = 0$ when $k = 2, \cdots, m$

According to (2.34), (2.35), (2.37) and (2.39), $B_k[u]+E_k[u]+F_k[u] = 0$ implies that

$$\begin{aligned} 0 =& - A_k[\phi]D_0[\phi] - A_1D_{k-1}[\phi] - 8kA_{k,0}\phi_1 - 8kA_{1,0}\phi_k + 8kA_{k,1}\phi_0 \\ &+ \widetilde{H}[\phi_0, \cdots, \phi_{k-1}] \\ =& - (A_{k,0} + A_{0,k})D_0 - A_1 2k(m-k-1)\phi_k \\ &- 8kA_{k,0}\phi_1 - 8kA_{1,0}\phi_k + 8kA_{k,1}\phi_0 + \widetilde{H}[\phi_0, \cdots, \phi_{k-1}] \\ =& - 4[(m-2) + 2k]A_{k,0}\phi_1 - 2k(m-k+1)A_1\phi_k \\ &+ 8kA_{k,1}\phi_0 + \widetilde{H}[\phi_0, \cdots, \phi_{k-1}], \end{aligned} \tag{2.43}$$

where $\widetilde{H}[\phi_0, \cdots, \phi_{k-1}]$ depends on $\phi_0, \cdots, \phi_{k-1}$. Then we multiply (2.43) by ϕ_0 and consider $\phi_1 = A_{1,0}\phi_0$ to find

$$\begin{aligned} &\langle\widetilde{H}[\phi_0, \cdots, \phi_{k-1}], \phi_0\rangle \\ =&4[(m-2) + 2k]A_{k,0}A_{1,0} + 4k(m-k+1)A_{1,0}A_{k,0} - 8kA_{k,1} \end{aligned}$$

and

$$A_{k,0} = \frac{\langle\widetilde{H}[\phi_0, \cdots, \phi_{k-1}], \phi_0\rangle}{4A_{1,0}(m-2+km-k^2+k)}. \tag{2.44}$$

If $0 < k < \frac{m+1+\sqrt{(m+1)^2+4(m-2)}}{2}$, $-k^2 + (m+1)k + m - 2 > 0$. This ensures that the denominator of (2.44) is greater than 0 when $k = 2, \cdots, m$.

Then

$$\phi_k = \frac{-4(m-2)A_{k,0}A_{1,0}\phi_0 + \widetilde{H}[\phi_0, \cdots, \phi_{k-1}]}{4k(m-k+1)A_{1,0}}, k = 2, \cdots, m. \tag{2.45}$$

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When $k = m + 1$, the coefficient of ϕ_k in Equation (2.43) is equal to zero. So we cannot get ϕ_{m+1}.

2.3.3 $B_{m+1,1}[u] + E_{m+1,1}[u] + F_{m+1,1}[u] = 0$.

We first calculate several expressions of $B_{m+1,1}[u]$, $E_{m+1,1}[u]$ and $F_{m+1,1}[u]$. For $B_{m+1,1}[u]$, we have

$$\begin{aligned} B_{m+1,1}[u] =& - A_1[\phi]D_{m,1}[u] - D_0[u]A_{m+1,1}[u] \\ =& 8(m+1)\langle\phi_0,\phi_1\rangle\psi_{m+1,1} - 4(m-2)\langle\phi_0,\psi_{m+1,1}\rangle\phi_1. \end{aligned}$$

Next, we calculate

$$\begin{aligned} F_{m+1,1}[u] =& - 4H_{m-1,1}[u] - 8H_{m,1}[u] \\ =& - 8(m+1)(\langle\phi_0,\phi_1\rangle\psi_{m+1,1} + \langle\phi_0,\psi_{m+1,1}\rangle\phi_1). \end{aligned}$$

Meanwhile, the calculation result of $E_{m+1,1}[u]$ is

$$E_{m+1,1}[u] = 2W_{m-1,1}[u] + 4W_{m,1}[u] = 4(m+1)\langle\phi_1,\psi_{m+1,1}\rangle\phi_0.$$

Based on these expressions above, we obtained from $B_{m+1,1}[u] + E_{m+1,1}[u] + F_{m+1,1}[u] = 0$ that

$$-3m\langle\phi_0,\psi_{m+1,1}\rangle\phi_1 + (m+1)\langle\phi_1,\psi_{m+1,1}\rangle\phi_0 = 0.$$

Starting from this equation, through further rearrangement and simplification, we derive

$$(1-2m)\langle\phi_0,\psi_{m+1,1}\rangle = 0.$$

Since $1 - 2m \neq 0$, it can be further determined that

$$\langle\phi_0,\psi_{m+1,1}\rangle = 0. \tag{2.46}$$

It is important to note that ϕ_0 and $\psi_{m+1,1}$ are functions that map from the $m-1$ dimensional sphere $\mathbb{S}^{m-1}$ to the $n-1$ dimensional sphere $\mathbb{S}^{n-1}$.

2.3.4 $B_k[u] + E_k[u] + F_k[u] = 0$ when $m + 1 \leq k < 2m + 1$

First, we calculate the expressions for $B_k[u]$, $F_k[u]$, and $E_k[u]$. For $B_k[u]$,

$$B_k[u] = -A_1[\phi]D_{k-1}[u] - D_0[\phi]A_k[\phi] + \widetilde{B}_{k-1}[\phi_p,\psi_{p,1} : p < k]. \tag{2.47}$$

Here, the term $\widetilde{B}_{k-1}[\phi_p,\psi_{p,1} : p < k]$ is a function that relies on $\phi_p,\psi_{p,1}$ for all $p < k$. When $m \leq k \leq 2m$,

$$\begin{aligned} D_k[u] =& D_k[\phi] - 2\xi_{k,1}[\psi_1] + \eta_{k,1}[\psi_1] - (m-1)\zeta_{k,1}[\psi_1] - 2\eta_{k,2}[\psi_2] \\ &+ 2(m-2)(\psi_{k-1,1} + 2\psi_{k,1} + \psi_{k+1,1}) \\ =& - 2k(k+1)\phi_{k+1} + 2(m-2)(k+1)\phi_{k+1} - 4(k+1)\psi_{k+1,1} \\ &+ 2\psi_{k+1,1} + 2(m-2)\psi_{k+1,1} + \widetilde{D}[\phi_p,\psi_{p,1} : p < k] \\ =& 2(k+1)(m-2-k)\phi_{k+1} - 2(2k-m+3)\psi_{k+1,1} \\ &+ \widetilde{D}[\phi_p,\psi_{p,1} : p < k]. \end{aligned} \tag{2.48}$$

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Similarly, $\widetilde{D}[\phi_p, \psi_{p,1} : p < k]$ is also a function that depends on $\phi_p, \psi_{p,1}, p < k$. Substituting (2.48) into (2.47), we can get

$$\begin{aligned} B_k[u] =& - A_1[\phi]D_{k-1}[u] - D_0[u]A_k[\phi] + B_{k-1}[\phi_p, \psi_{p,1} : p < k] \\ =& - 4k(m-k-1)A_{1,0}\phi_k + 4(2k-m+1)A_{1,0}\psi_{k,1} \\ & - 4(m-2)A_{k,0}\phi_1 + \widetilde{B}[\phi_p, \psi_{p,1} : p < k]. \end{aligned}$$

For $F_k[u]$, by calculating based on (2.28), we obtain

$$\begin{aligned} F_k[u] =& - 4H_{k-2}[u] - 8H_{k-1}[u] \\ =& - 8(\langle k\phi_k + \psi_{k,1}, \phi_0\rangle\phi_1 + \langle\phi_0, \phi_1\rangle(k\phi_k + \psi_{k,1})) + F[\phi_p, \psi_{p,1} : p < k] \\ =& - 8k\langle\phi_0, \phi_k\rangle\phi_1 - 8k\langle\phi_0, \phi_1\rangle\phi_k - 8\langle\phi_0, \psi_{k,1}\rangle\phi_1 - 8\langle\phi_0, \phi_1\rangle\psi_{k,1} \\ & + F[\phi_p, \psi_{p,1} : p < k]. \end{aligned}$$

For $E_k[u]$, by calculating based on (2.31), we obtain

$$\begin{aligned} E_k[u] =& 2W_{k-2}[u] + 4W_{k-1}[u] \\ =& 8k\langle\phi_k, \phi_1\rangle\phi_0 + 8\langle\psi_{k,1}, \phi_1\rangle\phi_0 + \widetilde{E}[\phi_p, \psi_{p,1} : p < k]. \end{aligned}$$

Therefore, $B_k[u] + F_k[u] + E_k[u] = 0$ yields

$$\begin{aligned} 0 =& B_k[u] + F_k[u] + E_k[u] \\ =& - 4k(m-k+1)A_{1,0}\phi_k + 4(2k-m-1)A_{1,0}\psi_{k,1} \\ & - 4(m-2)A_{k,0}\phi_1 + \Phi_{k-1}[\phi_p, \psi_{p,1} : p < k]. \end{aligned} \tag{2.49}$$

When $k = m+1$, we multiply equation (2.49) by ϕ_0 and recall $\langle\psi_{m+1,1}, \phi_0\rangle = 0$ to discover

$$0 = 4(m+1)A_{1,0}\psi_{m+1,1} - 4(m-2)A_{m+1,0}\phi_1 + \Phi_{k-1}[\phi_p, \psi_{p,1} : p < k].$$

This implies that when $m > 2$,

$$A_{m+1,0} = \frac{\langle\Phi_m[\phi_0, \cdots, \phi_m], \phi_0\rangle}{4(m-2)A_{1,0}}$$

So

$$\psi_{m+1,1} = \frac{\langle\Phi_m[\phi_0, \cdots, \phi_m], \phi_0\rangle\phi_0 - \Phi_m[\phi_0, \cdots, \phi_m]}{4(m+1)\langle\phi_0, \phi_1\rangle}$$

We have obtained $\phi_1, \cdots, \phi_m, \psi_{m+1,1}$ in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ which is only related to ϕ_0. Let

$$w = \sum_{k=0}^{m} \phi_k d(x)^k + \psi_{m+1,1} d^{m+1} \log(-d), d = \rho - 1.$$

Then, according to the above discussion, w satisfies the conclusion of Theorem 2.1.

