ON THE NUMBER OF TOPOLOGIES DEFINABLE FOR A FINITE SET

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No general rule for determining the number $N(n)$ of topologies definable for a finite set of cardinal n is known. In this note we relate $N(n)$ to a function $F_t(r_1, \cdots, r_{t+1})$ defined below which has a simple combinatorial interpretation. This relationship seems useful for the study of $N(n)$. In particular this can be used to calculate $N(n)$ for small values. For $n = 3, 4, 5, 6$ we find $N(3) = 29, N(4) = 355, N(5) = 7{,}181, N(6) = 145{,}807$.

Let T be a topology on a finite set E. Let S_1 be the collection of all non-empty sets in T which do not properly contain any non-empty set in T. It is clear that S_1 is a collection of disjoint subsets of E. If for any collection K of sets $P_\cup(K)$ denotes the set of all non-empty unions of sets in K then $P_\cup(S_1) \subseteqq T$. Let $\cup S_1$ be the union of all sets in S_1. Then every non-empty set in T is of the form $U \cup V$ where $V \in P_\cup(S_1)$ and U is a subset of $E - \cup S_1$. Let T_1 be the collection of all the sets U and the null set. It can be easily proved that T_1 is a topology on $E - \cup S_1$. We shall refer to S_1 and T_1 as "nucleus" and "orbital topology" of the topology T, respectively.

By a "reduced base" of a topology on a finite set we shall mean a base such that no base set is a union of other base sets.

THEOREM. *Let B_1 be a reduced base for T_1. Then there is a unique single-valued mapping $f : B_1 \to P_\cup(S_1)$ such that $B = \{X_1 \cup X_1 f,\ X_1 \in B_1\} \cup S_1$ is a reduced base for T. Also, f preserves the inclusion relation $\subseteqq$ for sets. Conversely if S_1 is a non-empty collection of disjoint non-empty subsets of E, T_1 is any topology on $E - \cup S_1$ and f is a single-valued mapping from a reduced base B_1 for T_1 into $P_\cup(S_1)$ which preserves $\subseteqq$ then $B = \{X_1 \cup X_1 f,$ $X_1 \in B_1\} \cup S_1$ is a reduced base for a topology T on E such that S_1, T_1 are respectively the nucleus and the orbital topology of T.*

PROOF. For any $X_1 \in B_1$, we define $X_1 f$ to be a member of $P_\cup(S_1)$ such that $X_1 \cup X_1 f \in T$ and $X_1 \cup V \notin T$ if $X_1 f \supset V$. $X_1 f$ exists because T_1 is the orbital topology of T. If $V^* \in P_\cup(S_1)$ has the property stated for $X_1 f$ then $V^* \supseteqq X_1 f$ and $X_1 f \supseteqq V^*$, so that $X_1 f = V^*$. Thus f is a mapping from B_1 into $P_\cup(S_1)$. We show that f is the mapping required by the first part of the theorem. Let $X_1 \subseteqq X_1'$; then

$$(X_1 \cup X_1 f) \cap (X_1' \cup X_1' f) = X_1 \cup (X_1 f \cap X_1' f) \in T,$$

since X_1, $X_1' f$ are disjoint for all X_1, $X_1' \in B_1$. We conclude from the definition of f that $X_1 f \cap X_1' f = X_1 f$ so that $X_1 f \subseteqq X_1' f$ and hence f preserves $\subseteqq$. Next let $Y \in T$ and let $Y = U \cup V$, where $U \in T_1$, $V \in P_\cup(S_1)$. Since B_1 is a base for T_1 we can write $U = \cup B_1'$ for some subcollection B_1' of B_1. If U is empty, Y is trivially a union of sets in

$$B = \{X_1 \cup X_1 f, X_1 \in B_1\} \cup S_1.$$

Hence we can suppose B_1' non-empty. Then $X_1' f \subseteqq V$ for every $X_1' \in B_1'$; for

$$X_1' \cup (V \cap X_1' f) = (U \cup V) \cap (X_1' \cup X_1' f) \in T$$

and therefore $V \cap X_1' f = X_1' f$. Hence $Y = \cup \{X_1' \cup X_1' f, X_1' \in B_1'\} \cup$ (union of sets in S_1). This proves that B is a base for T. That B is reduced follows directly from the definition of f and the assumption that B_1 is reduced. To prove the uniqueness of the mapping f suppose that f^* is another mapping satisfying the first part of the theorem. Then, for some $X_1 \in B_1$, $X_1 f \subset X_1 f^*$. But $X_1 \cup X_1 f \in T$ and therefore is a union of sets in $B^* = \{Y_1 \cup Y_1 f^*, Y_1 \in B_1\} \cup S_1$. Since B_1 is reduced this is impossible in view of $X_1 f \subset X_1 f^*$.

