

UPPER SEMI CONTINUITY OF ATTRACTORS OF DELAY DIFFERENTIAL EQUATIONS IN THE DELAY

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It is shown that if a retarded delay differential equation has a global attractor \mathcal{A}_{τ_0} in the space $C([- \tau_0, 0], \mathbb{R}^d)$ for a given nonzero constant delay τ_0 , then the equation has an attractor \mathcal{A}_τ in the space $C([- \tau, 0], \mathbb{R}^d)$ for nearby constant delays τ . Moreover the attractors \mathcal{A}_τ converge upper semi continuously to \mathcal{A}_{τ_0} in $C([- \tau_0, 0], \mathbb{R}^d)$ in the sense that they are identified through corresponding segments of entire trajectories in \mathbb{R}^d with nonempty compact subsets $I_{\tau_0}(\mathcal{A}_\tau)$ of $C([- \tau_0, 0], \mathbb{R}^d)$ which converge upper semi continuously to \mathcal{A}_{τ_0} in $C([- \tau_0, 0], \mathbb{R}^d)$.

1. INTRODUCTION

Consider a retarded delay differential equation

$$(1) \quad \dot{x}(t) = f(x(t), x(t - \tau)), \quad t \geq 0,$$

in \mathbb{R}^d with a constant delay $\tau > 0$ and assume that $f \in C_b^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$, where C_b^1 is the space of continuously differentiable functions from $\mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^d which are bounded and have bounded first derivatives. Then, for any given continuous function $\phi : [-\tau, 0] \rightarrow \mathbb{R}^d$ as initial value, the delay differential equations (1) possesses a unique solution $x(t, \phi)$ which exists for all future time (see, for example, Driver [1] or Hale and Verduyn Lunel [3]).

The solutions of initial value problems for the delay differential equations (1) can be reformulated as a semi-dynamical system or semi-flow on the Banach space \mathcal{C}_τ of continuous functions $\phi : [-\tau, 0] \rightarrow \mathbb{R}^d$ equipped with the norm

$$\|\phi\|_\tau = \max \{ \|\phi(s)\| \mid s \in [-\tau, 0] \}.$$

For any $y \in C([- \tau, \infty), \mathbb{R}^d)$ and fixed $t \geq 0$ define $y_t \in \mathcal{C}_\tau$ by $y_t(s) := y(t + s)$ for $s \in [-\tau, 0]$. Then, the initial value problem for the delay differential equations (1) with

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solution $x(t, \phi) \in \mathbb{R}^d$ for $t \geq 0$ generates a semi-flow $\Phi_t : C_\tau \rightarrow C_\tau$ with $\phi \rightarrow \Phi_t(\phi)$ defined by

$$(2) \quad \Phi_t(\phi)(s) := x(t + s, \phi), \quad s \in [-\tau, 0], \quad t \geq 0.$$

Let

$$(3) \quad M := \sup \left\{ \|f(x, y)\| \mid (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \right\} < \infty.$$

It is easy to see that $\Phi_t(\phi)$ is Lipschitz continuous with constant M for any initial function $\phi \in C_\tau$ and $t \geq 0$, which means the family $\{\Phi_t\}_{t \geq 0}$ defines a semi-flow on the subspace

$$\text{Lip}_{\tau,L} := \left\{ \psi \in C_\tau \mid |\psi(s_1) - \psi(s_2)| \leq L|s_1 - s_2|, \quad s_1, s_2 \in [-\tau, 0] \right\}$$

of C_τ for any $L \geq M$. The operator Φ_t is compact for each $t > 0$ by the Arzela–Ascoli theorem and the semi-flow will thus have an attractor if it has a closed and bounded absorbing set.

Since different delays will be considered in the sequel, we shall denote by $\Phi_t^{(\tau)}$ the semi-flow on C_τ corresponding to the τ -delay differential equations (1), that is, with delay $\tau > 0$. A nonempty compact subset \mathcal{A}_τ of C_τ is called the global attractor of the semiflow Φ_t of the delay differential equations (1) if it is invariant, that is, $\Phi_t^{(\tau)}(\mathcal{A}_\tau) = \mathcal{A}_\tau$ for all $t \geq 0$ and attracts all nonempty closed and bounded subsets B of C_τ , that is, for every such set B and each $\varepsilon > 0$ there exists $T_{\tau,B,\varepsilon} \geq 0$ such that

$$(4) \quad H_\tau^*(\Phi_t^{(\tau)}(B), \mathcal{A}_\tau) < \varepsilon \quad \text{for } t \geq T_{\tau,B,\varepsilon},$$

where H_τ^* is the Hausdorff semi-distance between closed and bounded subsets of C_τ . A global attractor is uniformly asymptotically stable [2] and can thus be characterised by a Lyapunov function (see Theorem 2.2 later).