Theorem 2.8 *Let $\phi_0 \in C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ be a boundary map with nowhere-vanishing energy density. There exist vector functions $\overline{\phi}_1 = \phi_1, \overline{\phi}_2, \cdots, \overline{\phi}_m, \overline{\psi}_{m+1,1}$*

in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ *such that the vector function*

$$\overline{w} = \phi_0 + \sum_{k=1}^{m} \overline{\phi}_k d(x)^k + \overline{\psi}_{m+1,1} d^{m+1} \log(-d), d = \rho - 1.$$

satisfies

$$(1)|\tau(\overline{w})| = O(d^{m+2}\log(-d)) \text{ as } \rho \to 1^-;$$
$$(2)\langle \phi_1, \phi_1\rangle = \frac{\langle \nabla\phi_0, \nabla\phi_0\rangle}{m-1}, \langle \phi_0, \phi_1\rangle = \sqrt{\frac{\langle \nabla\phi_0, \nabla\phi_0\rangle}{m-1}};$$
$$(3)\phi_1 = \sqrt{\frac{\langle \nabla\phi_0, \nabla\phi_0\rangle}{m-1}}\phi_0;$$
$$(4)\frac{1}{C} \le \frac{1-|\overline{w}|^2}{1-\rho} \le C \text{ for some constant } C > 0.$$

where $\overline{\phi}_k(\theta) = \overline{\phi}_k(s\theta), \overline{\psi}_{m+1,1}(\theta) = \overline{\psi}_{m+1,1}(s\theta)$, *for any* $s \in (0,1), \theta \in \mathbb{S}^{m-1}, k = 1, \cdots, m$.

Proof Assume

$$u(x) = \phi_0 + \sum_{k=1}^{\infty} \overline{\phi}_k(x) d^k + \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} \overline{\psi}_{k,l}(x) d^k (\log(-d))^l. \tag{2.50}$$

Similar to Proposition 2.7, we have

$$\rho\frac{1-|u|^2}{1-\rho^2}\tau(u)$$
$$= \sum_{k=1}^{\infty}(\overline{B}_k[u] + \overline{F}_k[u] + \overline{E}_k[u])d^k + \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}(\overline{B}_{k,l}[u] + \overline{F}_{k,l}[u] + \overline{E}_{k,l}[u])d^k(\log(-d))^l.$$

Since $a, b > -1$, we see that

$$\overline{B}_1[u] = B_1[u], \overline{F}_1[u] = F_1[u], \overline{E}_1[u] = E_1[u].$$

So $\overline{B}_1[u] + \overline{F}_1[u] + \overline{E}_1[u] = 0$ yields $\overline{\phi}_1 = \phi_1$. Hence we have proved (2), (3) and (4) in Theorem 2.8.

For $k \ge 2$, we have

$$\overline{F}_k[u] = F_k[u] + f_1(x)F_{k-1}[u], \overline{E}_k[u] = E_k[u] + f_2(x)E_{k-1}[u],$$

where $f_1(x), f_2(x) = O((1-\rho^2)^{1+\min\{a,b\}})$ and they depend on h_{ij} and $\overline{h}_{ij}$.

For $\overline{B}_k[u]$, $\overline{B}_k[u] = B_k[u] + f_3(x)\hat{B}_{k-1}[u]$, where $\hat{B}_{k-1}[u]$ depends on $\phi_0, \overline{\phi}_p, \overline{\psi}_{p,1}, p < k$. Therefore when $2 \le k \le m$, similarly to (2.43) and (2.45), $\overline{B}_k[u] + \overline{F}_k[u] + \overline{E}_k[u] = 0$ implies that

$$\begin{aligned} 0 = &-4[(m-2)+2k]\langle\overline{\phi}_k, \phi_0\rangle\phi_1 - 2k(m-k+1)A_1\overline{\phi}_k \\ &+ 8k\langle\overline{\phi}_k, \phi_1\rangle\phi_0 + \hat{H}[\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_{k-1}], \end{aligned} \tag{2.51}$$

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where $\hat{H}[\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_{k-1}]$ depends on $\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_{k-1}$. Then

$$\langle \overline{\phi}_k, \phi_0 \rangle = \frac{\langle \hat{H}[\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_{k-1}], \phi_0 \rangle}{4A_{1,0}(m-2+km-k^2+k)}. \tag{2.52}$$

and

$$\overline{\phi}_k = \frac{-4(m-2)\langle \overline{\phi}_k, \phi_0 \rangle A_{1,0}\phi_0 + \hat{H}[\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_{k-1}]}{4k(m-k+1)A_{1,0}}, k = 2, \cdots, m.$$

When $k = m+1$, the coefficient of $\overline{\phi}_k$ in equation (2.51) is equal to zero. So we cannot get $\overline{\phi}_{m+1}$.

Since $a > -1$ and $b > -1$, we have $\overline{B}_{m+1,1}[u] = B_{m+1,1}[u], \overline{E}_{m+1,1}[u] = E_{m+1,1}[u]$ and $\overline{F}_{m+1,1}[u] = F_{m+1,1}[u]$. This implies that

$$\langle \phi_0, \overline{\psi}_{m+1,1} \rangle = 0. \tag{2.53}$$

When $m+1 \le k < 2m+1$, similar to (2.49), $\overline{B}_k[u] + \overline{F}_k[u] + \overline{E}_k[u] = 0$ implies that

$$\begin{aligned} 0 =& -4k(m-k+1)A_{1,0}\overline{\phi}_k + 4(2k-m-1)A_{1,0}\overline{\psi}_{k,1} \\ &- 4(m-2)\langle \overline{\phi}_k, \phi_0 \rangle \phi_1 + \overline{\Phi}_{k-1}[\phi_0, \overline{\phi}_p, \overline{\psi}_{p,1} : 1 \le p < k]. \end{aligned} \tag{2.54}$$

When $k = m+1$, we obtain

$$\begin{aligned} 0 =& 4(m+1)A_{1,0}\overline{\psi}_{m+1,1} - 4(m-2)\langle \overline{\phi}_{m+1}, \phi_0 \rangle \phi_1 \\ &+ \overline{\Phi}_{k-1}[\phi_0, \overline{\phi}_p, \overline{\psi}_{p,1} : 1 \le p < k]. \end{aligned} \tag{2.55}$$

Since $\langle \overline{\psi}_{m+1,1}, \phi_0 \rangle = 0$, multiplying (2.55) by ϕ_0 , we can deduce that when $m > 2$,

$$\langle \overline{\phi}_{m+1}, \phi_0 \rangle = \frac{\langle \overline{\Phi}_m[\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_m], \phi_0 \rangle}{4(m-2)A_{1,0}}$$

So $\overline{\psi}_{m+1,1}$ can be solved by (2.55),

$$\overline{\psi}_{m+1,1} = \frac{\langle \overline{\Phi}_m[\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_m], \phi_0 \rangle \phi_0 - \overline{\Phi}_m[\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_m]}{4(m+1)\langle \phi_0, \phi_1 \rangle}$$

We have obtained $\overline{\phi}_1, \cdots, \overline{\phi}_m, \overline{\psi}_{m+1,1}$ in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ which is related to ϕ_0, h_{ij} and $\overline{h}_{ij}$. Let

$$\overline{w} = \phi_0 + \sum_{k=1}^{m} \overline{\phi}_k d(x)^k + \overline{\psi}_{m+1,1} d^{m+1} \log(-d), d = \rho - 1.$$

Then, according to the above discussion, $\overline{w}$ satisfies the conclusion of Theorem 2.8. ∎

3 $C^{m-1,\alpha}(\forall \alpha \in (0,1))$ Regularity Near the Boundary

In the following, we will generalize the proof of $C^{m-1,\alpha}(\forall\alpha \in (0,1))$ regularity of harmonic maps between hyperbolic spaces in [19] to harmonic maps between

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asymptotically hyperbolic spaces. The difference is that we will use $\overline{w}$ constructed by Theorem 2.8 as the initial value of the heat flow.

