COMBINATORIAL PROPERTIES OF ASSOCIATED ZONOTOPES

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1. Introduction. Let $S_1, \ldots, S_r$ be r line segments, each of non-zero length, in n-dimensional euclidean space $\mathbf{R}^n$. If a polytope Z is defined as the vector (Minkowski) sum

$$(1) \quad Z = S_1 + \ldots + S_r,$$

then the segments S_i will be called the *components* of Z. Since we do not wish to exclude the possibility that some of the components may be parallel, the polytope Z may be written in the form (1) in many different ways. For this reason it is convenient to define a *zonotope* to be the polytope Z together with some specified set of components $\{S_1, \ldots, S_r\}$. Figures 1, 2 and 3 show some zonotopes of 1, 2 and 3 dimensions with 4, 5 and 6 components.

We may assume, without loss of generality, that the origin o is the centre of each line segment S_i, and therefore also of Z. If

$$S_i = \mathrm{conv}\{-x_i, x_i\}, \qquad i = 1, \ldots, r,$$

for some set $X = \{x_1, \ldots, x_r\}$, then we write $P(Z)$ for the set of 2^r points

$$(2) \quad \pm x_1 \pm x_2 \pm \ldots \pm x_r$$

and Z may also be defined as $\mathrm{conv}\, P(Z)$. Yet another definition of Z is the following

$$Z = \{x \in \mathbf{R}^n | x = \lambda_1 x_1 + \ldots + \lambda_r x_r; -1 \leqq \lambda_i \leqq 1; i = 1, \ldots, r\}.$$

The elementary properties of zonotopes, such as the equivalence of the above definitions, are easily established, see [2; 3; 6], and the reader is referred to these publications for further information. In [2], Coxeter deals mainly with 3-dimensional zonotopes or 'zonohedra', and in [3] he introduces projective diagrams for a zonotope, an idea we shall make use of later. In [6] McMullen introduces the concept of an 'associated zonotope' which is fundamental in this paper. For convex polytopes in general we shall, for the most part, follow the notations and terminology of [4] and [7].
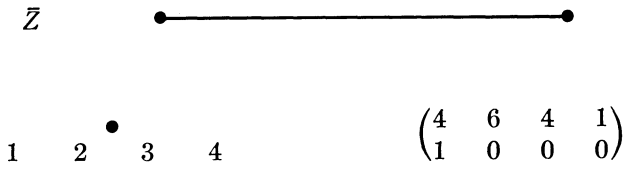
From now on it is convenient to assume that the set X linearly spans $\mathbf{R}^n$ so that Z is an n-polytope. If X is a linear basis of $\mathbf{R}^n$ (card $X = n$) then Z is called a *cube*. If all the components, with exactly one exception S_k, lie in an
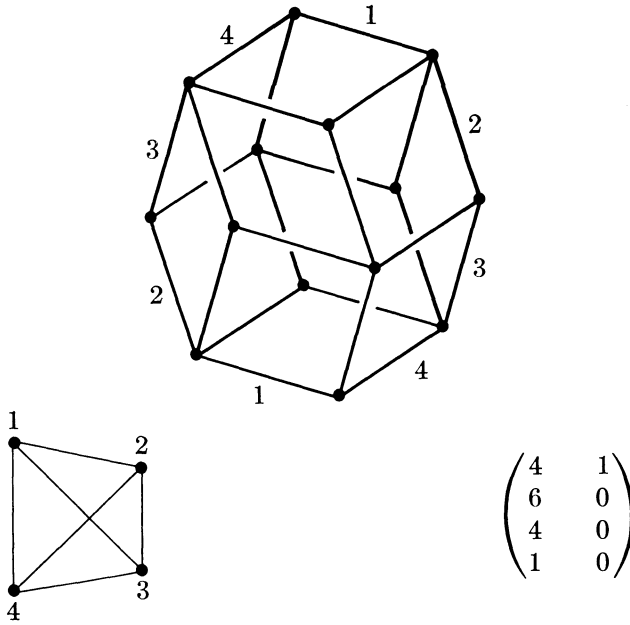
$$\begin{pmatrix} 4 & 1 \\ 6 & 0 \\ 4 & 0 \\ 1 & 0 \end{pmatrix}$$

$$v(Z) = 14 \qquad b(Z) = 0 \qquad i(Z) = 2$$

$$\begin{pmatrix} 4 & 6 & 4 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$v(\bar{Z}) = 2 \qquad b(\bar{Z}) = 0 \qquad i(\bar{Z}) = 14$$

FIGURE 1. $(n = 3, r = 4)$

$(n - 1)$-dimensional linear subspace of $\mathbf{R}^n$, then Z is called a *prism*, and S_k is called an *upright* of Z. A *cubical zonotope* is one whose proper faces are all cubes. (The zonotopes of Figure 1 are cubical; those of Figures 2 and 3 are not.)