Let $\mathcal{S}_\tau \subset C(\mathbb{R}, \mathbb{R}^d)$ denote the set of all entire bounded solutions of the τ -delay differential equations (1), that is, those which exist and are bounded for all $t \in \mathbb{R}$. An entire solution is obviously Lipschitz continuous with Lipschitz constant M . The corresponding entire trajectories of the semi-flow $\Phi_t^{(\tau)}$ thus belong to $\text{Lip}_{\tau,L}$ for any $L \geq M$ (we henceforth choose and then hold fixed such an L) and are contained in the global attractor \mathcal{A}_τ of the semi-flow, if it exists. Indeed, by invariance, there exists an entire bounded trajectory through every point of an attractor \mathcal{A}_τ , which is thus the image in $\text{Lip}_{\tau,L}$ of all of the entire bounded trajectories of the semi-flow $\Phi_t^{(\tau)}$. Consequently, \mathcal{A}_τ can be represented by or identified with a compact subset $I_{\tau_0}(\mathcal{A}_\tau)$ in the space C_{τ_0} for any other delay $\tau_0 \neq \tau$, where

$$I_{\tau_0}(\mathcal{A}_\tau) := \left\{ \phi \in C_{\tau_0} \mid \phi(s) = x^{(\tau)}(t + s), \quad s \in [-\tau_0, 0], \quad \text{for some } x^{(\tau)} \in \mathcal{S}_\tau \text{ and } t \in \mathbb{R} \right\}$$

2. THE MAIN RESULT

The following theorem is the main result of this paper. The underlying assumption that $f \in C_b^1$ is not a major restriction since the long term dynamics takes values in a closed and bounded subset of the function space C_τ , with the corresponding functions thus taking values in a closed and bounded, hence compact, subset of $\mathbb{R}^d \times \mathbb{R}^d$.

THEOREM 2.1. *Suppose that the semi-flow $\Phi_t^{(\tau_0)}$ has a global attractor \mathcal{A}_{τ_0} for some $\tau_0 > 0$. Then the semi-flow $\Phi_t^{(\tau)}$ has an attractor \mathcal{A}_τ (possibly only local) for all $\tau > 0$ with $|\tau - \tau_0|$ sufficiently small. Moreover, the \mathcal{A}_τ converge to \mathcal{A}_{τ_0} upper semi continuously in C_{τ_0} in the sense that*

$$(1) \quad H_{\tau_0}^*(I_{\tau_0}(\mathcal{A}_\tau), \mathcal{A}_{\tau_0}) \rightarrow 0 \quad \text{as } \tau \rightarrow \tau_0.$$

The upper semi continuous convergence (1) is obviously equivalent to the upper semi continuous convergence of the compact \mathcal{S}_τ to \mathcal{S}_{τ_0} in the Hausdorff semi metric based on the metric

$$d_{unif}(x, y) := \sum_{j=-\infty}^{\infty} 2^{-|j|} \max_{-\tau_0 \leq s \leq 0} \|x(j\tau_0 + s) - y(j\tau_0 + s)\|$$

on $C(\mathbb{R}, \mathbb{R}^d)$, that is, corresponding to uniform convergence on compact subsets. (The choice of the length τ_0 of the intervals here is suggestive, but not critical – it need only be finite).

2.1. PROOF OF THEOREM 2.1: Theorem 2.1 is essentially a consequence of the total stability of the global attractor \mathcal{A}_{τ_0} , in particular of the fact that the uniform asymptotical stability of such attractor can be characterised by a Lipschitz continuous Lyapunov function. The proof which follows is similar to those, in different contexts, in Kloeden and Lorenz [4, Theorem 1.1] and in Kloeden and Siegmund [6, Theorem 1].