By Theorem 5.2 of [18], there exists a solution $u(x,t)$ for the parabolic equation of harmonic maps with initial data $\overline{w}$

$$\begin{aligned} \partial_t u(x,t) &= \tau(u)(x,t), \quad (x,t) \in M \times (0,+\infty) \\ u(x,0) &= \overline{w}(x) \;\; x \in M \times \{t=0\}, \end{aligned}$$

such that $\lim\limits_{t\to\infty} u(x,t) = u_\infty(x)$ for some harmonic map u_∞ with bounded energy density from M to N. Moreover,

$$\lim_{x\to\infty} \hat{d}(u_\infty(x), \overline{w}(x)) = 0, \tag{3.1}$$

where $\hat{d}$ is the distance function of N. Expressing $u_\infty = (u_\infty^1, \cdots, u_\infty^n)$, in terms of rectangular coordinates, let us denote $|u_\infty|^2 = \sum\limits_{p=1}^{n} (u_\infty^p)^2$. Then by equation (3.1) and condition (4) of Theorem 2.8 for $\overline{w}$, we easily see that there is a constant $C > 0$ such that

$$\frac{1}{C} \le \frac{1-|u_\infty|}{1-\rho} \le C. \tag{3.2}$$

***Lemma* 3.1** *Let*

$$\begin{aligned} \Delta_M = &\sum_{i,j=1}^{m} \Big(\frac{(1-\rho^2)^2}{4}\delta_{ij} + O((1-\rho^2)^{4+a})\Big)\frac{\partial^2}{\partial x^i \partial x^j} \\ &+ \sum_{i=1}^{m} \frac{(m-2)}{2}((1-\rho^2) + O((1-\rho^2)^{3+a})x^i \frac{\partial}{\partial x^i}. \end{aligned}$$

For $0 < s \le m-1$, there is a constant $\delta \in (0,1)$ such that $(1-\rho)^s$ is a superharmonic function of M at $B_1 \setminus B_\delta$.

Proof Notice that $\Delta_{\mathbb{H}^m} = \frac{(1-\rho^2)^2}{4}\Delta_0 + \sum\limits_{i=1}^{m} \frac{(m-2)(1-\rho^2)}{2} x^i \frac{\partial}{\partial x^i}$, then

$$\Delta_M (1-\rho)^s = \Delta_{\mathbb{H}^m}(1-\rho)^s + O((1-\rho)^{s+2+a}).$$

According to Lemma 3.1 of [19],

$$\Delta_M(1-\rho)^s = \frac{s}{4}(1+\rho)(1-\rho)^{s+1}\Big(-1-\frac{m-1}{\rho}\Big) + O((1-\rho)^{s+2+a}).$$

Hence there is a constant $\delta \in (0,1)$ such that

$$\Delta_M(1-\rho)^s \le 0 \text{ in } B_1 \setminus B_\delta.$$

∎

Similar to the proof of Lemma 3.2 and Lemma 3.3 of [19], we have

$$d(u_\infty(x), \overline{w}(x)) = O((1-\rho)^{m-1}\log(\frac{1}{1-\rho})).$$

and

$$|u_\infty(x) - \overline{w}(x)| = O((1-\rho)^{m}\log(\frac{1}{1-\rho})).$$

From now on, since there is no ambiguity, we can simply denote u_∞ by u.

***Lemma* 3.2** *Assume the function $v(\rho,\theta)$ on the B^m satisfies $v = 0$ at the boundary and for any $\varepsilon > 0$, v is $C^{m-1,1-\varepsilon}$ up to the boundary. Also, for any $\varepsilon > 0$, $j \geq 0$,*

$$|\nabla^j v| = O((1-\rho)^{m-j-\varepsilon}) \text{ as } \rho \to 1.$$

Then for any $s, j \geq 0$, we have

$$|\nabla_\theta^s \nabla^j v| = O((1-\rho)^{m-j-\varepsilon}) \text{ as } \rho \to 1.$$

Proof Let $\varepsilon > 0$ be fixed and define $k := m - \varepsilon$. Consequently, we have $|v| = O((1-\rho)^k)$. Suppose $|\nabla v| = O((1-\rho)^{\overline{k}})$. By Lemma 4.1 of [19], it follows that

$$\sup\{\overline{k} : \ |\nabla v| = O((1-\rho)^{\overline{k}})\} = k - 1. \tag{3.3}$$

Let $d = 1-\rho$. Assume that $|\partial_\theta v| = O((1-\rho)^r)$ and define

$$s = \sup\{r : \ |\partial_\theta v| = O((1-\rho)^r)\}. \tag{3.4}$$

By the definition of supremum, $s - \delta < r \leq s, \forall \delta > 0$.

We aim to prove $s \geq k - \delta$ for all $\delta > 0$. We will proceed by contradiction. Assume $s < k - \delta_0$ for some $\delta_0 > 0$, then it follows that $r < k - \delta_0$.

Consider the case when $m = 2$, v has a Fourier series expansion

$$v = \frac{a_0}{2} + \sum_{l=1}^{\infty} a_l \cos(l\theta) + b_l \sin(l\theta),$$

where the Fourier coefficients are given by

$$a_l = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\cos l\theta \mathrm{d}\theta, l = 0, 1, 2, \cdots$$
$$b_l = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\sin l\theta \mathrm{d}\theta, l = 1, 2, \cdots$$

Given the decay properties of v, we have

$$a_l = O((1-\rho)^k), l = 0, 1, 2, \cdots$$
$$b_l = O((1-\rho)^k), l = 1, 2, \cdots$$

Differentiating v with respect to θ, we obtain

$$\frac{\partial v}{\partial \theta} = \sum_{l=1}^{\infty} -l a_l \sin(l\theta) + l b_l \cos(l\theta).$$

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Since $|\partial_\theta v| = O((1-\rho)^r)$, combining with (3.4) and $s < k - \delta_0$, there must exist some a_l or b_l (without loss of generality, assume it is a_l) such that $la_l = O((1-\rho)^r)$ and $la_l \neq O((1-\rho)^k)$. This implies that l satisfies $l = O((1-\rho)^{r-k})$. Therefore, v includes a term of the form $a_l \sin(h(\rho)(1-\rho)^{r-k}\theta)$, where $h \in L^\infty(\overline{B}) \cap C^\infty(B \backslash \{0\})$. And then $\frac{\partial v}{\partial \rho} = O((1-\rho)^{r-1})$. However $r - 1 < k - \delta_0 - 1$, which contradicts (3.3).

When $m \geq 3$, $v = v(\rho, \theta^1, \cdots, \theta^{m-1})$. Fix $\theta^2, \cdots, \theta^{m-1}$ and regard v as a function of ρ and θ^1. Then we can prove, similar to the case of $m = 2$, that $\frac{\partial v}{\partial \theta^1}$ satisfies $|\frac{\partial v}{\partial \theta^1}| = O((1-\rho)^{k-\delta})$ for any $\delta > 0$. Similarly, the partial derivative of v with respect to $\theta^2, \cdots, \theta^{m-1}$ also has the same conclusion.

For any $s \geq 0$, $g := (1-\rho)^s v = O((1-\rho)^{m+s-\varepsilon})$ and $g \in C^{m+s-1,1-\varepsilon}(\overline{B^m} \setminus \{0\})$. So we can use the method of Fourier expansion to prove $|\nabla_\theta^s g| = O((1-\rho)^{m+s-\varepsilon})$, that is $|\nabla_\theta^s v| = O((1-\rho)^{m-\varepsilon})$.

■

***Lemma* 3.3** *Let $\overline{w}$ be given by Theorem 2.8. Then we have for any $\varepsilon > 0$, $j \geq 0$ that*

$$|\nabla_0^j (u(x) - \overline{w}(x))| = O((1-\rho)^{m-j-\varepsilon}) \text{ as } \rho \to 1, \tag{3.5}$$

and for any $s, j \geq 0$, we have

$$|\nabla_\theta^s \nabla_0^j (u(x) - \overline{w}(x))| = O((1-\rho)^{m-j-\varepsilon}) \text{ as } \rho \to 1. \tag{3.6}$$

Proof Let $v := u - \overline{w}$ and define $d = 1 - \rho$. According to Lemma 3.3 and Theorem 4.2 of [19], we can assert that for any $\varepsilon > 0$ and integers j satisfying $0 \leq j \leq m-1$,$|v| = O(d^{m-\varepsilon})$ and $|\nabla_0^j v| = O(d^{m-j-\varepsilon})$. In our current analysis, instead of estimating $\Delta_0 v$ as was done in [19], we need to focus on estimating $\sum\limits_{i,j=1}^{m} (\delta_{ij} + O((1-\rho^2)^{2+a})) \frac{\partial^2 v}{\partial x^i \partial x^j}$. It is the second derivative part of Δ_M and thus it is an elliptic operator. Let $\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_m, \overline{\psi}_{m+1,1}$ in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ and $\overline{w}$ be given by Theorem 2.8. According to Theorem 2.8,

$$\begin{aligned} & O(d^{m+2}\log(-d)) = \tau(\overline{w}) \\ = & \sum_{i,j=1}^{m} (\frac{(1-\rho^2)^2}{4}\delta_{ij} + O((1-\rho^2)^{4+a})) \frac{\partial^2 \overline{w}}{\partial x^i \partial x^j} \\ & + \sum_{i=1}^{m} \frac{(m-2)}{2}((1-\rho^2) + O((1-\rho^2)^{3+a}) x^i \frac{\partial \overline{w}}{\partial x^i} \\ & + \frac{(1-\rho^2)^2 + O((1-\rho^2)^{4+a})}{2} (\frac{1}{1-|\overline{w}|^2} + O((1-|\overline{w}|^2)^{b+1}) \\ & \times (2\langle \overline{w}, \nabla_0 \overline{w} \rangle \nabla_0 \overline{w} - \langle \nabla_0 \overline{w}, \nabla_0 \overline{w} \rangle \overline{w}). \end{aligned} \tag{3.7}$$