Other special types of zonotope mentioned in the literature are 'equilateral zonotopes' and 'polar zonotopes', see [**2**, p. 29]. We shall not discuss these here since their definitions and special properties are essentially metrical, whereas here we are primarily concerned with combinatorial properties. Some photographs of beautiful models of equilateral zonotopes appear in [**2**, Plate II]. These have been constructed in such a way as to maximize the orders of their
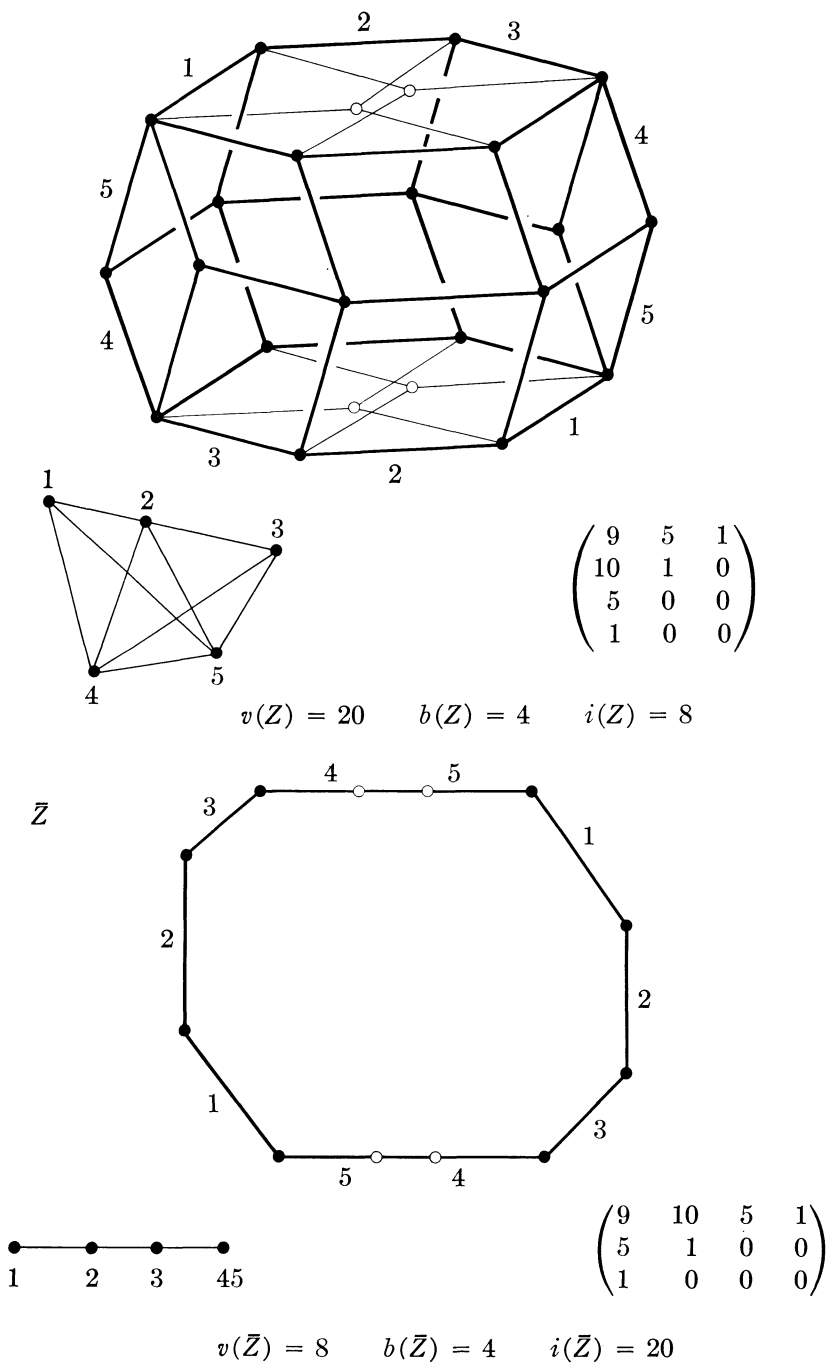
$$\begin{pmatrix} 9 & 5 & 1 \\ 10 & 1 & 0 \\ 5 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$v(Z) = 20 \qquad b(Z) = 4 \qquad i(Z) = 8$$

$\bar{Z}$

$$\begin{pmatrix} 9 & 10 & 5 & 1 \\ 5 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$v(\bar{Z}) = 8 \qquad b(\bar{Z}) = 4 \qquad i(\bar{Z}) = 20$$

FIGURE 2. $(n = 3, r = 5)$

$$\begin{pmatrix} 15 & 14 & 6 & 1 \\ 15 & 5 & 1 & 0 \\ 6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$v(Z) = 24 \qquad b(Z) = 20 \qquad i(\bar{Z}) = 20$$

$$\begin{pmatrix} 15 & 15 & 6 & 1 \\ 14 & 5 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