Yoshizawa [7] provides various necessary and sufficient conditions which characterise the uniform asymptotical stability of a compact sets in terms of Lyapunov functions for ordinary differential equations. The following theorem is a straightforward generalisation of Yoshizawa in [7, Theorem 22.5] to retarded delay differential equations, see also [7, Theorem 35.2] as well as [5]. □

THEOREM 2.2. *Suppose that the solution flow $\Phi_t^{(\tau_0)}(\phi)$ in C_{τ_0} of τ_0 -delay differential equations (1) has a global attractor \mathcal{A}_{τ_0} in C_{τ_0} , which is thus uniformly asymptotically stable. Then there exists a function $V : C_{\tau_0} \rightarrow [0, \infty)$ satisfying*

- (1) $|V(\phi) - V(\psi)| \leq L_V \|\phi - \psi\|_{\tau_0}$ for all $\phi, \psi \in C_{\tau_0}$
- (2) There exist strongly monotone increasing functions $a, b : [0, \infty) \rightarrow [0, \infty)$ with $a(0) = b(0) = 0$ and $a(\tau) < b(\tau)$ for $\tau > 0$ such that

$$a(\text{dist}_{\tau_0}(\phi, \mathcal{A}_{\tau_0})) \leq V(\phi) \leq b(\text{dist}_{\tau_0}(\phi, \mathcal{A}_{\tau_0})), \quad \phi \in C_{\tau_0}.$$

(3) There exists a positive real number c such that

$$V(\Phi_t^{(\tau_0)}(\phi)) \leq e^{-ct} V(\phi), \quad t \geq 0, \phi \in \mathcal{C}_{\tau_0}.$$

Here $\text{dist}_{\tau_0}(\phi, \mathcal{B})$ denotes the distance between a point ϕ and a closed and bounded subset \mathcal{B} in \mathcal{C}_{τ_0} .

Consider the solution $x^{(\tau)}(t, \phi_0) \in \mathbb{R}^d$ for $t \geq -\tau$ of the τ -delay differential equations (1) with initial value $\phi_0 \in \mathcal{C}_\tau$. If $\tau < \tau_0$, extend $x^{(\tau)}(t, \phi_0)$ to $t \geq -\tau_0$ by defining $x^{(\tau)}(t, \phi_0) = \phi_0(-\tau)$ for $-\tau_0 \leq t \leq -\tau$. For $\phi_0 \in \mathcal{C}_\tau$ and $t \geq 0$ define

$$S_t^{(\tau, \tau_0)}(\phi_0)(-s) := x^{(\tau)}(t - s, \phi_0), \quad 0 \leq s \leq \tau_0,$$

so $S_t^{(\tau, \tau_0)}(\phi_0) \in \mathcal{C}_{\tau_0}$ for all $t \geq 0$ with $S_0^{(\tau, \tau_0)}(\phi_0) = E_\tau^{\tau_0}(\phi_0)$, where

$$E_\tau^{\tau_0}(\phi_0) := \begin{cases} \phi_0(-s) & \text{for } 0 \leq s \leq \min\{\tau, \tau_0\} \\ \phi_0(-\tau) & \text{for } \tau \leq s \leq \tau_0 \text{ if } \tau < \tau_0 \end{cases}$$

for each $\phi_0 \in \mathcal{C}_\tau$. Note that $S_t^{(\tau, \tau_0)}$ is not a semi-flow.

The following Lyapunov inequality, which will be proved in the appendix, is fundamental to the proof.

LEMMA 2.3. Suppose that $|\tau - \tau_0| \leq \min\{\tau, \tau_0\}$. There exists a constant K independent of τ and τ_0 such that

$$(2) \quad V(S_{s+t}^{(\tau, \tau_0)}(\phi_0)) \leq e^{-cs} V(S_t^{(\tau, \tau_0)}(\phi_0)) + L_V K |\tau - \tau_0|^2$$

for all $s \in [0, |\tau - \tau_0|]$, $t \geq 0$ and all $\phi_0 \in \text{Lip}_{\tau_0, L}$.