Let u be a harmonic map from M to N so that $u \in C^1$ as a map from $\overline{B}^m$ to $\overline{B}^n$. Then

$$\tau(u) = 0. \tag{3.8}$$

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Note that we have the following expression for $\overline{w}$

$$\overline{w} = \phi_0 + \sum_{k=1}^{m} \overline{\phi}_k d(x)^k + \overline{\psi}_{m+1,1} d^{m+1} \log(-d). \tag{3.9}$$

For any $\varepsilon > 0$, since $u = \overline{w} + (u - \overline{w})$, we get

$$u = \overline{w} + (u - \overline{w}) = \phi_0 + \phi_1 d + O(d^{2-\varepsilon}). \tag{3.10}$$

Given that $|\phi_0|^2 = 1$, we can derive

$$|u|^2 - 1 = 2\langle \phi_0, \phi_1 \rangle d + O(d^{2-\varepsilon})$$

and

$$\frac{1 - |u|^2}{1 - \rho} = 2\langle \phi_0, \phi_1 \rangle + O(d^{1-\varepsilon}).$$

Thus, we have

$$\frac{(1-\rho^2)^2}{2(1-|u|^2)} = \frac{(1+\rho)^2(1-\rho)}{4\langle \phi_0, \phi_1 \rangle + O(d^{1-\varepsilon})}. \tag{3.11}$$

Additionally, consider the expression

$$\begin{aligned} &\frac{(1-\rho^2)^2 + O((1-\rho^2)^{4+a})}{2} \left(\frac{1}{1-|u|^2} + O((1-|u|^2)^{b+1}\right) \\ =&\frac{(1-\rho^2)^2}{2(1-|u|^2)}(1 + O(1-\rho)). \end{aligned} \tag{3.12}$$

By subtracting (3.7) from (3.8), we obtain

$$\begin{aligned} &O(d^{m+2}\log(-d)) = \tau(u) - \tau(\overline{w}) \\ =&\Delta_M v \\ &+ \frac{(1-\rho^2)^2}{2(1-|u|^2)}(1 + O(1-\rho)) \\ &\times [(2\langle u, \nabla_0 u\rangle \nabla_0 u - \langle \nabla_0 u, \nabla_0 u\rangle u) - (2\langle \overline{w}, \nabla_0 \overline{w}\rangle \nabla_0 \overline{w} - \langle \nabla_0 \overline{w}, \nabla_0 \overline{w}\rangle \overline{w})] \\ &+ \left(\frac{(1-\rho^2)^2}{2(1-|u|^2)} - \frac{(1-\rho^2)^2}{2(1-|\overline{w}|^2)}(1 + O(1-\rho)\right) \\ &\times (2\langle \overline{w}, \nabla_0 \overline{w}\rangle \nabla_0 \overline{w} - \langle \nabla_0 \overline{w}, \nabla_0 \overline{w}\rangle \overline{w}). \end{aligned} \tag{3.13}$$

The calculation of $(2\langle u, \nabla_0 u\rangle \nabla_0 u - \langle \nabla_0 u, \nabla_0 u\rangle u) - (2\langle \overline{w}, \nabla_0 \overline{w}\rangle \nabla_0 \overline{w} - \langle \nabla_0 \overline{w}, \nabla_0 \overline{w}\rangle \overline{w})$ in a certain expansion process yields

$$\begin{aligned} &O(d^{m+2}\log(-d)) \\ =&\Delta_M v + \frac{(1-\rho^2)^2}{2(1-|u|^2)}(1 + O(1-\rho))[2(\langle u, \nabla_0 u\rangle \nabla_0 v + (\langle v, \nabla_0 u\rangle \\ &+ \langle \overline{w}, \nabla_0 v\rangle)\nabla_0 \overline{w}) - (\langle \nabla_0 v, \nabla_0 u\rangle u + \langle \nabla_0 \overline{w}, \nabla_0 v\rangle u + \langle \nabla_0 \overline{w}, \nabla_0 \overline{w}\rangle v)] \\ &+ \left(\frac{(1-\rho^2)^2}{2(1-|u|^2)} - \frac{(1-\rho^2)^2}{2(1-|\overline{w}|^2)}\right)(1 + O(1-\rho)) \\ &\times (2\langle \overline{w}, \nabla_0 \overline{w}\rangle \nabla_0 \overline{w} - \langle \nabla_0 \overline{w}, \nabla_0 \overline{w}\rangle \overline{w}), \end{aligned} \tag{3.14}$$

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where $\Delta_M = \sum_{i,j=1}^{m}(\frac{(1-\rho^2)^2}{4}\delta_{ij} + O((1-\rho^2)^{4+a}))\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{m}\frac{(m-2)}{2}((1-\rho^2) + O((1-\rho^2)^{3+a})x^i\frac{\partial}{\partial x^i}$. For any $l > 0$, given that $|\nabla_0 v| = O((1-\rho)^{m-\varepsilon-1})$, we can deduce that $\sum_{i,j=1}^{m}(\delta_{ij} + O((1-\rho^2)^{2+a}))\frac{\partial^2}{\partial x^i \partial x^j}((1-\rho)^l v) = O((1-\rho)^{m+l-\varepsilon-2})$ and must be $C^{m+l-3,1-\varepsilon}$ up to the boundary. Together with the fact that $(1-\rho)^l v = 0$ at the boundary, this implies that $(1-\rho)^l v$ is $C^{m+l-1,1-\varepsilon}$ up to the boundary. In particular, $|\nabla^{m+l-1}((1-\rho)^l v)| = O((1-\rho)^{1-\varepsilon})$. Therefore we have

$$|\nabla_0^{m+l-1} v| = O((1-\rho)^{1-\varepsilon-l}),$$

and subsequently,

$$|\nabla_0^j v| = O(d^{m-j-\varepsilon}), \forall \varepsilon > 0, j \geq 0.$$

Finally, by applying Lemma 3.2, we can obtain (3.6).

■

***Theorem* 3.4** *Let M and N be asymptotically hyperbolic spaces with metrics* (1.2) *and* (1.3) *respectively in the Poincaré disk model. Let u be a harmonic map from M to N so that $u \in C^1$ as a map from $\overline{B^m}$ to $\overline{B^n}$. Suppose that the boundary map ϕ_0 of u, when restricted to $\mathbb{S}^{m-1}$, is in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$, and has nowhere-vanishing energy density with respect to the stanard metrics. Then $u \in C^{m-1,\alpha}(\overline{B^m}, \overline{B^n})$ for all* $0 < \alpha < 1$.

Proof Notice that $u - \overline{w} = 0$ on ∂B^m. According to Lemma 3.3 and Theorem 4.2 of [19], $u - \overline{w} \in C^{m-1,\alpha}(\overline{B^m}, \overline{B^n})$ and then $u \in C^{m-1,\alpha}(\overline{B^m}, \overline{B^n})$ for all $0 < \alpha < 1$. ■

4 Optimal Regularity Near the Boundary

4.1 Equations of $v = u - \overline{w}$

First, note the following expression for $\overline{w}$,

$$\overline{w} = \phi_0 + \sum_{k=1}^{m} \overline{\phi}_k d(x)^k + \overline{\psi}_{m+1,1} d^{m+1} \log(-d). \tag{4.1}$$

Differentiating $\overline{w}$ with respect to $x^i (i = 1, \cdots, m)$, we obtain

$$\frac{\partial \overline{w}}{\partial x^i} = \frac{\partial \phi_0}{\partial x^i} + \phi_1 \frac{x^i}{\rho} + O(d), i = 1, \cdots, m. \tag{4.2}$$

It follows that for any $\varepsilon > 0$,

$$\begin{aligned} u =& \overline{w} + (u - \overline{w}) = \phi_0 + \phi_1 d + O(d^{2-\varepsilon}), \\ \frac{\partial u}{\partial x^i} =& \frac{\partial \overline{w}}{\partial x^i} + \frac{\partial (u - \overline{w})}{\partial x^i} = \frac{\partial \phi_0}{\partial x^i} + \phi_1 \frac{x^i}{\rho} + O(d^{1-\varepsilon}), i = 1, \cdots, m. \end{aligned} \tag{4.3}$$