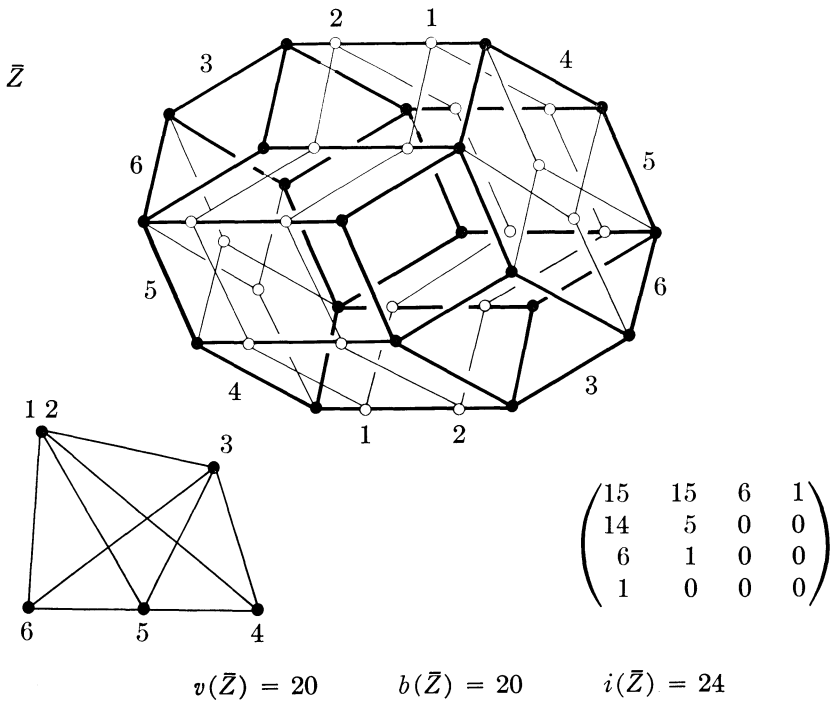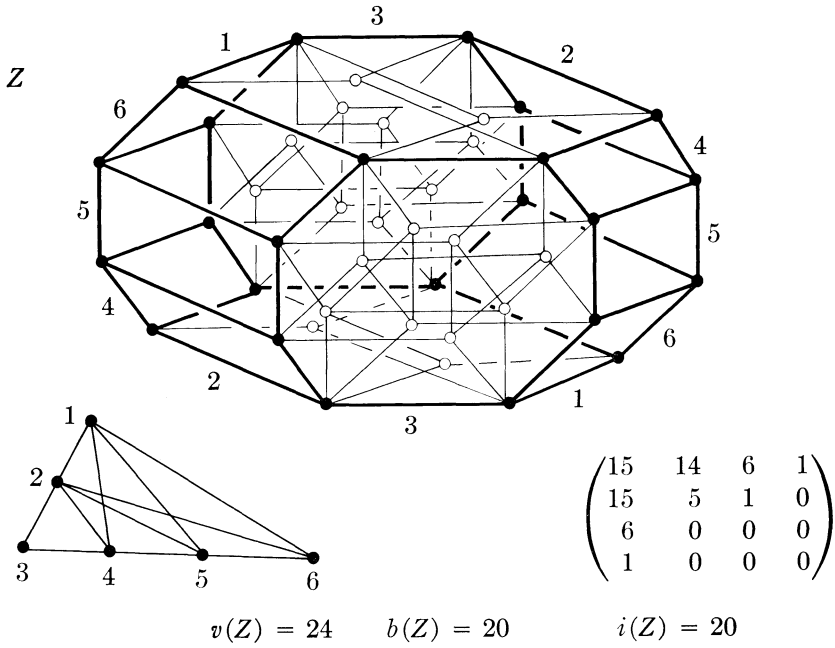$$v(\bar{Z}) = 20 \qquad b(\bar{Z}) = 20 \qquad i(\bar{Z}) = 24$$

FIGURE 3. $(n = 3, r = 6)$

symmetry groups, and consequently, except in one case, the points $P(Z)$ are not all distinct. Here we shall, on the contrary, always assume that $P(Z)$ consists of 2^r *distinct* points. The reason for this assumption is purely technical. If we did not make it then it would be necessary to label each point with a 'multiplicity' according to the number of different ways in which it could be expressed in the form (2), and this would lead to trivial, but tiresome, complications.

For any given Z the set $P(Z)$ may be partitioned into three subsets. First we have the *vertices* of Z, and the number of these will be denoted by $v(Z)$. Secondly we have the *interior points* of Z, and the number of these will be denoted by $i(Z)$. Finally we have those points which lie in the boundary of Z, but are not vertices of Z. These will be called *boundary points*, and their number will be denoted by $b(Z)$. Thus

$$(3) \quad v(Z) + b(Z) + i(Z) = 2^r.$$

In Figures 2 and 3, boundary points are marked as small open circles.