From Lemma 2.3 we have

$$V(S_{(n+1)\Delta}^{(\tau, \tau_0)}(\phi_0)) \leq e^{-c\Delta} V(S_{n\Delta}^{(\tau, \tau_0)}(\phi_0)) + L_V K \Delta^2$$

for $n \geq 1$, where $\Delta := |\tau - \tau_0| > 0$ with $|\tau - \tau_0| \leq \min\{\tau, \tau_0\}$. Define

$$\eta(\Delta) := \frac{2L_V K \Delta^2}{1 - e^{-c\Delta}} \approx \frac{2L_V K}{c} \Delta$$

and

$$\Lambda_{\tau_0}[\eta(\Delta)] := \{\phi \in \text{Lip}_{\tau_0, L} \mid V(\phi) \leq \eta(\Delta)\},$$

Then, as in the proof of [4, Theorem 1.1] it can be shown that the closed and bounded set $\Lambda_{\tau_0}[\eta(\Delta)]$ is positively invariant and absorbing for the sequence of mappings $S_{n\Delta}^{(\tau, \tau_0)}$, provided Δ is sufficiently small, in the sense that $S_{n\Delta}^{(\tau, \tau_0)}(\phi_0) \in \Lambda_{\tau_0}[\eta(\Delta)]$ for $n \geq 1$ and $\phi_0 \in \mathcal{C}_\tau$ with $E_\tau^{\tau_0}(\phi_0) \in \Lambda_{\tau_0}[\eta(\Delta)]$ and that for any finite $R_0 > 0$ there exists an $N_{R_0, \tau, \Delta} \in \mathbb{N}$ such that

$$S_{n\Delta}^{(\tau, \tau_0)}(\phi_0) \in \Lambda_{\tau_0}[\eta(\Delta)]$$

for $n \geq N_{R_0, \tau, \Delta}$ and all $\phi_0 \in \mathcal{C}_\tau$ with $E_\tau^{\tau_0}(\phi_0) \in \mathcal{B}_{\tau_0}[\mathcal{A}_{\tau_0}, R_0]$, the closed and bounded ball of radius R_0 about the compact subset \mathcal{A}_{τ_0} in $\text{Lip}_{\tau_0, L}$ defined by

$$\mathcal{B}_{\tau_0}[\mathcal{A}_{\tau_0}, R_0] := \{\phi \in \text{Lip}_{\tau_0, L} \mid \text{dist}_{\tau_0}(\phi, \mathcal{A}_{\tau_0}) \leq R_0\}.$$

Note that

$$\Lambda_{\tau_0}[\eta(\Delta)] \subset \mathcal{B}_{\tau_0}[\mathcal{A}_{\tau_0}, a^{-1}(\eta(\Delta))]$$

by property (2) of the Lyapunov function.

To show positive invariance it suffices to consider a single iterate $S_\Delta^{(\tau, \tau_0)}(\phi_0)$ for $\phi_0 \in \mathcal{C}_\tau$ with $E_\tau^{\tau_0}(\phi_0) \in \Lambda_{\tau_0}[\eta(\Delta)]$. Then

$$\begin{aligned} V(S_\Delta^{(\tau, \tau_0)}(\phi_0)) &\leq e^{-c\Delta} V(S_0^{(\tau, \tau_0)}(\phi_0)) + L_V K \Delta^2 \\ &\leq e^{-c\Delta} \eta(\Delta) + \frac{1}{2}(1 - e^{-c\Delta}) \eta(\Delta) \\ &= \frac{1}{2}(1 + e^{-c\Delta}) \eta(\Delta) \leq \eta(\Delta), \end{aligned}$$

since $V(S_0^{(\tau, \tau_0)}(\phi_0)) \leq \eta(\Delta)$. Hence $S_\Delta^{(\tau, \tau_0)}(\phi_0) \in \Lambda_{\tau_0}[\eta(\Delta)]$.

To show the absorbing property first note that there exists a $\gamma > 0$ such that

$$1 + e^{-c\gamma} = 2e^{-c\gamma/2} \quad \text{and} \quad 1 + e^{-c\Delta} < 2e^{-c\Delta/2}$$

for all $0 < \Delta < \gamma$. Suppose that

$$\Delta = |\tau - \tau_0| < \Delta^* := \min\{\tau, \tau_0, \gamma\}$$

and consider $S_\Delta^{(\tau, \tau_0)}(\phi_0)$ for $\phi_0 \in \text{Lip}_{\tau, L}$ with $E_\tau^{\tau_0}(\phi_0) \notin \Lambda_{\tau_0}[\eta(\Delta)]$. Then

