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ON SUBSPACES OF REPLETE AND MEASURE REPLETE SPACES

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ABSTRACT. The concepts of repleteness and more generally measure repleteness are investigated for set-theoretic lattices on specific subspaces of a lattice space. These general results are then applied to specific topological spaces, and we obtain as special cases some known theorems as well as some new results concerning for example, $\propto$-completeness, realcompactness, measure compactness and Borel-measure compactness.

The notion of repleteness, and more generally measure repleteness, are investigated in this paper for set-theoretic lattices on certain subspaces of a space X to which topologically related properties can be applied. We obtain as special cases some known results as well as some new results concerning for example, α-completeness, realcompactness, measure-compactness and Borel-measure compactness.

1. **Introduction.** Measure replete spaces have been investigated in special topological settings by Varadarajan [14], Moran [11], Gardner [7], and others, and in the abstract lattice setting by Bachman and Sultan [4], and Szeto [13]. In this paper, we let $\mathbf{L}$ be a lattice of subsets of the set X such that $\mathbf{L}$ is measure replete and we investigate measure repleteness with respect to various lattices of subsets of a subset E of X. In particular, we establish in Section 3 two general theorems, establishing under suitable conditions, the measure repleteness of a variety of lattices for certain subsets E of X. Special cases of these theorems in topological settings yield results of Kirk [10] and Moran [11] as well as new Borel-measure compact results for analytic spaces.

In Section 4 we turn to the special case of replete lattices and here a further improvement of one of the general theorems of Section 3 is possible. We then consider some equivalent characterizations of repleteness under suitable conditions for the lattice on X. Special cases of these results are given in topological spaces leading to theorems of Mrówka [12], Frolik [6] and Wenjen [16] pertaining to realcompactness while new results concerning α-completeness are obtained.

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We begin by introducing the notations that will be used throughout this paper; the lattice terminology that will be used is standard and can be found, for example, in [1], [2], [13], and [15].

2. **Notation and definitions.** We use the same lattice definitions used in [2] and we assume that the reader is familiar with the results of that paper. Throughout this paper $\mathbf{L}$ will denote a lattice of subsets of an abstract set X such that the empty set and X are lattice elements. $\tau\mathbf{L}$ (resp. $\delta\mathbf{L}$) is the lattice obtained from $\mathbf{L}$ by taking arbitrary (resp. countable) intersections of elements of $\mathbf{L}$, and $s(\mathbf{L})$ will be the Souslin sets obtained from $\mathbf{L}$. $\mathbf{A}(\mathbf{L})$ and $\sigma(\mathbf{L})$ denote the algebra and σ-algebra respectively generated by $\mathbf{L}$. If X is a topological space, $\mathbf{F}_X$ or $\mathbf{F}(X)$ will be the closed sets of X and $\mathbf{z}_X$ or $\mathbf{z}(X)$, the lattice of zero sets of continuous functions on X. $M_R(\mathbf{L})$ is the set of all finitely additive $\mathbf{L}$-regular measures defined on $\mathbf{A}(\mathbf{L})$ while $M_R^\sigma(\mathbf{L})$ and $M_R^\tau(\mathbf{L})$ are subsets of $M_R(\mathbf{L})$ consisting of measures which are also σ-smooth and τ-smooth, respectively. $I_R(\mathbf{L})$ is the subset of $M_R(\mathbf{L})$ consisting of 0–1 valued $\mathbf{L}$-regular measures. $I_R^\sigma(\mathbf{L})$ is defined analogously. Without loss of generality, we will assume that all measures under consideration are non-negative. Also, if $\mu \in M_R^\sigma(\mathbf{L})$, μ can be uniquely extended to $\sigma(\mathbf{L})$ and is $\delta\mathbf{L}$-regular there. We will assume in this case that μ is defined on $\sigma(\mathbf{L})$.

$S(\mu)$ will denote the *support* of the measure μ. If $S(\mu) \neq \emptyset$ for each $\mu \in M_R^\sigma(\mathbf{L})$, $\mu \neq 0$ (resp. $\mu \in I_R^\sigma(\mathbf{L})$) then $\mathbf{L}$ is said to be *measure replete* (resp. *replete*). If X is a Tychonoff space and if $\mathbf{z}_X$ is measure replete (resp. replete) then X is called *measure compact* (resp. *realcompact*). If X is a T_1 topological space and if $\mathbf{F}_X$ is measure replete (resp. replete) then X is called *Borel-measure compact* [7] (resp. *α-complete* [5]).