For the converse, let B be as defined in the theorem. Then $E = \cup B = (\cup B_1) \cup (\cup S_1)$. Let Y, Y^* be any two members of B and write $Y = X_1 \cup X_1 f$, $Y^* = X_1^* \cup X_1^* f$. Since f preserves $\subseteqq$,

$$\begin{aligned} Y \cap Y^* &= (X_1 \cap X_1^*) \cup (X_1 f \cap X_1^* f) \\ &= (X_1 \cap X_1^*) \cup (X_1 \cap X_1^*) f \cup \text{ (union of sets in } S_1). \end{aligned}$$

Now X_1, $X_1^* \in B_1$ and $X_1 \cap X_1^* = \cup B_1'$, where B_1' is a subcollection of B_1. Since $Z_1' f \subseteqq (X_1 \cap X_1^*) f$ for every $Z_1' \in B_1'$, this gives

$$Y \cap Y^* = \cup \{Z_1' \cup Z_1' f, Z_1' \in B_1'\} \cup \text{ (union of members of } S_1);$$

so that $Y \cap Y^*$ is a union of members of B. In case one or both of Y, Y^* are members of S_1 and therefore not expressible in the form $X \cup Xf$, $Y \cap Y^*$ is trivially a union of sets in B. Hence the intersection of any two members of B is a union of members of B and therefore B is a base for a topology T on E. The rest of the theorem now follows directly.

For any topology T on a finite set E we can form the sequence $T_0 = T$, (S_1, T_1), (S_2, T_2), $\cdots$, (S_t, T_t), S_{t+1}, where S_k, T_k are respectively the nucleus and the orbital topology of T_{k-1} for $t \geqq k \geqq 1$ and S_{t+1} is a reduced base as well as the nucleus of T_t, so that $T_t = P_\cup(S_{t+1})$. By the above theorem there is a unique sequence of mappings $f_1, \cdots, f_t$ such that for

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$1 \leqq i \leqq t$, f_i maps B_i into $P_\cup(S_i)$, where B_i is a reduced base for T_i and is defined by

$$B_t = S_{t+1},\ B_i = \{X_{i+1} \cup X_{i+1}f_{i+1},\ X_{i+1} \in B_{i+1}\} \cup S_{i+1}, \quad \text{for } 0 \leqq i \leqq t.$$

By our theorem, every topology on E can be obtained as follows: Partition E into any number, say r, of disjoint and collectively exhaustive classes $E_1, \cdots, E_r$ and then partition, in an arbitrary way, the set $\{E_1, \cdots, E_r\}$ into disjoint and collectively exhaustive classes, say, $S_1, \cdots, S_{t+1}$. Let $f_1, \cdots, f_t$ be any mappings such that

(i) f_t maps $B_t = S_{t+1}$ into $P_\cup(S_t)$,

(ii) f_{t-i} maps B_{t-i} into $P_\cup(S_{t-i})$ where

$$B_{t-i} = \{X \cup Xf_{t-i+1},\ X \in B_{t-i+1}\} \cup S_{t-i+1},$$

(iii) each of the mappings $f_1, \cdots, f_t$ preserves the inclusion relation $\subseteqq$ for sets.

Then $B = B_0 = \{X_1 \cup X_1 f_1,\ X_1 \in B_1\} \cup S_1$ is a base for a topology on E and every topology on E is obtained in this way.

In view of this we can express the number $N(n)$ of topologies definable for a finite set of cardinal n as follows:

$$(1) \qquad N(n) = \sum_{r=1}^{n} \left[M_{n,r}\, r! \sum_{r_1+\cdots r_{t+1}=r} \left\{ [F_t(r_1, \cdots, r_{t+1})/r_1! \cdots r_{t+1}!] \right\} \right]$$

where $M_{n,r}$ is the number of ways a set of order n can be partitioned into r unordered classes and $F_t(r_1, \cdots, r_{t+1})$ is the number of sequences of mappings $f_1, \cdots, f_t$ described above when $S_1, \cdots, S_{t+1}$ have $r_1, \cdots, r_{t+1}$ members respectively. The summation in curly brackets extends over all finite sequences $r_1, \cdots, r_{t+1}$ of positive integers satisfying $r_1 + \cdots + r_{t+1} = r$.

The following recurrence relation holds for $M_{n,r}$:

$$(2) \qquad M_{n+1,r} = rM_{n,r} + M_{n,r-1}.$$

The function $F_t(r_1, \cdots, r_{t+1})$ has a simple combinatorial interpretation which we explain by taking $t = 3$ and by referring to the figure below.