Taking into account equation (4.3) and the fact that $|\phi_0|^2 \equiv 1$, we can deduce that the inner product $\langle \phi_0, \frac{\partial \phi_0}{\partial x^i} \rangle = 0$ for all $i = 1, \cdots, m$. Taking into account (3) of

Theorem 2.8, we see that $\langle \phi_1, \frac{\partial \phi_0}{\partial x^i} \rangle = 0$ for all $i = 1, \cdots, m$ and then

$$2\langle u, \nabla_0 u \rangle \nabla_0 = 2 \sum_{i=1}^{m} (\langle \phi_0, \phi_1 \frac{x^i}{\rho} \rangle + O(d^{1-\varepsilon})) \frac{\partial}{\partial x^i}. \tag{4.4}$$

We can apply (3.11) and (4.4) to deduce

$$\begin{aligned}\frac{(1-\rho^2)^2}{(1-|u|^2)} \langle u, \nabla_0 u \rangle \nabla_0 =& \frac{(1+\rho)^2(1-\rho)}{2\langle \phi_0, \phi_1 \rangle + O(d^{1-\varepsilon})} \sum_{i=1}^{m} (\langle \phi_0, \phi_1 \frac{x^i}{\rho} \rangle + O(d^{1-\varepsilon})) \frac{\partial}{\partial x^i} \\ =& [\frac{(1+\rho)(1-\rho^2)}{2\rho} + O(d^{2-\varepsilon})] \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i} \\ =& [(1-\rho^2) + O(d^{2-\varepsilon})] \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i}.\end{aligned} \tag{4.5}$$

Define linear operator

$$\begin{aligned} L :=& \Delta_H + (1-\rho^2) \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i} \\ =& \frac{(1-\rho^2)^2}{4} \Delta_0 + \sum_{i=1}^{m} \frac{m(1-\rho^2)}{2} x^i \frac{\partial}{\partial x^i}. \end{aligned} \tag{4.6}$$

The right side of the equation (3.14) can be written in two parts. The first part is

$$\begin{aligned}&\Delta_M v + \frac{(1-\rho^2)^2}{2(1-|u|^2)} (1 + O(1-\rho)) [2(\langle u, \nabla_0 u \rangle \nabla_0 v + (\langle v, \nabla_0 u \rangle + \langle \overline{w}, \nabla_0 v \rangle) \nabla_0 \overline{w}) \\ &- (\langle \nabla_0 v, \nabla_0 u \rangle u + \langle \nabla_0 \overline{w}, \nabla_0 v \rangle u + \langle \nabla_0 \overline{w}, \nabla_0 \overline{w} \rangle v)] \\ =& Lv + \frac{(1-\rho^2)^2}{2(1-|u|^2)} [2\langle \overline{w}, \nabla_0 v \rangle \nabla_0 \overline{w} - (\langle \nabla_0 v, \nabla_0 u \rangle u + \langle \nabla_0 \overline{w}, \nabla_0 v \rangle u)] \\ &+ O(d^3)|\nabla_0^2 v| + O(d^{2-\varepsilon})|\nabla_0 v| + O(d)|v|.\end{aligned}$$

In the m-dimensional spherical coordinates $(\rho, \theta^1, \theta^2, \cdots, \theta^{m-1})$, the expression for the gradient operator ∇_0 is

$$\nabla_0 = \frac{\partial}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \sum_{k=1}^{m-1} \left(\frac{1}{\prod_{j=1}^{k-1} \sin \theta^j} \frac{\partial}{\partial \theta^k} \mathbf{e}_{\theta^k} \right), \tag{4.7}$$

where $\mathbf{e}_\rho$ is the radial unit vector and $\mathbf{e}_{\theta^k}$ are the angular unit vectors corresponding to the coordinates θ^k. $\prod_{j=1}^{k-1} \sin \theta^j$ denotes the product of $\sin \theta^j$ from $j = 1$ to $k-1$, with the convention that $\prod_{j=1}^{0} \sin \theta_j = 1$. Then (4.1), (4.2) and (4.3) imply

$$\begin{aligned}&2\langle \overline{w}, \nabla_0 v \rangle \nabla_0 \overline{w} - (\langle \nabla_0 v, \nabla_0 u \rangle u + \langle \nabla_0 \overline{w}, \nabla_0 v \rangle u). \\ =& 2\langle \phi_0, \partial_\rho v \rangle \phi_1 - 2\langle \partial_\rho v, \phi_1 \rangle \phi_0 + O(d^{1-\varepsilon})|\partial_\rho v| + O(1)|\nabla_\theta v|\end{aligned}$$

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Since (3) of Theorem 2.8, $\langle \phi_0, \partial_\rho v\rangle \phi_1 = \langle \partial_\rho v, \phi_1\rangle \phi_0$. And then

$$\begin{aligned}&\Delta_M v + \frac{(1-\rho^2)^2}{2(1-|u|^2)}(1+O(1-\rho))[2(\langle u, \nabla_0 u\rangle \nabla_0 v + (\langle v, \nabla_0 u\rangle \\ &+ \langle \overline{w}, \nabla_0 v\rangle)\nabla_0 \overline{w}) - (\langle \nabla_0 v, \nabla_0 u\rangle u + \langle \nabla_0 \overline{w}, \nabla_0 v\rangle u + \langle \nabla_0 \overline{w}, \nabla_0 \overline{w}\rangle v)] \\ =& Lv + O(d^3)|\nabla_0^2 v| + O(d^{2-\varepsilon})|\nabla_0 v| + O(d)(|v| + |\nabla_\theta v|).\end{aligned} \tag{4.8}$$

The second part of (3.14) is

$$\begin{aligned}&(\frac{(1-\rho^2)^2}{2(1-|u|^2)} - \frac{(1-\rho^2)^2}{2(1-|\overline{w}|^2)})(1+O(1-\rho)) \\ &\times (2\langle \overline{w}, \nabla_0 \overline{w}\rangle \nabla_0 \overline{w} - \langle \nabla_0 \overline{w}, \nabla_0 \overline{w}\rangle \overline{w}) \\ =&\frac{(2-m)(1+\rho)^2}{4}\langle v, \phi_0\rangle \phi_0 + O(d^{1-\varepsilon})v.\end{aligned} \tag{4.9}$$

In fact, we can calculate

$$\begin{aligned}\frac{(1-\rho^2)^2}{2(1-|u|^2)} - \frac{(1-\rho^2)^2}{2(1-|\overline{w}|^2)} =& \frac{(1-\rho^2)^2(u+\overline{w})v}{2(1-|u|^2)(1-|\overline{w}|^2)} \\ =& \frac{(1-\rho^2)^2\langle \phi_0, v\rangle}{(1-|u|^2)(1-|\overline{w}|^2)} + O(d^{1-\varepsilon})v.\end{aligned}$$

Since $\langle \phi_0, \phi_0\rangle \equiv 1$, $\langle \phi_0, \frac{\partial \phi_0}{\partial x^i}\rangle = 0$, for any $i = 1, \cdots, m$. (2.41), (4.1) and (4.2) imply that

$$\begin{aligned}&2\langle \overline{w}, \nabla_0 \overline{w}\rangle \nabla_0 \overline{w} - \langle \nabla_0 \overline{w}, \nabla_0 \overline{w}\rangle \overline{w} \\ =& 2\langle \phi_0, \phi_1\rangle \phi_1 - \langle \nabla_0 \phi_0, \nabla_0 \phi_0\rangle \phi_0 - \langle \phi_1, \phi_1\rangle \phi_0 + O(d) \\ =& (2-m)A_{1,0}^2 \phi_0 + O(d),\end{aligned}$$

where $A_{1,0}^2 = \frac{\langle \nabla \phi_0, \nabla \phi_0\rangle}{m-1}$.

We conclude from (3.14), (4.8) and (4.9) that

$$\begin{aligned}&Lv - \frac{(m-2)(1+\rho)^2}{4}\langle v, \phi_0\rangle \phi_0 \\ =& O(d^3)|\nabla_0^2 v| + O(d^{2-\varepsilon})|\nabla_0 v| + O(d)(|v| + |\nabla_\theta v|) + O(d^{m+2}\log(-d)).\end{aligned}$$

Notice that $\frac{(1+\rho)^2}{4} - 1 = \frac{(\rho+3)(\rho-1)}{4}$. Then v satisfies

$$\begin{aligned}&Lv - (m-2)\langle v, \phi_0\rangle \phi_0 \\ =& O(d^3)|\nabla_0^2 v| + O(d^{2-\varepsilon})|\nabla_0 v| + O(d)(|v| + |\nabla_\theta v|) + O(d^{m+2}\log(-d)).\end{aligned} \tag{4.10}$$

4.2 Estimate of $\langle u - \overline{w}, \phi_0\rangle$ and $u - \overline{w}$

When $m \geq 3$, we need to deal with $(m-2)\langle v, \phi_0\rangle \phi_0$.