We let $W(A) = \{\mu \in I_R(\mathbf{L}) \mid \mu(A) = 1,\ A \in \mathbf{A}(\mathbf{L})\}$ and $W(\mathbf{L}) = \{W(L) \mid L \in \mathbf{L}\}$. The topology on $I_R(\mathbf{L})$ will always be $\tau W(\mathbf{L})$ (i.e., the Wallman topology). Similarly, $W_\sigma(A) = W(A) \cap I_R^\sigma(\mathbf{L})$, $A \in \sigma(\mathbf{L})$, and $W_\sigma(\mathbf{L}) = \{W_\sigma(L) \mid L \in \mathbf{L}\}$. We note the well-known result that if $\mathbf{L}$ is a separating and disjunctive lattice on X then $\mathbf{L}$ is replete iff $I_R^\sigma(\mathbf{L}) = X$ where X is identified with its homeomorphic image under the mapping $x \to \mu_x$ on which X is given the $\tau\mathbf{L}$ topology and $I_R^\sigma(\mathbf{L})$ the Wallman topology restricted.

3. **Measure replete subspaces.** In this section we obtain two theorems which generalize results of Moran [11] and Kirk [10] on measure repleteness of certain subspaces of particular topological spaces. We are also able to obtain some new results concerning measure repleteness on specific subspaces of an analytic space (i.e., the continuous image of a complete, separable, metric space) (see [9] for details).

We begin with the following lemmas, the proofs of which may be found in Szeto [13].

LEMMA 3.1. *If $\mu \in M_R(\mathbf{L})$ where $\mathbf{L}$ is a lattice of subsets of the set X and if $T(\mu)=\{x \in X \mid \text{if } x \in L', \mu(L')>0, L \in \mathbf{L}\}$ then $S(\mu)=T(\mu)$.*

Remark. The above lemma is true even if μ is not $\mathbf{L}$-regular.

LEMMA 3.2. *Suppose $\mathbf{L}$ is a δ-lattice of subsets of the set X. Then $M_R^\sigma(\mathbf{L})=M_R^\tau(\mathbf{L})$ iff $S(\mu)\neq\emptyset$ for all $\mu\in M_R^\sigma(\mathbf{L})$ $(\mu\neq 0)$.*

LEMMA 3.3. *Let $\mathbf{L}$ be a δ-lattice of subsets of the set X. Then $\mu\in M_R^\tau(\mathbf{L})$ iff $\mu^*(\bigcap_\alpha L_\alpha)=\inf \mu(L_\alpha)$ whenever $L_\alpha\downarrow$, $L_\alpha\in\mathbf{L}$.*

Using the above results we are now able to state and prove our first theorem.

THEOREM 3.1. *Let $\mathbf{L}$ be a δ-lattice of subsets of the set X such that $\sigma(\mathbf{L})\subset s(\mathbf{L})$ and let $E\in\sigma(\mathbf{L})$. If $\mathbf{L}$ is measure replete then $\mathbf{L}_E$ is measure replete where $\mathbf{L}_E$ is any lattice of subsets of E such that $\mathbf{L}\cap E\subset\mathbf{L}_E\subset\tau(\mathbf{L}\cap E)$.*

Proof. Let $\mu\in M_R^\sigma(\mathbf{L}_E)$, $\mu\neq 0$. Define $\nu(A)=\mu(A\cap E)$ where $A\in\sigma(\mathbf{L})$. Clearly, ν is σ-smooth on $\sigma(\mathbf{L})$. Since $\mathbf{L}$ is a δ-lattice and $\sigma(\mathbf{L})\subset s(\mathbf{L})$, $\nu\in M_R^\sigma(\mathbf{L})$. Therefore, $S(\nu)\neq\emptyset$. From Lemma 3.2 we see that $\nu\in M_R^\tau(\mathbf{L})$.

