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92.27 The number of HH's in a coin-tossing experiment and the Fibonacci sequence

In the study of some coin-tossing experiments, an appearance of the Fibonacci sequence is known. While analysing a new coin-tossing experiment also, we came across the Fibonacci sequence. This is described in the article.

*Introduction*

*Example:* We consider the following coin-tossing experiment.

Toss an unbiased coin $n$ times and obtain a string of Heads (H) and Tails (T). For example, if $n = 10$, we may get the following string:

\[ \text{HHTHHHTTHT}. \]

In the above string, we have 3 HHs, namely at toss numbers (1,2), (4,5) and (5,6).

In the above experiment, the following result, where Fibonacci numbers are involved, is well known.

**Result:** If an unbiased coin is tossed $n$ times, then the probability that there are no two successive heads is given by $F_{n+2}/2^n$, where $F_{n+2}$ is the $(n + 2)$th Fibonacci number. [1]

The analysis that we carried out is discussed in the next section.

*Analysis*

Let $X$ denote the number of HH (double heads) in the string of $n$ tosses of an unbiased coin, where overlapping is allowed.

It may be recalled that, when an unbiased coin is tossed $n$ times, there are $2^n$ possible strings of Hs and Ts in all. Of these $2^n$ strings, let $S(n, r)$ denote the total number of strings where $X = r \ (r = 0,1,2, \ldots, n - 1)$.

We further divide these $S(n, r)$ strings into two categories of substrings as follows.
Notation: Let $ST(n, r)$ and $SH(n, r)$ denote the number of substrings of $n$ tosses with $X = r$ and the $n$th toss (last toss) resulting in T and H respectively. 

It is obvious that

$$\sum_{r=0}^{n-1} S(n, r) = 2^n.$$  

For example, when $n = 3$, there are 8 possible strings of 3 tosses as follows:

$$\{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}.$$  

Now, $S(3,0) = 5 = \text{Number of strings with no double Hs in 3 tosses, namely the substrings TTT, HTH, HTT, TTH, THT.}$ Also $S(3,1) = 2$ corresponding to $[THH, HHT]$ and $S(3,2) = 1$ to $[HHH]$.  

Further, $ST(3,0) = 3$, namely the sub-strings TTT, HTT, TTH and $SH(3,0) = 2$, namely the substrings HTH and TTH.  

Similarly $ST(3,1) = 1$, $SH(3,1) = 1$, whereas $SH(3,2) = 1$ and $ST(3,2) = 0$.  

Thus we observe that $S(3, r) = ST(3, r) + SH(3, r)$ for $r = 0, 1, 2$.  

While studying the patterns of substrings $ST(n, r)$ and $SH(n, r)$, we observe the following results, which we put in the form of a lemma.

Lemma 1: For $n \geq 2$ and $0 \leq r \leq n - 1$,

(a) $ST(n, r) = ST(n - 1, r) + SH(n - 1, r)$  
(b) $SH(n, r) = ST(n - 1, r) + SH(n - 1, r - 1)$  
(c) $S(n, r) = ST(n, r) + SH(n, r)$  

where $S(n, -1) = SH(n, -1) = ST(n, -1) = 0$, for all $n$.  

The explanation of (a) and (b) follows.  

In $n$ trials, substrings with exactly $r$ HHs and the $n$th trial resulting in T are formed from the substrings of $(n - 1)$ tosses with $r$ HHs in $(n - 1)$ tosses, and the $(n - 1)$th toss resulting in either H or T. Hence we have (a).  

Similarly, a sub-string with exactly $r$ HHs in $n$ trials and the last trial resulting in H is formed from a substring with $r$ HHs in the $(n - 1)$ tosses and the last toss resulting in T or a substring that has $(r - 1)$ HHs in the $(n - 1)$ tosses, with the $(n - 1)$th toss being an H and an H in the last toss. Hence we get (b). Also (c) is already noted.  

From the above lemma, we get an interesting recurrence relation in terms of $S(n, r)$. This relation we present in the following theorem.