***Lemma* 4.1** *Let u be a harmonic map from M to N so that $u \in C^1$ as a map from $\overline{B^m}$ to $\overline{B^n}$. Suppose that the boundary map ϕ_0 of u, when restricted to $\mathbb{S}^{m-1}$, is in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$, and has nowhere-vanishing energy density with respect to the*

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stanard metrics. Let $\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_m, \overline{\psi}_{m+1,1}$ *in* $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ *and* $\overline{w}$ *be given by Theorem 2.8. Then* $\langle u - \overline{w}, \phi_0 \rangle = O((1-\rho)^{m+1+\varepsilon_0})$ *and* $u - \overline{w} = O((1-\rho)^{m+1})$*, where* $\varepsilon_0 = \frac{(m+1)+\sqrt{(m+1)^2+4(m-2)}}{2} - m - 1$*. In addition, for any* $\varepsilon > 0$, $j \geq 0$,

$$|\nabla_0^j \langle u - \overline{w}, \phi_0 \rangle| = O((1-\rho)^{m+1+\varepsilon_0 - j - \varepsilon}) \text{ as } \rho \to 1^-, \tag{4.11}$$

and for any $s, j \geq 0$ *, we have*

$$|\nabla_\theta^s \nabla_0^j \langle u - \overline{w}, \phi_0 \rangle| = O((1-\rho)^{m+1+\varepsilon_0 - j - \varepsilon}) \text{ as } \rho \to 1^-. \tag{4.12}$$

Proof Set $v := u - \overline{w}$. Multiplying (4.10) by ϕ_0, we have

$$\begin{aligned} &L\langle v, \phi_0 \rangle - (m-2)\langle v, \phi_0 \rangle \\ =&O(d^3)|\nabla_0^2 v| + O(d^{2-\varepsilon})|\nabla_0 v| + O(d)(|v| + |\nabla_\theta v|) + O(d^{m+2}\log(-d)). \end{aligned} \tag{4.13}$$

Define $v_0 := \langle v, \phi_0 \rangle$. By Proposition 3.3, for any $\varepsilon > 0$ and non-negative integers j, s, we have $|\nabla_\theta^s \nabla^j v_0| = O((1-\rho)^{m-j-\varepsilon})$.

In the m-dimensional spherical coordinates $(\rho, \theta^1, \theta^2, \cdots, \theta^{m-1})$, the expression for the gradient operator ∇_0 is (4.7) and the expression for the Laplace operator Δ_0 is

$$\Delta_0 = \frac{\partial^2}{\partial \rho^2} + \frac{m-1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\Delta_{\mathbb{S}^{m-1}}, \tag{4.14}$$

where $\Delta_{\mathbb{S}^{m-1}}$ is the Laplace operator on the $(m-1)$-dimensional sphere.

Hence, by (4.13), along the direction ρ, $v_0 := \langle v, \phi_0 \rangle$ satisfies the ODEs

$$\frac{(1-\rho^2)^2}{4}\frac{\partial^2 v_0}{\partial \rho^2} + \frac{m(1-\rho^2)}{2}\rho\frac{\partial v_0}{\partial \rho} - (m-2)v_0 = O(d^{m+1-2\varepsilon}). \tag{4.15}$$

Define $r := \frac{1-\rho}{1+\rho}$. We compute

$$\begin{aligned} \frac{\partial v_0}{\partial \rho} &= \frac{\partial v_0}{\partial r}\frac{\partial r}{\partial \rho} = -\frac{2}{(1+\rho)^2}\frac{\partial v_0}{\partial r} = -\frac{(1+r)^2}{2}\frac{\partial v_0}{\partial r}, \\ \frac{\partial^2 v_0}{\partial \rho^2} &= \frac{\partial(\frac{\partial v_0}{\partial \rho})}{\partial r}\frac{\partial r}{\partial \rho} = \frac{(1+r)^4}{4}\frac{\partial^2 v_0}{\partial r^2} + \frac{(1+r)^3}{2}\frac{\partial v_0}{\partial r}. \end{aligned}$$

Based on the above calculations, we can derive

$$r^2\frac{\partial^2 v_0}{\partial r^2} - mr\frac{\partial v_0}{\partial r} - (m-2)v_0 =: \eta = O(r^{m+1-2\varepsilon}). \tag{4.16}$$

Let $t = \ln r$ and then $r = e^t$. We proceed to compute the derivatives of v_0 with respect to r in terms of t

$$\begin{aligned} \frac{\partial v_0}{\partial r} =& \frac{\partial v_0}{\partial t}\frac{\partial t}{\partial r} = e^{-t}\frac{\partial v_0}{\partial t}, \\ \frac{\partial^2 v_0}{\partial r^2} =& \frac{\partial(\frac{\partial v_0}{\partial r})}{\partial t}\frac{\partial t}{\partial r} = e^{-2t}\frac{\partial^2 v_0}{\partial t^2} - e^{-2t}\frac{\partial v_0}{\partial t}. \end{aligned} \tag{4.17}$$

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Substituting (4.17) into (4.16), we obtain the following second-order linear ordinary differential equation

$$\frac{\partial^2 v_0}{\partial t^2} - (m+1)\frac{\partial v_0}{\partial t} - (m-2)v_0 = \eta. \tag{4.18}$$

It has two characteristic roots

$$\mu_1 = \frac{(m+1) - \sqrt{(m+1)^2 + 4(m-2)}}{2}, \ \mu_2 = \frac{(m+1) + \sqrt{(m+1)^2 + 4(m-2)}}{2}.$$

We have an explicit formula for v_0,

$$v_0 = C_1(\theta)e^{\mu_1 t} + C_2(\theta)e^{\mu_2 t} + G_0(\eta),$$

where

$$G_0(\eta) = \frac{1}{\mu_2 - \mu_1}[e^{\mu_2 t}\int_{t_1}^{t} e^{-\mu_2\tau}\eta(\tau,\theta)\mathrm{d}\tau - e^{\mu_1 t}\int_{-\infty}^{t} e^{-\mu_1\tau}\eta(\tau,\theta)\mathrm{d}\tau], t_1 \in (-\infty, t).$$

Note that $r = e^t$.

$$v_0 = C_1(\theta)r^{\mu_1} + C_2(\theta)r^{\mu_2} + G_0(\eta)$$

with

$$G_0(\eta) = \frac{1}{\mu_2 - \mu_1}[r^{\mu_2}\int_{e^{t_1}}^{r} s^{-1-\mu_2}\eta(s,\theta)\mathrm{d}s - r^{\mu_1}\int_0^r s^{-1-\mu_1}\eta(s,\theta)\mathrm{d}s].$$

From (4.16), there exists a positive number $C > 0$ such that $|\eta(s,\theta)| \le Cs^{m+1-2\varepsilon}$. Therefore

$$|G_0(\eta)| \le Cr^{m+1-2\varepsilon} + \widetilde{C}r^{\mu_2}.$$

Since we have known $v = O(r^{m-\varepsilon})$, $C_1 = 0$. So

$$v_0 = C_2(\theta)r^{\mu_2} + G_0(\eta). \tag{4.19}$$

In view of $G_0(\eta) = O(r^{m+1-2\varepsilon})$, we have $v_0 = O(r^{m+1-2\varepsilon})$.

Owing to (4.10) and (4.19), v satisfies equation

$$Lv = \frac{(1-\rho^2)^2}{4}\Delta_0 v + \sum_{i=1}^{m}\frac{m(1-\rho^2)}{2}x^i\frac{\partial v}{\partial x^i} = O(d^{m+1-2\varepsilon}).$$

By Proposition 3.3, v satisfies the ODEs

$$\frac{(1-\rho^2)^2}{4}\frac{\partial^2 v}{\partial \rho^2} + \frac{m(1-\rho^2)}{2}\rho\frac{\partial v}{\partial \rho} = O(d^{m+1-2\varepsilon}). \tag{4.20}$$

Notice that $r = \frac{1-\rho}{1+\rho}$, v satisfies

$$r^2\frac{\partial^2 v}{\partial r^2} - mr\frac{\partial v}{\partial r} =: \overline{\eta} = O(r^{m+1-2\varepsilon}). \tag{4.21}$$

Let $t = \ln r$ and then $r = e^t$. We compute

$$\frac{\partial^2 v}{\partial t^2} - (m+1)\frac{\partial v}{\partial t} = \overline{\eta}. \tag{4.22}$$

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It has two characteristic roots

$$\overline{\mu}_1 = 0,\ \overline{\mu}_2 = m + 1.$$

We have an explicit formula for v,

$$v = \overline{C}_1(\theta)e^{\overline{\mu}_1 t} + \overline{C}_2(\theta)e^{\overline{\mu}_2 t} + \overline{G}_0(\eta)$$

where

$$\overline{G}_0(\eta) = \frac{1}{\overline{\mu}_2 - \overline{\mu}_1}[e^{\overline{\mu}_2 t}\int_{\bar{t}_1}^{t} e^{-\overline{\mu}_2 \tau}\overline{\eta}(\tau,\theta)\mathrm{d}\tau - e^{\overline{\mu}_1 t}\int_{-\infty}^{t} e^{-\overline{\mu}_1 t}\overline{\eta}(\tau,\theta)\mathrm{d}\tau], \bar{t}_1 \in (-\infty, t).$$

Notice that $r = e^t$,

$$v = \overline{C}_1(\theta)r^{\overline{\mu}_1} + \overline{C}_2(\theta)r^{\overline{\mu}_2} + \overline{G}_0(\eta),$$

with

$$\overline{G}_0(\eta) = \frac{1}{\overline{\mu}_2 - \overline{\mu}_1}[r^{\overline{\mu}_2}\int_{e^{\bar{t}_1}}^{r} s^{-1-\mu_2}\overline{\eta}(s,\theta)\mathrm{d}s - r^{\overline{\mu}_1}\int_{0}^{r} s^{-1-\overline{\mu}_1}\overline{\eta}(s,\theta)\mathrm{d}s].$$

From (4.16), there exists a positive number $C > 0$ such that $|\overline{\eta}(s,\theta)| \leq Cs^{m+1-2\varepsilon}$. Therefore

$$|\overline{G}_0(\eta)| \leq Cr^{m+1-2\varepsilon} + \hat{C}r^{\overline{\mu}_2}.$$

Since we have known $v = O(r^{m-\varepsilon})$, $\overline{C}_1 = 0$. So

$$v = (\overline{C}_2\theta)r^{\overline{\mu}_2} + \overline{G}_0(\eta) \tag{4.23}$$

In view of $\overline{G}_0(\eta) = O(r^{m+1-2\varepsilon})$, we have $v = O(r^{m+1-2\varepsilon})$.