Suppose $S(\nu)\cap E=\emptyset$. If $S(\nu)=\bigcap_\alpha L_\alpha$ where $\nu(L_\alpha)=\nu(X)$ and $L_\alpha\in\mathbf{L}$, then it follows from Lemma 3.3 that $\nu^*(S(\nu))=\nu(X)$, or equivalently, $\nu_*(S(\nu)')=0$. Thus $\nu(E)=\mu(E)=0$ which contradicts our assumption that $\mu\neq 0$. Let $x\in S(\nu)\cap E$ and let $x\in E-L_E$, $L_E\in\mathbf{L}_E$. Then $E-L_E=\bigcup_\beta L'_\beta\cap E$, $L_\beta\in\mathbf{L}$. Therefore, $x\in L'\cap E\subset E-L_E$ for some $L\in\mathbf{L}$. Thus $\mu(E-L_E)\geq\mu(L'\cap E)=\nu(L')$. Since $x\in S(\nu)$, it follows from Lemma 3.1 that $\nu(L')>0$. Therefore $x\in S(\mu)$. □

We obtain the following corollary due to Moran [11].

COROLLARY 3.1. *Suppose X is a Tychonoff space. If X is measure compact then any Baire set of X is measure compact.*

Proof. Let $\mathbf{L}=\mathbf{z}_X$ and $\mathbf{L}_E=\mathbf{z}_E$ where $E\in\sigma(\mathbf{z}_X)$. □

As an additional corollary we have the following new result.

COROLLARY 3.2. *If X is an analytic space which is Borel-measure compact then any Borel set of X is measure compact.*

Proof. Let $\mathbf{L}=\mathbf{F}_X$ and let $\mathbf{L}_E=\tau(\mathbf{L}\cap E)=\mathbf{F}_E$ where $E\in\sigma(\mathbf{F}_X)$. □

The following lemma is a direct consequence of a general extension theorem due to Bachman and Sultan (see [3]).

LEMMA 3.4. *Suppose $\mathbf{L}$ is a separating and disjunctive lattice of subsets of the set X. Then any $\nu\in M_R^\tau(\mathbf{L})$ can be extended to $\rho\in M_R^\tau(\tau\mathbf{L})$ and $\rho(\bigcap_\alpha L_\alpha)=\inf\rho(L_\alpha)$, $L_\alpha\in\mathbf{L}$, $L_\alpha\downarrow$.*

Using this lemma and our previous results we obtain a theorem similar to Theorem 3.1.

THEOREM 3.2. *Let $E \in \tau\mathbf{L}$ where $\mathbf{L}$ is a separating, disjunctive and δ-lattice of subsets of the set X such that $\sigma(\mathbf{L}) \subset s(\mathbf{L})$. Assume $\mathbf{L} \cap E \subset \mathbf{L}_E \subset \tau(\mathbf{L} \cap E)$. Then $\mathbf{L}$ measure replete implies $\mathbf{L}_E$ is measure replete.*

Proof. Let $\mu \in M_R^\sigma(\mathbf{L}_E)$, $\mu \neq 0$. Define $\nu(A) = \mu(A \cap E)$ where $A \in \sigma(\mathbf{L})$. Arguing exactly as in the proof of Theorem 3.1 we obtain that $\nu \in M_R^\tau(\mathbf{L})$. Using Lemma 3.4, extend ν to $\rho \in M_R^\tau(\tau\mathbf{L})$. If $S(\nu) \cap E = \varnothing$ then $\rho(S(\nu)) = \rho(X)$ and thus $\rho(E) = 0$. Since $E = \bigcap_\alpha L_\alpha$, $L_\alpha \in \mathbf{L}$, $\rho(E) = \inf \rho(L_\alpha) = \inf \nu(L_\alpha) = \mu(E)$ which implies that $\mu = 0$, a contradiction. Proceeding as in the proof of Theorem 3.1 completes the proof. □

COROLLARY 3.3 (Kirk [10]). *Suppose X is a Tychonoff space. If X is measure compact then any closed subspace of X is measure compact.*

Proof. Take $\mathbf{L} = \mathbf{z}_X$. If we let $E \in \tau\mathbf{z}_X = \mathbf{F}_X$ and $\mathbf{L}_E = \mathbf{z}_E$ the result follows from the above theorem. □

We have another new result concerning Borel-measure compactness of a closed subspace of a Borel-measure compact analytic space.

COROLLARY 3.4. *Any closed subspace of a Borel-measure compact and analytic space is Borel-measure compact.*

Proof. Let $\mathbf{L} = \mathbf{F}_X$ and $\mathbf{L}_E = \mathbf{F}_E$ where $E \in \mathbf{F}_X$. □

4. **Repleteness.** We now consider the concept of repleteness and are able to obtain a theorem stronger than Theorem 3.2 of the last section for this specific case. We will then consider and show some equivalent characterizations of repleteness for specific types of lattices.