It is easy to show that every face F of Z can be written in the form

$$(4) \quad S_{\sigma(1)} + \ldots + S_{\sigma(s)} + \epsilon_{\sigma(s+1)} x_{\sigma(s+1)} + \ldots + \epsilon_{\sigma(r)} x_{\sigma(r)}$$

where $0 \leqq s \leqq r$, σ is a permutation of $(1, \ldots, r)$, and each $\epsilon_i = \pm 1$. In fact (see [**6**, § 2]), if u is the outward normal to a hyperplane H which supports Z in F, then

$$(5) \quad \langle u, x_{\sigma(i)} \rangle = 0 \quad \text{for } i = 1, \ldots, s,$$

$$\langle u, \epsilon_{\sigma(i)} x_{\sigma(i)} \rangle = \mu_{\sigma(i)} > 0 \quad \text{for } i = s+1, \ldots, r,$$

and the equation of H is

$$\langle u, x \rangle = \epsilon_{\sigma(s+1)} \mu_{\sigma(s+1)} + \ldots + \epsilon_{\sigma(r)} \mu_{\sigma(r)}.$$

Thus every face of a zonotope is itself a zonotope, and, in particular, the 2-faces are centrally symmetric polygons. The converse statement is also true; if Z is any polytope with centrally symmetric 2-faces, then Z must be a zonotope [**8**].

Every set which can be written in the form (4) will be called a *cell C* of Z, and $S_{\sigma(1)}, \ldots, S_{\sigma(s)}$ will be called the *components* of C. Let us denote by $[C]$ the *family* of 2^{r-s} cells

$$(6) \quad S_{\sigma(1)} + \ldots + S_{\sigma(s)} \pm x_{\sigma(s+1)} \pm \ldots \pm x_{\sigma(r)}.$$

There is just one family of 0-dimensional cells, namely $P(X)$, and the family $[Z]$ contains only one cell. There are 2^r families of cells in all. Every face F of Z is a cell of Z, and, unless $F = Z$, the family $[F]$ always contains at least one face of Z other than F, namely the reflection of F in the origin. On the other hand, not every family of cells contains faces of Z. The following is easily established.

(7)   *The family* (6) *of cells of Z contains* (*proper*) *faces of Z if and only if
there exists a* (*proper*) *linear subspace L of R^n such that*

$$L \cap X = \{x_{\sigma(1)}, \ldots, x_{\sigma(s)}\}.$$

In the case of a cubical zonotope, every family of cells of Z of dimension
strictly less than n contains faces of Z, and every cell of Z of dimension
less than n is a cube.

Coxeter's *first projective diagram* for Z [**3**, p. 141] consists of the set of points
X, regarded as lying in a projective space P_{n-1}, together with all the projective
subspaces of P_{n-1} that are spanned by subsets of X. It is thus a projective
configuration and may be denoted by $D(X)$. In Figures 1, 2 and 3 we reproduce
the configurations corresponding to each of the six zonotopes illustrated.
Clearly there is a one-to-one correspondence between the subsets of X and
the families of cells of Z, and also (on account of (7)) between the (proper)
projective subspaces of $D(X)$ and the families of (proper) faces of Z of dimen-
sion greater than zero.

It is a simple consequence of Euler's Theorem [**2**, p. 9; **4**, p. 130] that every
3-polytope must have 2-faces which are k-gons for $k < 6$ [**4**, p. 254] and there-
fore every 3-dimensional zonotope must have 2-faces which are parallelo-
grams. This implies that every projective configuration in the plane must
contain an 'ordinary' line, that is to say, one which contains exactly two points
of the configuration. As Coxeter points out [**3**, p. 142] this provides a proof,
in fact a very elegant proof, of Sylvester's Theorem. First Motzkin, and later
Hansen [**5**] extended Sylvester's Theorem to projective spaces of $m \geqq 3$
dimensions, and proved that every projective configuration must contain an
'ordinary' hyperplane, that is to say, a subspace of $m - 1$ dimensions in which
the points of the configuration, with exactly one exception, lie in a subspace
of $m - 2$ dimensions. In terms of zonotopes, the interpretation of this asser-
tion is the following.

(8)   *At least two facets of an n-dimensional zonotope are* $(n - 1)$-*dimensional
prisms.*

The name 'zonotope' arises from the fact that the faces of Z lie in 'zones'.
The kth *zone* of Z is defined to be the set of all faces of which S_k is a component.
The combinatorial properties which we shall consider in this paper concern
the relationships between the cell structure of Z, its faces, and its zones. We
begin, in § 2, defining an 'associated zonotope' $\bar{Z}$ whose properties are, in a
remarkable way, complementary to those of Z.

2. Associated zonotopes. Let $X = \{x_1, \ldots, x_r\} \subset \mathbf{R}^n$ be defined as before,
and let $\bar{X} = \{\bar{x}_1, \ldots, \bar{x}_r\} \subset \mathbf{R}^{r-n}$ be a linear representation of X. For the
definition of a linear representation and its properties see [**9**] or [**6**]. Here we
repeat these briefly in order to make the exposition self-contained. Let us

construct an $r \times r$ matrix

$$(9) \quad M(X) = \begin{bmatrix} x_1 & \bar{x}_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ x_r & \bar{x}_r \end{bmatrix}$$

where each x_i represents a $1 \times n$ block of elements, namely the coordinates of x_i relative to some given coordinate system in $\mathbf{R}^n$, and each $\bar{x}_i$ represents a $1 \times (r - n)$ block of elements. $M(X)$ must be constructed in such a way that it is non-singular and also so that each of its first n columns is orthogonal to each of its last $r - n$ columns. This can always be done by taking the last $r - n$ columns to be a basis of the $(r - n)$-dimensional linear space of linear dependences of the set X [6, § 3]. The set $\bar{X} = \{\bar{x}_1, \ldots, \bar{x}_r\} \subset \mathbf{R}^{r-n}$ is called a *linear representation* of X. It should be noticed that