$$\begin{aligned} V(S_\Delta^{(\tau, \tau_0)}(\phi_0)) &\leq e^{-c\Delta} V(S_0^{(\tau, \tau_0)}(\phi_0)) + L_V K \Delta^2 \\ &= e^{-c\Delta} V(S_0^{(\tau, \tau_0)}(\phi_0)) + \frac{1}{2}(1 - e^{-c\Delta}) \eta(\Delta) \\ &\leq \frac{1}{2}(1 + e^{-c\Delta}) V(S_0^{(\tau, \tau_0)}(\phi_0)) \\ &\leq e^{-c\Delta/2} V(S_0^{(\tau, \tau_0)}(\phi_0)), \end{aligned}$$

since $V(S_0^{(\tau, \tau_0)}(\phi_0)) > \eta(\Delta)$. Repeating this argument,

$$V(S_{n\Delta}^{(\tau, \tau_0)}(\phi_0)) < e^{-cn\Delta/2} V(S_0^{(\tau, \tau_0)}(\phi_0))$$

as long as $S_0^{(\tau, \tau_0)}(\phi_0), S_\Delta^{(\tau, \tau_0)}(\phi_0), \dots, S_{(n-1)\Delta}^{(\tau, \tau_0)}(\phi_0) \notin \Lambda_{\tau_0}[\eta(\Delta)]$. Now

$$V(S_0^{(\tau, \tau_0)}(\phi_0)) \leq b(\text{dist}_{\tau_0}(S_0^{(\tau, \tau_0)}(\phi_0), \mathcal{A}_{\tau_0})) \leq b(R_0),$$

so

$$V(S_{n\Delta}^{(\tau,\tau_0)}(\phi_0)) \leq e^{-cn\Delta/2} b(R_0).$$

Finally, define $N_{R_0,\tau,\Delta}$ to be the smallest integer n for which

$$e^{-cn\Delta/2} b(R_0) \leq \eta(\Delta) < e^{-c(n-1)\Delta/2} b(R_0).$$

This proves that $\Lambda_{\tau_0}[\eta(\Delta)]$ is an absorbing set for the $S_{n\Delta}^{(\tau,\tau_0)}$.

Hence the semi-flow $\Phi_t^{(\tau)}(\phi_0)$ will be contained in a corresponding positive invariant, closed and bounded subset \mathcal{B}_τ of \mathcal{C}_τ for all $t \geq N_{R_0,\tau,\Delta}\Delta$. Since the operators $\Phi_t^{(\tau)}$ are compact for $t > 0$, the semi-flow $\Phi^{(\tau)}$ has an attractor (possibly only local) \mathcal{A}_τ in \mathcal{B}_τ .

By invariance, for any $\phi_0 \in \mathcal{A}_\tau$ and $T > 0$, there is a $\psi_0 \in \mathcal{A}_\tau$ such that $\Phi_T^{(\tau)}(\psi_0) = \phi_0$. Taking

$$T = n\Delta \geq \max\{\tau, \tau_0, N_{R_0,\tau,\Delta}\Delta\},$$

it follows that

$$S_{n\Delta}^{(\tau,\tau_0)}(\psi_0) \in \Lambda_{\tau_0}[\eta(\Delta)],$$

which means that

$$\text{dist}_{\tau_0}(S_{n\Delta}^{(\tau,\tau_0)}(\psi_0), \mathcal{A}_{\tau_0}) \leq a^{-1}(\eta(\Delta))$$

for any $\psi_0 \in \mathcal{A}_\tau$. Since $S_{n\Delta}^{(\tau,\tau_0)}(\psi_0)$ corresponds in $\text{Lip}_{\tau_0,L}$ to part of an entire solution of the τ -delay differential equations through $\phi_0 = \Phi_T^{(\tau)}(\psi_0) \in \mathcal{A}_\tau$, this implies that

$$H_{\tau_0}^*(I_{\tau_0}(\mathcal{A}_\tau), \mathcal{A}_{\tau_0}) \leq a^{-1}(\eta(\Delta)) \rightarrow 0 \text{ as } \Delta \rightarrow 0,$$

which completes the proof of Theorem 2.1. □

3. APPENDIX: PROOF OF LEMMA 2.3

Consider the semi-flow representation $\Phi_s^{(\tau_0)}(S_t^{(\tau,\tau_0)}(\phi_0))$ solution of the τ_0 -delay differential equations starting at $S_t^{(\tau,\tau_0)}(\phi_0)$, where $\phi_0 \in \text{Lip}_{\tau,L}$. Then