The following theorem improves Theorem 3.2 with respect to repleteness.

THEOREM 4.1. *Suppose $\mathbf{L}$ is a δ- and disjunctive lattice of subsets of X such that $\sigma(\mathbf{L}) \subset s(\mathbf{L})$ and $\mathbf{L} \cap E \subset \mathbf{L}_E \subset \tau(\mathbf{L} \cap E)$ where E is $\mathbf{G}_\delta$-closed with respect to $\tau\mathbf{L}$. If $\mathbf{L}$ is replete then $\mathbf{L}_E$ is replete.*

Proof. Let $\mu \in I_R^\sigma(\mathbf{L}_E)$. If $\nu(A) = \mu(A \cap E)$, $A \in \sigma(\mathbf{L})$, then $\nu \in I_R^\sigma(\mathbf{L})$ and $\nu = \mu_x$ on $\mathbf{L}$ since $\mathbf{L}$ is disjunctive. Suppose $x \notin E$ where E is $\mathbf{G}_\delta$-closed. Then there exists a set $G = \bigcap_{n=1}^\infty O_n$, $O_n' \in \tau\mathbf{L}$ such that $x \in G$ and $G \cap E = \varnothing$. Since $\mathbf{L}$ is disjunctive, $x \in \tilde{L}_n \subset L_n' \subset O_n$ for all n, where $\tilde{L}_n$, $L_n \in \mathbf{L}$. Therefore, if $\hat{L} = \bigcap_{n=1}^\infty \tilde{L}_n$, then $\nu(\hat{L}) = 0$. Since ν is $\mathbf{L}$-regular, $\nu(L) = 1$ for some $L \in \mathbf{L}$ and $L \subset \hat{L}'$. Thus $x \in \hat{L}'$ which is a contradiction. Therefore, $x \in S(\nu) \cap E$. Continuing as in the proof of Theorem 3.1 we see that $S(\mu) \neq \varnothing$. □

The following two lemmas can be easily proved and hence their proofs are omitted.

LEMMA 4.1. *Suppose* $\mathbf{L}$ *is a separating and disjunctive lattice of subsets of* X. *Then* $\bigcap_{n=1}^{\infty} A_n = \emptyset$ *iff* $\bigcap_{n=1}^{\infty} W(A_n) \subset I_R(\mathbf{L}) - X$ *where* $A_n \in \mathbf{A}(\mathbf{L})$.

LEMMA 4.2. $W_\sigma\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} W_\sigma(A_n)$, $A_n \in \sigma(\mathbf{L})$.

LEMMA 4.3. *If* $\mathbf{L}$ *is a normal, separating, disjunctive, and countably paracompact lattice of subsets of* X *and if* $\bigcap_{n=1}^{\infty} W(L_n) \subset I_R(\mathbf{L}) - X$, $L_n \in \mathbf{L}$, *then there exists* $K_0 \in \mathbf{z}(I_R(\mathbf{L}))$ *such that* $\bigcap_{n=1}^{\infty} W(L_n) \subset K_0 \subset I_R(\mathbf{L}) - X$.

Proof. Let $\bigcap_{n=1}^{\infty} W(L_n) \subset I_R(\mathbf{L}) - X$ where $L_n \in \mathbf{L}$. From Lemma 4.1, $\bigcap_{n=1}^{\infty} L_n = \emptyset$. Since $\mathbf{L}$ is countably paracompact, $\bigcap_{n=1}^{\infty} W(L_n) \subset \bigcap_{n=1}^{\infty} W(\tilde{L}_n)' \subset I_R(\mathbf{L}) - X$ where $L_n \subset \tilde{L}_n'$, $\bigcap_{n=1}^{\infty} \tilde{L}_n' = \emptyset$ and $\tilde{L}_n \in \mathbf{L}$ for all n. But $\bigcap_{n=1}^{\infty} W(L_n)$ is compact, since $\mathbf{L}$ is normal and hence $I_R(\mathbf{L})$ is compact Hausdorff. Therefore, there exists a compact $\mathbf{G}_\delta$ set K_n such that $\bigcap_{n=1}^{\infty} W(L_n) \subset K_n \subset W(\tilde{L}_n)'$ for any n. (See Halmos [8], p. 218 for details.) Let $K_0 = \bigcap_{n=1}^{\infty} K_n$. $\square$

LEMMA 4.4. *If* $\mathbf{L}$ *is a lattice of subsets of* X *and if* $Z \in \mathbf{z}(I_R(\mathbf{L}))$ *then* $Z = \bigcap_{n=1}^{\infty} W(L_n)'$, $L_n \in \mathbf{L}$.