(a) $\bar{X}$ is only determined within a non-singular linear transformation of $\mathbf{R}^{r-n}$,

(b) $\bar{X}$ spans $\mathbf{R}^{r-n}$ linearly,

(c) there is a canonical one-to-one correspondence between X and $\bar{X}$, corresponding elements having the same subscript,

(d) the relationship between X and $\bar{X}$ is symmetrical, and

(e) if no $r - 1$ of the points of X lie in an $(n - 1)$-dimensional linear subspace of $\mathbf{R}^n$, then no vector of $\bar{X}$ is the zero vector.

Let us now define

$$\bar{Z} = \bar{S}_1 + \ldots + \bar{S}_r \subset \mathbf{R}^{r-n}$$

where

$$\bar{S}_i = \operatorname{conv}\{-\bar{x}_i, \bar{x}_i\}, \qquad i = 1, \ldots, r.$$

If Z is not a prism then property (e) shows that $\bar{Z}$ is a properly defined $(r - n)$-dimensional zonotope with components $\bar{S}_1, \ldots, \bar{S}_r$. We say that $\bar{Z}$ is *associated* with Z, and note that, because of property (d), Z is also associated with $\bar{Z}$. Property (a) shows that the construction does not define $\bar{Z}$ uniquely, but that its combinatorial type is determined. If Z is cubical so is $\bar{Z}$. Further, two components $\bar{S}_i$ and $\bar{S}_j$ of $\bar{Z}$ are parallel if and only if there exists an $(n - 1)$-dimensional linear subspace of $\mathbf{R}^n$ containing the set $X \backslash \{x_i, x_j\}$. Figures 1, 2 and 3 show three pairs of associated zonotopes.

Although the above algebraic definition of associated zonotopes is convenient for most purposes, the following geometrical interpretation, which is due to P. McMullen, is of interest. In $\mathbf{R}^r$ take any regular r-dimensional cube W and any pair of complementary orthogonal subspaces, L_n and L_{r-n}. Then the image of W under orthogonal projection on to L_n, and the image of W under orthogonal projection onto L_{r-n}, are associated zonotopes. In fact the cells of these zonotopes are the images, under the parallel projections, of the faces of W.

H. S. M. Coxeter has pointed out the following elegant example of this geometrical interpretation in the case $r = 4$, $n = 2$. For a suitable pair of complementary two-dimensional subspaces, the images Z, $\bar{Z}$ of a four-dimensional cube W are octagons. Such an octagon Z is illustrated in Figure 4, where the cells of Z are also indicated. In this diagram two circuits of one-dimensional cells are indicated by thicker lines, namely the boundary of Z and an octagram
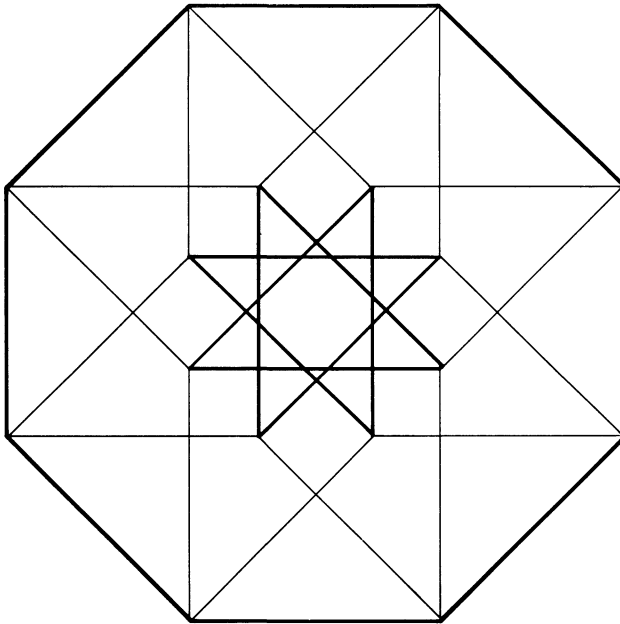

FIGURE 4

lying in the interior of Z. The inverse images of these circuits are edge-circuits of W, each containing 8 edges. They are *complementary Petrie polygons*. By 'complementary' we mean that each vertex of W belongs to precisely one of the edge-circuits, and by 'Petrie polygon' we mean that no s consecutive edges belong to the same $(s - 1)$-face of W ($1 \leqq s \leqq 4$), [**2**, pp. 223, 244]. Further, in the associated octagon $\bar{Z}$, the roles of these two Petrie polygons are interchanged; the one that projects into the boundary of Z projects into the octagram in the interior of $\bar{Z}$, and vice versa.