Repeating the process above, we can get $\langle v, \phi_0\rangle = O(r^{\min\{\mu_2, m+2-3\varepsilon\}}) = O(r^{\mu_2})$, ($\mu_2 = \frac{(m+1)+\sqrt{(m+1)^2+4(m-2)}}{2}$) and $v = O(r^{m+1})$.

According to (4.13), $v_0 = \langle v, \phi_0\rangle$ satisfies the conditions of Lemma 4.1 of [19] if the general elliptic operator is substituted for Δ_0. Similar to the proof of Proposition 3.3(Replace m in Prop 3.3 with $m + 1 + \varepsilon_0$.), for any $\varepsilon > 0$ and non-negative integers j, s, we have $|\nabla_\theta^s \nabla^j v_0| = O((1-\rho)^{m+1+\varepsilon_0-j-\varepsilon})$.

■

4.3 The coefficient function $\overline{\phi}_{m+1}$ and the rest of the proof

The next step in the proof of Theorem 1.1 is to find $W_{m+1} = \overline{w} + \overline{\phi}_{m+1}d^{m+1}$ where $\overline{w}$ is given by Theorem 2.8 such that $u - W_{m+1} = o(d^{m+1})$.

We have already proved $u - \overline{w} = O(d^{m+1})$ in Lemma 4.1, but since we have reached the conclusion by solving ordinary differential equations, the coefficient $\frac{u-\overline{w}}{d^{m+1}}$ is not yet certain. Therefore, the following lemma proves this coefficient by the convergence method.

Define

$$c(\rho,\theta) = \frac{u - \overline{w}}{d^{m+1}}(\rho,\theta) \tag{4.24}$$

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***Lemma* 4.2** *Let u be a harmonic map from M to N so that $u \in C^1$ as a map from $\overline{B^m}$ to $\overline{B^n}$. Suppose that the boundary map ϕ_0 of u, when restricted to $\mathbb{S}^{m-1}$, is in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$, and has nowhere-vanishing energy density with respect to the stanard metrics. Let $\phi_0, \overline{\phi}_1, \cdots, \overline{\phi}_m, \overline{\psi}_{m+1,1}$ in $C^\infty(\mathbb{S}^{m-1}, \mathbb{S}^{n-1})$ and $\overline{w}$ be given by Theorem 2.8. Let c be defined by* (4.24)*. Then there exists $\overline{\phi}_{m+1} \in C^\infty(\mathbb{S}^{m-1}; \mathbb{S}^{n-1})$ such that c converges to $\overline{\phi}_{m+1}$ in $C^j(\mathbb{S}^{m-1}; \mathbb{S}^{n-1})$ for all $j \geq 0$ as $\rho \to 1$, and for any $s \geq 0$,*

$$|\nabla^s_\theta(c - \overline{\phi}_{m+1})| \leq C(1-\rho)^{1-\varepsilon}, \tag{4.25}$$

where ε can take any real number in $(0, 1)$.

Proof Set $v := u - \overline{w}$. By Proposition 3.3 and Lemma 4.1, we know for any $\varepsilon > 0$ and non-negative integers j, s, $|\nabla^j \nabla^s_\theta v| = O((1-\rho)^{m+1-j-\varepsilon})$.

Define function $\varphi := \nabla^s_\theta v$ and $\varphi_0 := \nabla^s_\theta \langle v, \phi_0 \rangle$. Differentiating (4.13) with respect to θ s times, we know

$$L\varphi_0 - (m-2)\varphi_0 = O((1-\rho)^{m+2-\varepsilon}). \tag{4.26}$$

Set $r = \frac{1-\rho}{1+\rho}$ and $\mu_2 = \frac{(m+1)+\sqrt{(m+1)^2+4(m-2)}}{2}$. So for any fixed $\theta_0 \in \mathbb{S}^{m-1}$, we have

$$r^2 \frac{\partial^2 \varphi_0(\theta_0, r)}{\partial r^2} - mr \frac{\partial \varphi_0(\theta_0, r)}{\partial r} - (m-2)\varphi_0 = O(r^{m+2-\varepsilon}).$$

Introducing $\xi_0(r) := \frac{\varphi_0(\theta_0, r)}{r^{\mu_2}}$, we have

$$\begin{aligned} & r^{\mu_2+2} \frac{\partial^2 \xi_0}{\partial r^2} + (2\mu_2 - m) r^{\mu_2+1} \frac{\partial \xi_0}{\partial r} \\ =& r^2 \frac{\partial^2 \varphi_0}{\partial r^2} - mr \frac{\partial \varphi_0}{\partial r} - \mu_2(\mu_2 - m - 1)\varphi_0 = O(r^{m+2-\varepsilon}), \\ & (r^{\mu_2+2}\xi_0'(r))' + (\mu_2 - m - 2) r^{\mu_2+1} \xi_0'(r) = O(r^{m+2-\varepsilon}), \end{aligned}$$

where we have used that μ_2 is a solution of $\mu_2^2 - (m+1)\mu_2 - (m-2) = 0$.

Set $y_0 := r^{\mu_2+2}\xi_0'(r)$ and notice that $\mu_2 < m+2$. Then y_0 satisfies the following ODE and

$$r y_0'(r) + (\mu_2 - m - 2) y_0 =: \pi_0 = O(r^{m+3-\varepsilon})$$

has solution

$$y_0(r) = r^{m+2-\mu_2}\left(\int_{r_0}^r \pi_0(R) R^{\mu_2-m-3} \mathrm{d}R + \widetilde{C}\right),$$

where $r_0 \in (0, r)$. Since $y_0 = O(r^{\mu_2+1-\varepsilon})$, $\widetilde{C} = 0$ and $y_0 = O(r^{m+3-\varepsilon})$. Therefore $\xi_0'(r) = O(r^{m+1-\mu_2-\varepsilon})$, which gives $\xi_0(r) = O(r^{m+2-\mu_2-\varepsilon})$ and $\varphi_0 = O(r^{m+2-\varepsilon})$.

Differentiating (4.10) with respect to θ s times, we have

$$L\varphi = \frac{(1-\rho^2)^2}{4}\Delta_0\varphi + \sum_{i=1}^m \frac{m(1-\rho^2)}{2} x^i \frac{\partial \varphi}{\partial x^i} = O((1-\rho)^{m+2-\varepsilon}).$$

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Similar to the estimate of φ_0, we can get

$$r^2\frac{\partial^2\varphi(\theta_0,r)}{\partial r^2}-mr\frac{\partial\varphi(\theta_0,r)}{\partial r}=O(r^{m+2-\varepsilon}).$$

Introducing $\xi(r):=\frac{\varphi(\theta_0,r)}{r^{m+1}}$, we have

$$r^{m+3}\frac{\partial^2\xi}{\partial r^2}+(m+2)r^{m+2}\frac{\partial\xi}{\partial r}=r^2\frac{\partial^2\varphi}{\partial r^2}-mr\frac{\partial\varphi}{\partial r}=O(r^{m+2-\varepsilon}),$$
$$(r^{m+3}\xi'(r))'-r^{m+2}\xi'(r)=O(r^{m+2-\varepsilon}).$$

Set $y=r^{m+3}\xi'(r)$, then

$$ry'(r)-y=:\pi=O(r^{m+3-\varepsilon})$$

has solution

$$y(r)=r(\int_0^r R^{-2}\pi(R)\mathrm{d}R+\widetilde{C}).$$

Since $y=O(r^{m+2-\varepsilon})$, $\widetilde{C}=0$ and $y=O(r^{m+3-\varepsilon})$. Therefore $\xi'(r)=O(r^{-\varepsilon})$, which gives

$$|\frac{\nabla_\theta^s v(\theta_0,r)}{r^{m+1}}-\frac{\nabla_\theta^s v(\theta_0,\overline{r})}{\overline{r}^{m+1}}|=|\xi(r)-\xi(\overline{r})|\le C|r^{1-\varepsilon}-\overline{r}^{1-\varepsilon}|$$

and

$$|\nabla_\theta^s(\frac{v(\theta_0,\rho)}{(1-\rho)^{m+1}}-\frac{v(\theta_0,\overline{\rho})}{(1-\overline{\rho})^{m+1}})|\le C|(1-\rho)^{1-\varepsilon}-(1-\overline{\rho})^{1-\varepsilon}|.$$

Letting $\overline{\rho}\to 1$, the conclusion follows.