Proof. Follows immediately from the fact that Z is a compact, $\mathbf{G}_\delta$ set in $I_R(\mathbf{L})$. $\square$

In what follows, $\bar{X}^\delta$ denotes the $\mathbf{G}_\delta$-closure of X in $I_R(\mathbf{L})$.

LEMMA 4.5. *If* $\mathbf{L}$ *is a separating and disjunctive lattice of subsets of* X *then* $I_R^\sigma(\mathbf{L}) \subset \bar{X}^\delta$.

Proof. Suppose $\mu \in I_R^\sigma(\mathbf{L})$ and $\mu \notin \bar{X}^\delta$. Then there exists a set $G = \bigcap_{n=1}^{\infty} O_n$, $O_n' \in \tau W(\mathbf{L})$, such that $\mu \in G$ and $G \cap X = \emptyset$. Let $O_n = \bigcup_\alpha W(L_\alpha^{(n)})'$, $L_\alpha^{(n)} \in \mathbf{L}$. Then $\mu \in \bigcap_{n=1}^{\infty} W(L_\alpha^{(n)})' \subset G \subset I_R(\mathbf{L}) - X$ and so by Lemma 4.1, $\bigcap_{n=1}^{\infty} L_\alpha^{(n)\prime} = \emptyset$. Since $\mu(L_\alpha^{(n)\prime}) = 1$ for all n we have a contradiction. $\square$

LEMMA 4.6. *If* $\mathbf{L}$ *is a lattice of subsets of* X *which is separating, disjunctive, normal, and countably paracompact, then* $I_R^\sigma(\mathbf{L}) = \bar{X}^\delta$.

Proof. We shall prove that $\bar{X}^\delta \subset I_R^\sigma(\mathbf{L})$. Suppose $\mu \in \bar{X}^\delta$ and $\mu \in I_R(\mathbf{L}) - I_R^\sigma(\mathbf{L})$. Then there exists $L_n \in L$, $L_n \downarrow \emptyset$ such that $\mu(L_n) = 1$ for all n. Therefore $\mu \in \bigcap_{n=1}^{\infty} W(L_n) \subset I_R(\mathbf{L}) - X$. From Lemma 4.3 $\mu \in \bigcap_{n=1}^{\infty} W(L_n) \subset K_0 \subset I_R(\mathbf{L}) - X$ for some $K_0 \in \mathbf{z}(I_R(\mathbf{L}))$. But K_0 is a $\mathbf{G}_\delta$ set in $I_R(\mathbf{L})$ which contradicts the assumption that $\mu \in \bar{X}^\delta$. $\square$

We relate the concepts of repleteness and the $\mathbf{G}_\delta$-closure of a set X through the following theorem.

THEOREM 4.2. *Suppose* $\mathbf{L}$ *is a separating, disjunctive, normal, and countably paracompact lattice of subsets of* X. *Then* $\mathbf{L}$ *is replete iff* X *is* $\mathbf{G}_\delta$*-closed in* $I_R(\mathbf{L})$.

Proof. $\mathbf{L}$ is a replete iff $I_R^\sigma(\mathbf{L}) = X$. The result follows immediately from Lemma 4.6. □

COROLLARY 4.1 (Mrówka [12]). *Suppose X is a Tychonoff space. Then X is realcompact iff X is $\mathbf{G}_\delta$-closed in βX, the Stone-Cech compactification of X.*

Proof. Take $\mathbf{L} = \mathbf{z}_X$. Then $I_R(\mathbf{L}) = \beta X$. □

Theorem 4.2 gives us the following additional new application.

COROLLARY 4.2. *Suppose X is a T_1 topological space which is normal and countably paracompact. Then X is α-complete iff X is $\mathbf{G}_\delta$-closed in wX, the Wallman compactification of X.*

Proof. Take $\mathbf{L} = \mathbf{F}_X$. Then $wX = I_R(\mathbf{L})$. □

We now show that under the hypotheses of Theorem 4.2 repleteness is equivalent to expressing X as an arbitrary intersection of complements of zero sets or as an arbitrary intersection of $\mathbf{F}_\sigma$ sets in $I_R(\mathbf{L})$. We begin with the following lemma.