We now describe some of the geometrical relationships between Z and $\bar{Z}$. A very brief account of these appears in [**6**, § 5]. Throughout this section we shall assume that every zonotope under consideration is *not* a prism so that an associated zonotope is properly defined.

Two one-to-one correspondences turn out to be of importance. The *first correspondence* is between cells of Z and cells of $\bar{Z}$, mapping

$$(10) \quad C = S_{\sigma(1)} + \ldots + S_{\sigma(s)} + \epsilon_{\sigma(s+1)} x_{\sigma(s+1)} + \ldots + \epsilon_{\sigma(r)} x_{\sigma(r)}$$

on to

(11) $\bar{C} = \bar{S}_{\sigma(1)} + \ldots + \bar{S}_{\sigma(s)} + \epsilon_{\sigma(s+1)}\bar{x}_{\sigma(s+1)} + \ldots + \epsilon_{\sigma(r)}\bar{x}_{\sigma(r)}.$

In particular it maps each point

(12) $q = \epsilon_1 x_1 + \ldots + \epsilon_r x_r$

of $P(Z)$ on to the point

(13) $\bar{q} = \epsilon_1 \bar{x}_1 + \ldots + \epsilon_r \bar{x}_r$

of $P(\bar{Z})$. The *second correspondence* is between families of cells of Z and families of cells of $\bar{Z}$. With C defined as in (10), it maps $[C]$ on to $[\tilde{C}]$, where

(14) $\tilde{C} = \bar{x}_{\sigma(1)} + \ldots + \bar{x}_{\sigma(s)} + \bar{S}_{\sigma(s+1)} + \ldots + \bar{S}_{\sigma(r)}.$

Thus it maps the family $P(Z)$ on to $[\bar{Z}]$, and $P[\bar{Z}]$ on to $[Z]$. From now on we shall use the notations $\bar{C}$ and $\tilde{C}$ in the above senses without any further explanation.

(15)   THEOREM. *If the point q defined in* (12) *is a vertex, boundary point, or interior point of Z, then the point $\bar{q}$, defined in* (13), *is an interior point, boundary point, or vertex of $\bar{Z}$, respectively.*

In the case of cubical zonotopes (which have no boundary points), this theorem is due to McMullen [**6**]. The proof given here is essentially the same as in [**6**]. This theorem shows that our *first correspondence* interchanges vertices and interior points between Z and $\bar{Z}$, and so may be thought of as 'turning Z inside out' to obtain $\bar{Z}$.

For simplicity of notation let us write

(16) $p = x_1 + \ldots + x_r$

and, to begin with, assume that p is an interior point of Z. Then there is no supporting hyperplane of Z containing p, and so, by (5), there exists no vector $u \in \mathbf{R}^n$ such that $\langle u, x_i \rangle \geqq 0$ for $1 \leqq i \leqq r$. In other words the set X does not lie in any closed half-space of $\mathbf{R}^n$ bounded by a hyperplane through the origin, and this, in turn, implies that

(17) $o \in \operatorname{int} \operatorname{conv} X$

Since the algebra is reversible we have established the following.

(18)   *Condition* (17) *is necessary and sufficient for p to be an interior point of Z.*

On the other hand, suppose that p is a relatively interior point of some proper face F of Z. Again, for simplicity of notation write

(19) $F = S_1 + \ldots + S_s + x_{s+1} + \ldots + x_r.$

Then p is a vertex of Z if $s = 0$, and a boundary point of Z if $s \geqq 1$. Let u be

the outward normal to a hyperplane that supports Z in F so that, by (5),

$$\langle u, x_i \rangle = 0 \quad \text{for } i = 1, \ldots, s$$
$$\langle u, x_i \rangle = \mu_i > 0 \quad \text{for } i = s + 1, \ldots, r.$$

If we multiply the first n columns of $M(X)$, defined in (9), by the n components of u, respectively, and add, we obtain a vector

$$(0, 0, \ldots, 0, \mu_{s+1}, \ldots, \mu_r)^T$$

which is a linear dependence of $\bar{X}$. Thus

$$\mu_{s+1}\bar{x}_{s+1} + \ldots + \mu_r\bar{x}_r = o,$$

and since each $\mu_i > 0$ we see that

(20) $o \in \text{rel int conv}\{\bar{x}_{s+1}, \ldots, \bar{x}_r\}.$

(In fact, see [6], (20) is a necessary and sufficient condition for F, defined by (19), to be a face of Z.) If $s \geqq 1$, then $p \in \text{rel int } F$ implies that

$$x_1 + \ldots + x_s \in \text{rel int } (S_1 + \ldots + S_s),$$

and so, by (18),

(21) $o \in \text{rel int conv}\{x_1, \ldots, x_s\}.$

As the algebra is reversible we obtain necessary and sufficient conditions, and we have established the following general statement.