■

Proof of Theorem 1.1. Let $\overline{\phi}_0,\cdots,\overline{\phi}_m,\overline{\psi}_{m+1,1}$ and $\overline{w}$ be given by Theorem 2.8 and $\overline{\phi}_{m+1}$ be given by Lemma 4.2. Set $v:=u-\overline{w}$. According to Lemma 4.1, we know $v=O(r^{m+1})$. By Theorem 4.2 of [19], $u\in C^\infty(B_1^m;B_1^n)\cap C^{m,\alpha}(\overline{B_1^m};\overline{B_1^n}),\forall\alpha\in(0,1)$.

Let

$$\overline{w}_{m+1}=\overline{w}+\overline{\phi}_{m+1}d^{m+1}.$$

By (4.25), for any $s\ge 0$, we have

$$|\nabla_\theta^s(u-\overline{w}_{m+1})|=O(d^{m+2-\varepsilon}),\forall\varepsilon>0.$$

Furthermore, by Theorem 4.2 of [19], for any non-negative integers j and s, it holds that

$$|\nabla^j\nabla_\theta^s(u-\overline{w}_{m+1})|=O(d^{m+2-j-\varepsilon}),\forall\varepsilon>0.$$

Let $\overline{w}_{m_0}=\phi_0+\sum\limits_{k=1}^{m_0}\overline{\phi}_k(x)d^k+\sum\limits_{k=m+1}^{m_0}\sum\limits_{l=1}^{[\frac{m_0-1}{m}]}\overline{\psi}_{k,l}(x)d^k(\log(-d))^l$. We assume, by induction, that the following inequality has been established for $m=m_0\ge m+1$ and for all non-negative integers j and s,

$$|\nabla^j\nabla_\theta^s(u-\overline{w}_{m_0})|=O(d^{m_0+1-j-\varepsilon}). \tag{4.27}$$

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In Theorem 2.8, solving the equation $\overline{B}_{m_0+1}[u] + \overline{F}_{m_0+1}[u] + \overline{E}_{m_0+1}[u] = 0$ is equivalent to finding $\overline{\phi}_{m_0+1}$ that satisfies this relationship. Similarly, the set of equations $\overline{B}_{m_0+1,l}[u] + \overline{F}_{m_0+1,l}[u] + \overline{E}_{m_0+1,l}[u] = 0$ for $l = 1, \cdots, [\frac{m_0-1}{m}]$ is used to determine the coefficients $\overline{\psi}_{m_0+1,l}$. We then define the function

$$\overline{w}_{m_0+1} := \phi_0 + \sum_{k=1}^{m_0+1} \overline{\phi}_k(x) d^k + \sum_{k=m+1}^{m_0} \sum_{l=1}^{[\frac{m_0-1}{m}]} \overline{\psi}_{k,l}(x) d^k (\log(-d))^l.$$

We have

$$\tau(\overline{w}_{m_0+1}) = O(d^{m_0+2}\log(-d)). \tag{4.28}$$

Based on equation (4.27), we can establish an estimate for the difference between u and $\overline{w}_{m_0+1}$. For any non-negative integers j and s, we have

$$\begin{aligned} &|\nabla^j \nabla_\theta^s (u - \overline{w}_{m_0+1})| \\ =&|\nabla^j \nabla_\theta^s (u - \overline{w}_{m_0} - \overline{\phi}_{m_0+1} d^{m_0+1} - \sum_{l=1}^{[\frac{m_0}{m}]} \overline{\psi}_{m_0+1,l}(x) d^{m_0+1} (\log(-d))^l)| \\ =&O(d^{m_0+1-j-\varepsilon}). \end{aligned} \tag{4.29}$$

Set $v_{m_0+1} := u - \overline{w}_{m_0+1}$ and $v^0_{m_0+1} := \langle u - \overline{w}_{m_0+1}, \phi_0\rangle$. Similar to the process used to derive equation (4.10), and by leveraging the result in (4.28), we can obtain

$$\begin{aligned} &Lv_{m_0+1} - (m-2)\langle v_{m_0+1}, \phi_0\rangle\phi_0 \\ =&O(d^3)|\nabla_0^2 v_{m_0+1}| + O(d^{2-\varepsilon})|\nabla_0 v_{m_0+1}| \\ &+ O(d)(|v_{m_0+1}| + |\nabla_\theta v_{m_0+1}|) + O(d^{m_0+2}\log(-d)), \end{aligned} \tag{4.30}$$

where linear operator L is defined in (4.6).

Multiplying both sides of equation (4.30) by the function ϕ_0, we get

$$\begin{aligned} &L\langle v_{m_0+1}, \phi_0\rangle - (m-2)\langle v_{m_0+1}, \phi_0\rangle \\ =&O(d^3)|\nabla_0^2 v_{m_0+1}| + O(d^{2-\varepsilon})|\nabla_0 v_{m_0+1}| \\ &+ O(d)(|v_{m_0+1}| + |\nabla_\theta v_{m_0+1}|) + O(d^{m_0+2}\log(-d)). \end{aligned} \tag{4.31}$$

Define $\varphi_{m_0+1} := \nabla_\theta^s(u - \overline{w}_{m_0+1})$. Differentiating (4.31) with respect to θ, we know $\varphi^0_{m_0+1} := \nabla_\theta^s\langle u - \overline{w}_{m_0+1}, \phi_0\rangle$ satisfies

$$\begin{aligned} &L\varphi^0_{m_0+1} - (m-2)\varphi^0_{m_0+1} \\ =&O(d^3)|\nabla_0^2 \varphi_{m_0+1}| + O(d^{2-\varepsilon})|\nabla_0 \varphi_{m_0+1}| \\ &+ O(d)(|\varphi_{m_0+1}| + |\nabla_\theta \varphi_{m_0+1}|) + O(d^{m_0+2}\log(-d)). \end{aligned} \tag{4.32}$$

As in the proof of Lemma 4.1, we know that $\varphi^0_{m_0+1} = O((1-\rho)^{m_0+2-\varepsilon})$. So we can conclude that

$$L\varphi_{m_0+1} = O((1-\rho)^{m_0+2-\varepsilon}). \tag{4.33}$$

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Finally, by using equation (4.29), we can derive

$$\frac{(1-\rho^2)^2}{4}\frac{\partial^2\varphi_{m_0+1}}{\partial\rho^2}+\sum_{i=1}^{m}\frac{m(1-\rho^2)}{2}\rho\frac{\partial\varphi_{m_0+1}}{\partial\rho}=O(d^{m_0+2-\varepsilon}). \tag{4.34}$$

Define $r := \frac{1-\rho}{1+\rho}$, we can get

$$r^2\frac{\partial^2\varphi_{m_0+1}}{\partial r^2}-mr\frac{\partial\varphi_{m_0+1}}{\partial r}=:\eta=O(r^{m_0+2-\varepsilon}). \tag{4.35}$$

Next, we introduce a new function $\xi_{m_0+1}(r) := \frac{\varphi_{m_0+1}(\theta_0,r)}{r^{m+1}}$. By substituting $\varphi_{m_0+1}=r^{m+1}\xi_{m_0+1}(r)$ into the previous equation, we obtain

$$r^{m+3}\frac{\partial^2\xi_{m_0+1}}{\partial r^2}+(m+2)r^{m+2}\frac{\partial\xi_{m_0+1}}{\partial r}=r^2\frac{\partial^2\varphi_{m_0+1}}{\partial r^2}-mr\frac{\partial\varphi_{m_0+1}}{\partial r}=O(r^{m_0+2-\varepsilon}),$$
$$(r^{m+3}\xi'_{m_0+1}(r))'-r^{m+2}\xi'_{m_0+1}(r)=O(r^{m_0+2-\varepsilon}).$$

Set $y_{m_0+1}=r^{m+3}\xi'_{m_0+1}(r)$, then

$$ry'_{m_0+1}(r)-y_{m_0+1}=:\pi_{m_0+1}=O(r^{m_0+3-\varepsilon})$$

has solution

$$y_{m_0+1}(r)=r(\int_0^r R^{-2}\pi(R)\mathrm{d}R+\widetilde{C}).$$

Therefore $y_{m_0+1}=O(r^{m_0+3-\varepsilon})$ and $\xi_{m_0+1}=O(r^{m_0-m+1-\varepsilon})$, which gives $\varphi_{m_0+1}=O(r^{m_0+2-\varepsilon})$.

We may then argue as Theorem 4.2 of [19] to obtain For any non-negative integers j and s, $|\nabla_\theta^s\nabla_\rho^j(u-\overline{w}_{m_0+1})|=O(d^{m_0+2-j-\varepsilon}), \forall\varepsilon\in(0,1)$.

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