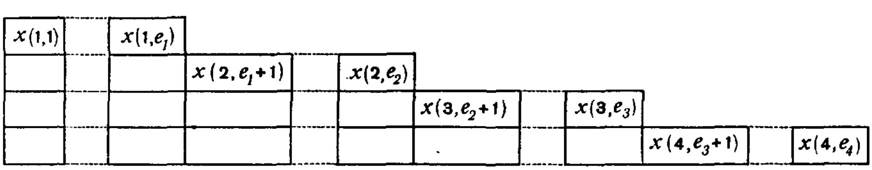

Figure 1

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In this figure we have taken $e_1 = r_4$, $e_2 = r_3+r_4$, $e_3 = r_2+r_3+r_4$, $e_4 = r_1+r_2+r_3+r_4$. Every one of the r_4 squares in the first row is given to be occupied with just one of the symbols $x(1, 1), \cdots, x(1, e_1)$ that are labels for sets in S_4. In the second row only the last r_3 squares on the right are given to be initially occupied, each by just one of the r_3 symbols $x(2, e_1+1), \cdots, x(2, e_2)$ that similarly stand for sets in S_3; and so on. Let us refer to the jth square from the left in the ith row from the top as $\sigma(i, j)$. In what follows we shall not explicitly mention the restrictions on the ranges of the variables $i, j, k, \cdots$. Write $\Sigma(i, j)=\{x(i, j)\}$ if $\sigma(i, j)$ is not initially empty. The combinatorial problem now is to place in every *empty* square $\sigma(i, j)$ a non-empty set $\Sigma(i, j)$ of symbols such that

(iv) $\Sigma(i, j) \subseteqq \{x(i, e_{i-1}+1), \cdots, x(i, e_i)\}$,

(v) $x(i, k) \in \Sigma(i, j)$ implies $\Sigma(i+1, k) \subseteqq \Sigma(i+1, j)$.

Thus, for example, the conditions (iv), (v) compel us to place in the empty squares of the third row in Fig. 1 symbols chosen from $x(3, e_2+1), \cdots, x(3, e_3)$, and if $x(3, e_3)$ has been placed in $\sigma(3, e_2)$ (the square immediately below the one containing $x(2, e_2)$) then $x(3, e_3)$ will have to occur in any set of symbols to be placed in a square of the third row which comes directly under a square containing $x(2, e_2)$. Let $Y(i, k) = \bigcup_{l=1}^{i} \Sigma(l, k)$. Then it is easily seen that if we let B_{4-i} be the set of all $Y(i, k)$ for fixed i and write $Y(i, k)f_{4-i} = \Sigma(i+1, k)$ then B_{4-i}, f_{4-i} satisfy (i), (ii), (iii) for $t = 3$.[1] It follows that $F_3(r_1, r_2, r_3, r_4)$ is the number of ways of placing the symbols $x(i, j)$ in the empty squares of Fig. 1 such that (iv) and (v) are satisfied.

We can use this interpretation of $F_t(r_1, \cdots, r_{t+1})$ to prove the following formulae.

(3) $F(r_1) = 1$,

(4) $F_1(r_1, r_2) = (2^{r_1}-1)^{r_2}$,

(5) $$F_2(r_1, 1, r_3) = \sum_{l=1}^{r_1} \binom{r_1}{l} 2^{(r_1-l)r_3},$$

(6) $$F_2(1, r_1, r_2) = \sum_{l=1}^{r_1} \sum_{m=1}^{r_2} 2^{r_2-m} \binom{r_1}{l} \binom{r_2}{m} (2^{r_2}-1)^{r_1-l}\{(2^m-1)^l - m(2^{m-1}-1)^l\},$$

(7) $$F_t(1, 1, \cdots, 1, r_{t+1}) = \sum_{j_1>0,\, j_1+\cdots+j_t \leq r_{t+1}} \binom{r_{t+1}}{j_1}\binom{r_{t+1}-j_1}{j_2} \cdots \binom{r_{t+1}-(j_1+\cdots+j_{t-1})}{j_t}.$$

[1] Strictly speaking, members of B_{4-i} must be taken as the unions $\cup\, Y(i, k)$ of all sets represented by the x's in $Y(i, k)$, but since x's represent disjoint sets this will not effect our conclusion about $F_3(r_1, \cdots, r_4)$.

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As an illustration we prove (5). We have to consider the number of ways some of the $x(i, j)$ can be placed in the empty squares in fig. 2 below such that (iv), (v) are satisfied.

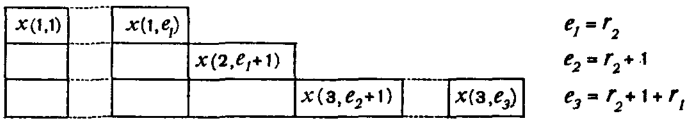

Figure 2

In every empty square of the second row of this figure we must put just $x(2, e_1+1)$. In the square $\sigma(3, e_1+1)$ under $x(2, e_1+1)$ we can place any subset $\Sigma(3, e_2+1)$ of $\{x(3, e_2+1), \cdots, x(3, e_3)\}$. In the remaining empty squares of the third row we must put every symbol in $\Sigma(3, e_2+1)$ in addition to some other symbols arbitrarily selected from

$$\{x(3, e_2+1), \cdots, (3\ e_3)\} - \Sigma(3, e_2+1).$$

The formula (5) is now obvious.

We have employed formulae (1)—(7) in calculating $N(n)$ for $n = 3, 4, 5, 6$.

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