(22) *A point $q = \epsilon_1 x_1 + \ldots + \epsilon_r x_r$ $(\epsilon_i = \pm 1)$ is a boundary point of Z if and only if*

(23) $o \in \text{rel int conv}\{\epsilon_{\sigma(1)}x_{\sigma(1)}, \ldots, \epsilon_{\sigma(s)}x_{\sigma(s)}\}$ *and*
     $o \in \text{rel int conv}\{\epsilon_{\sigma(s+1)}\bar{x}_{\sigma(s+1)}, \ldots, \epsilon_{\sigma(r)}\bar{x}_{\sigma(r)}\}$

for some $1 \leqq s \leqq r - 1$ and some permutation σ of $(1, \ldots, r)$. If $s = 0$ or r then one of the two conditions (23) is vacuous and the other condition is necessary and sufficient for q to be a vertex, or interior point, of Z, respectively.

Clearly this statement implies Theorem (15) which is thus proved. The following consequence is immediate.

(24) *If Z and $\bar{Z}$ are associated zonotopes, then*
     $v(Z) = i(\bar{Z}), \qquad b(Z) = b(\bar{Z}), \qquad i(Z) = v(\bar{Z}).$

In Figures 1, 2 and 3, the values of these quantities are indicated below the diagrams.

Statement (22) implies a number of other geometrical relationships between associated zonotopes, such as the following.

(25) *If a boundary point q of $P(Z)$ lies on the ith zone of Z, then the corresponding boundary point $\bar{q}$ does not lie on the ith zone of $\bar{Z}$.*

By 'q lies on the ith zone of Z' we mean that q lies in the relative interior of some proper face of Z of which S_i is a component.

(26)   *If $b_i(Z)$ boundary points lie on the ith zone of Z, then $b(Z) - b_i(Z)$ boundary points lie on the ith zone of $\bar{Z}$.*

Theorem (15) can be generalized to cells of dimension greater than zero. Extending the notation introduced earlier, for a cell C defined by (10), write $P(C)$ for the set of 2^s points

$$(27) \quad \{\pm x_{\sigma(1)} \pm \ldots \pm x_{\sigma(s)} + \epsilon_{\sigma(s+1)} x_{\sigma(s+1)} + \ldots + \epsilon_{\sigma(r)} x_{\sigma(r)}\}.$$

Then the cells of Z may be partitioned into three subsets in the following way. The cell C will be called a *vertex-cell* if $P(C)$ contains vertices, but no interior points of Z. If C is a cube then it is a vertex-cell if and only if it is a face of Z. The cell C will be called an *interior-cell* if $P(C)$ contains interior points, but no vertices, of Z. The third class consists of all the remaining cells; those for which $P(C)$ contains both vertices and interior points, or neither. These will be called *boundary-cells*. The terminology is not very appropriate but it is introduced temporarily so that the statement of the following theorem is closely analogous to that of (15).

(27)   THEOREM. *If a cell C is a vertex-cell, boundary-cell or interior-cell of Z, then $\bar{C}$ is an interior-cell, boundary-cell or vertex-cell of $\bar{Z}$, respectively.*

The proof is, of course, immediate from (15). Theorems (15) and (17) enable us to derive numerical data about zonotopes in a very simple way. We give only one example, since more general results will be given later.

(28)   *Let Z be an n-dimensional zonotope with $n + 1$ components, which is not a prism. If we denote by f_k the number of k-faces of Z, then*

$$f_k = \binom{n+1}{k}(2^{n+1-k} - 2) \quad (0 \leqslant k \leqslant n - 1).$$

We note that $\bar{Z}$ is one-dimensional and so has 2 vertices and $2^{n+1} - 2$ interior points. By Theorem (15) Z has $2^{n+1} - 2$ vertices and therefore the above expression holds for $k = 0$. For $k \geq 1$ we see that $\bar{Z}$ has $\binom{n+1}{k} 2^{n+1-k}$ cells with k components, of which $2\binom{n+1}{k}$ contain a vertex of $\bar{Z}$. From these facts, Theorem (27) immediately yields the stated value of f_k.

One problem of particular interest is that of determining the number of faces of Z that belong to any given family of cells. Our first result in this direction is the following.

(29)   *If a face $F \in [C]$ contains b_F boundary points of Z, then the family $[\tilde{C}]$ (related to $[C]$ by the second correspondence) contains b_F faces of $\bar{Z}$.*

Thus in Figure 3, the face $x_1 + x_2 + S_3 + S_4 + S_5 + S_6$ contains eight boundary points and so the family $\bar{S}_1 + \bar{S}_2 \pm \bar{x}_3 \pm \bar{x}_4 \pm \bar{x}_5 \pm \bar{x}_6$ contains eight faces (edges) of $\bar{Z}$. In particular (29) implies that if C is a cube then $[\tilde{C}]$ contains no proper faces of Z. To prove (29) we need the following lemma, which is of independent interest.

(30)   *Let Z be defined by* (1) *and Z_1 by*

$$Z_1 = S_{\sigma(1)} + \ldots + S_{\sigma(s)}$$

where $0 \leqq s \leqq r$ and σ is a permutation of $(1, \ldots, r)$. Then the image of $\bar{Z}$, under projection in the direction $\lin\{\bar{x}_{\sigma(s+1)}, \ldots, \bar{x}_{\sigma(r)}\}$ on to a complementary subspace, is a zonotope $\bar{Z}_1$, which is associated with Z_1.