$$\begin{aligned} V(S_{s+t}^{(\tau,\tau_0)}(\phi_0)) &\leq V(\Phi_s^{(\tau_0)}(S_t^{(\tau,\tau_0)}(\phi_0))) + \left| V(S_{s+t}^{(\tau,\tau_0)}(\phi_0)) - V(\Phi_s^{(\tau_0)}(S_t^{(\tau,\tau_0)}(\phi_0))) \right| \\ &\leq e^{-cs} V(S_t^{(\tau,\tau_0)}(\phi_0)) + L_V \left\| S_{s+t}^{(\tau,\tau_0)}(\phi_0) - \Phi_s^{(\tau_0)}(S_t^{(\tau,\tau_0)}(\phi_0)) \right\|_{\tau_0} \\ &\leq e^{-cs} V(S_t^{(\tau,\tau_0)}(\phi_0)) + L_V K |\tau - \tau_0|^2. \end{aligned}$$

Here we have used the exponential decay of the Lyapunov function along the semi-flow $\Phi^{(\tau_0)}$ and the Lipschitz property of the Lyapunov function on \mathcal{C}_{τ_0} . In the last line we have used the following lemma,

LEMMA 3.1. *Suppose that $|\tau - \tau_0| \leq \min\{\tau, \tau_0\}$. There exists a constant K independent of τ and τ_0 such that*

$$(3) \quad \left\| S_{s+t}^{(\tau, \tau_0)}(\phi_0) - \Phi_s^{(\tau_0)}(S_t^{(\tau, \tau_0)}(\phi_0)) \right\|_{\tau_0} \leq K|\tau - \tau_0|^2$$

for all $s \in [0, |\tau - \tau_0|]$, $t \geq 0$ and all $\phi_0 \in \text{Lip}_{\tau, L}$.

PROOF: Let $x(t)$ be the solution of the τ -delay differential equations and $y(t)$ be the solution of the τ_0 -delay differential equations for an initial value which is the same on the interval $[-\min\{\tau, \tau_0\}, 0]$, so in particular $x(0) = y(0)$. Subtracting interval versions of the differential equations gives

$$x(t) - y(t) = \int_0^t (f(x(s), x(s - \tau)) - f(y(s), y(s - \tau_0))) ds$$

and hence

$$|x(t) - y(t)| \leq L_f \int_0^t (|x(s) - y(s)| + |x(s - \tau) - y(s - \tau_0)|) ds,$$

where L_f is the Lipschitz constant of the vector field function f .

Assume that $\tau_0 \leq \tau$ (otherwise interchange the roles of x and y in what follows). Then

$$\begin{aligned} |x(t) - y(t)| &\leq L_f \int_0^t |x(s) - y(s)| ds + L_f \int_0^t |x(s - \tau) - y(s - \tau_0)| ds, \\ &\leq L_f \int_0^t |x(s) - y(s)| ds + L_f \int_0^t |x(s - \tau_0) - y(s - \tau_0)| ds \\ &\quad + L_f \int_0^t |x(s - \tau) - x(s - \tau_0)| ds, \\ &\leq L_f \int_0^t |x(s) - y(s)| ds + L_f \int_0^t 0 ds + L_f \int_0^t |x(s - \tau) - x(s - \tau_0)| ds, \\ &\leq L_f M |\tau - \tau_0|^2 + L_f \int_0^t |x(s) - y(s)| ds \end{aligned}$$

for $0 \leq t \leq |\tau - \tau_0|$.

Here we have used the facts that $x(s - \tau_0) = y(s - \tau_0)$ for $0 \leq s \leq \tau_0$ and

$$|x(s - \tau) - x(s - \tau_0)| \leq L|\tau - \tau_0|$$

for all $s \geq 0$ and initial values $\phi_0 \in \text{Lip}_{\tau, L}$. The latter follows from the inequality

$$|x(s - \tau) - x(s - \tau_0)| = \left| \int_{s-\tau_0}^{s-\tau} f(x(s), x(s - \tau)) \right| ds \leq M|\tau - \tau_0| \leq L|\tau - \tau_0|$$

for all $s \geq \tau$ and by the L -Lipschitz property of ϕ_0 for $0 \leq s \leq \tau$ (we omit the details).

An application of the Gronwall inequality and the reinterpretation of the solutions as elements of the space \mathcal{C}_{τ_0} yield the asserted inequality (3). □

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