LEMMA 4.7. *Suppose $\mathbf{L}$ is a separating and disjunctive lattice of subsets of a set X. If $X = \bigcap_\alpha F_\alpha$ where each F_α is an $\mathbf{F}_\sigma$ set in $I_R(\mathbf{L})$ then $\mathbf{L}$ is replete.*

Proof. If $X = \bigcap_\alpha F_\alpha$ where each F_α is an $\mathbf{F}_\sigma$ set, then $I_R(\mathbf{L}) - X = \bigcup_\alpha F'_\alpha$. Arguing as in the proof of Lemma 4.5, we see that if $\mu \in I_R(\mathbf{L}) - X$ and hence $\mu \in F'_\alpha$ for some α, then $\mu \in \bigcap_{n=1}^\infty W(L'_n) \subset F'_\alpha \subset I_R(\mathbf{L}) - X$ where $L_n \in \mathbf{L}$ from which it follows that $I_R^\sigma(\mathbf{L}) = X$. □

THEOREM 4.3. *Suppose $\mathbf{L}$ is a separating, disjunctive, normal, and countably paracompact lattice of subsets of X. Then $\mathbf{L}$ is replete iff $X = \bigcap_\alpha Z'_\alpha$ where $Z_\alpha \in \mathbf{z}(I_R(\mathbf{L}))$.*

Proof. Since each Z'_α is an $\mathbf{F}_\sigma$ set it follows from Lemma 4.7 that $\mathbf{L}$ is replete. Conversely, suppose $\mathbf{L}$ is replete. Let $\mu \in I_R(\mathbf{L}) - X$ and let $L_n \in \mathbf{L}$ be such that $L_n \downarrow \varnothing$ and $\mu(L_n) = 1$ for all n. Then $\mu \in \bigcap_{n=1}^\infty W(L_n) \subset I_R(L) - X$. From Lemma 4.3 there exists $Z_\mu \in \mathbf{z}(I_R(\mathbf{L}))$ such that $\mu \in \bigcap_{n=1}^\infty W(L_n) \subset Z_\mu \subset I_R(\mathbf{L}) - X$. Thus $X \subset \bigcap_\mu Z'_\mu$ where μ runs over $I_R(\mathbf{L}) - X$. But clearly $I_R(\mathbf{L}) - X \subset \bigcup_\mu Z_\mu$ and hence $X = \bigcap_\mu Z'_\mu$. □

COROLLARY 4.3 (Frolik [6]). *Suppose X is a Tychonoff space. Then X is realcompact iff $X = \bigcap_\alpha Z'_\alpha$ where each Z_α is a zero set in βX.*

Proof. Again take $\mathbf{L} = \mathbf{z}_X$. □

COROLLARY 4.4. *Suppose X is a T_1 topological space which is normal and countably paracompact. Then X is α-complete iff $X = \bigcap_\alpha Z'_{\alpha'}$ $Z_\alpha \in \mathbf{z}_{wX}$.*

Proof. Again let $\mathbf{L} = \mathbf{F}_X$. □

THEOREM 4.4. *Suppose* $\mathbf{L}$ *is a separating, disjunctive, normal, and countably paracompact lattice of subsets of* X. *Then* $\mathbf{L}$ *is replete iff* $X = \bigcap_\alpha F_\alpha$ *where each* F_α *is an* $\mathbf{F}_\sigma$ *set in* $I_R(\mathbf{L})$.

Proof. Follows immediately from Lemma 4.7 and Theorem 4.3. □

COROLLARY 4.5 (Wenjen [16]). *Suppose* X *is Tychonoff space. Then* X *is realcompact iff* $X = \bigcap_\alpha F_\alpha$ *where each* F_α *is an* $\mathbf{F}_\sigma$ *set in* βX.

Proof. Let $\mathbf{L} = \mathbf{z}_X$ in Theorem 4.4. □

If we take $\mathbf{L} = \mathbf{F}_X$ in Theorem 4.4, we obtain our final new result.

COROLLARY 4.6. *Suppose* X *is a* T_1 *topological space which is normal and countably paracompact. Then* X *is* α*-complete iff* $X = \bigcap_\alpha F_\alpha$ *where each* F_α *is an* $\mathbf{F}_\sigma$ *set in* wX.

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