Theorem 1: For $n \geq 2$ and $0 \leq r \leq n - 1$,

$$S(n, r) = S(n - 1, r - 1) + S(n - 1, r) + S(n - 2, r) - S(n - 2, r - 1) \quad (1)$$
Proof: From result (b) of Lemma 1, we have,
\[ ST(n, r) = SH(n + 1, r) - SH(n, r - 1). \]
Also from result (a) of Lemma 1
\[ ST(n, r) = ST(n - 1, r) + SH(n - 1, r) = SH(n - 1, r) + SH(n, r) - SH(n - 1, r - 1). \]
Equating these expressions and replacing \( n \) by \( n - 1 \) gives
\[ SH(n, r) = SH(n - 1, r - 1) + SH(n - 1, r) + SH(n - 2, r) - SH(n - 2, r - 1). \quad (2) \]
Using similar arguments, we get
\[ ST(n, r) = ST(n - 1, r - 1) + ST(n - 1, r) + ST(n - 2, r) - ST(n - 2, r - 1). \quad (3) \]
Adding (2) and (3) and using (c) of Lemma 1, we get
\[ S(n, r) = S(n - 1, r - 1) + S(n - 1, r) + S(n - 2, r) - S(n - 2, r - 1). \]
Using \( S(2,0) = 3 \) and \( S(2,1) = 1 \) and together with a computer program, we obtain a bivariate frequency table for \( S(n, r) \). The output is given below.

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<td>1</td>
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</tbody>
</table>

**TABLE 1:** \( S(n, r) \) for \( 3 \leq n \leq 11 \) and \( 0 \leq r \leq n - 1 \)

As stated in the introduction, the first column of the above table, \( S(n,0) \) contains elements of the Fibonacci sequence, since we note that \( S(n,0) = F_{n+2} \).

Alternatively, by putting \( x = 0 \) in result (1) of Theorem 1, we get
\[ S(n,0) = S(n - 1,0) + S(n - 2,0) \quad (4) \]
The result then follows from the initial conditions.
Concluding remarks

Using the recurrence relation of Theorem 1 and equation (4), we get,
\[ S(n + 2, 1) - S(n + 1, 1) - S(n, 1) = F_{n + 1}, \]
whose solution is
\[ S(n, 1) = \frac{nF_{n + 1} - F_n + nF_{n - 1}}{5}. \]

However expressions for \( S(n, 2), S(n, 3), \ldots \) are complicated. We conclude this paper with the following expression for \( S(n, 2) \) which we have derived. (This derivation is quite lengthy and hence omitted.)

\[ S(n, 2) = nF_{n - 2} - 3F_{n - 4} + \sum_{i=0}^{n-8} F_{i+2}S(n - i - 4, 1), \text{ for } n \geq 8. \]

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References

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92.28 Angles in croquet

In Association Croquet, when you hit another ball, you pick up your own ball and place it touching the other ball. You then hit your own ball with your mallet to make both balls move. This is known as a roquet shot.

When I was learning to play croquet, I was taught that the other ball moves off along the line joining the centre of the two balls (the \( x \)-axis in Figure 1 below). So you should decide which direction to send the other ball in and place your own ball accordingly. I was then taught to decide which direction to send my own ball in and to bisect the angle between these two directions; this would give the direction in which to swing my mallet (the \( p \)-axis in Figure 1 below).

This note attempts to model this situation and determine to what extent these rules for aiming are reliable. We shall be ignoring the effects of spin, which may be significant. We shall be treating the shot as a sequence of separate collisions, though in reality they happen simultaneously.

Let \( m \) be the mass of each ball and \( km \) be the mass of the mallet, where \( k > 1 \). Let \( e \) be the coefficient of restitution, which we shall assume is the same between the two balls as between the mallet and a ball. Let \( \theta \) be the angle between the \( x \)-axis and the \( p \)-axis. Let \( u \) be the initial velocity of the mallet.