The cases $s = 0, r$ are trivial, so in the proof we assume that $1 \leqq s \leqq r - 1$. We simplify the notation by taking σ to be the identity permutation. Let $Z_1 = S_1 + \ldots + S_s$ be t-dimensional $(1 \leqq t \leqq s)$. We construct the matrix $M(X)$ of (9) by writing down the first n columns, which are determined by the points x_i, and then the next $s - t$ columns are chosen to be a basis of the linear space of linear dependences of $\{x_1, \ldots, x_s\}$. Finally we complete $M(X)$ by adjoining a further $(r - n) - (s - t)$ columns in the usual way. Hence we see that the elements m_{ij} of $M(X)$ are zero for $s + 1 \leqq i \leqq r$, $n + 1 \leqq j \leqq n + s - t$ and that we can obtain a linear representation of $\{x_1, \ldots, x_s\}$ by deleting the last $r - n - s + t$ coordinates of each of $\bar{x}_1, \ldots, \bar{x}_s$. Because the first $s - t$ coordinates of each of $\bar{x}_{s+1}, \ldots, \bar{x}_r$ are zero, this is equivalent to projecting $\bar{x}_1, \ldots, \bar{x}_s$ in the direction $\lin\{\bar{x}_{s+1}, \ldots, \bar{x}_r\}$ on to complementary $(s - t)$-dimensional subspace. Hence the associated zonotope $\bar{Z}_1$ is obtained from $\bar{Z}$ in the manner stated. In addition we also deduce the following which will be required in the next section.

(31)   *If $t = \dim C$ and C has s components, then*

$$\dim \tilde{C} = r - n - s + t.$$

Using (30) it is easy to prove (29). Continuing with the same notation we may suppose that C is a translate of Z_1. Then the number of faces of Z in the family $[\tilde{C}]$ is equal to the number of vertices of the zonotope which arises by projecting $\bar{Z}$ in the direction $\lin\{\bar{x}_{\sigma(s+1)}, \ldots, \bar{x}_{\sigma(r)}\}$ on to a complementary subspace, that is to say, to the number of vertices of $\bar{Z}_1$. But by (15) this is equal to the number of relatively interior points of Z_1, and this is equal to b_F. Thus (29) is proved.

3. Deficiency and excess. For each cell C of Z we define two constants, as follows. The *deficiency* $d(C)$ of C is defined by

(32) $d(C) = n - \dim C$

and the *excess* $e(C)$ of C is defined by

(33) $e(C) = s - \dim C$

where s is the number of components of C. Both $d(C)$ and $e(C)$ are measures of the 'dependence' of the components of C. Thus $e(C) = 0$ if and only if C is a cube, and

$$e(Z) = \dim \bar{Z}, \qquad e(\bar{Z}) = \dim Z, \qquad d(Z) = d(\bar{Z}) = 0.$$

Since every cell in a given family $[C]$ has the same deficiency and excess we define $d[C]$ and $e[C]$ in the obvious manner.

(34)   THEOREM. *For any cell C of Z*

$$d(C) = e(\tilde{C}), \qquad e(C) = d(\tilde{C}).$$

Using (31) we see that

$$d(C) = n - \dim C = (r - s) - \dim \tilde{C} = e(\tilde{C}),$$

and

$$e(C) = s - \dim C = (n - r) - \dim \tilde{C} = d(\tilde{C}).$$

(In fact, of course, each equality follows from the other because of the symmetry between Z and $\bar{Z}$.)

Statement (34), along with (30), shows that if C is a translate of Z_1, then $e(C) = \dim \bar{Z}_1$. Also, if C is a cube $d(\tilde{C}) = 0$, so $\tilde{C}$ is $(r - n)$-dimensional and therefore $[\tilde{C}]$ contains no proper faces of $\bar{Z}$, thus confirming the observation made after statement (29).

(35)   *For any zonotope Z,*

$$\sum (-1)^{d[C]+e[C]} = 0$$

where the summation is over the 2^r families of cells $[C]$ of Z.

This is an Euler-type relation for the cell-families of Z. To prove it we note that the definitions imply

(36) $e(C) - d(C) = (s - \dim C) - (n - \dim C) = s - n.$

But $s - n$ is even for exactly half of the cell families and is odd for the other half. Hence

$$0 = \sum (-1)^{s-n} = \sum (-1)^{e[C]-d[C]} = \sum (-1)^{d[C]+e[C]}.$$

(37)   THEOREM. *For any zonotope Z*

$$v(Z) = \sum (-1)^{e[C]}, \qquad b(Z) = 2\sum (1 - (-1)^{d[C]\,e[C]}),$$
$$i(Z) = \sum (-1)^{d[C]},$$

where, in each case, summation is over the 2^r cell-families $[C]$ of Z.

To prove the first equality we construct a set of r hyperplanes

(38) $\mathscr{H} = \{H_1, \ldots, H_r\}$

through the origin $o \in \mathbf{R}^n$ with

$$H_i = \{x \in \mathbf{R}^n | \langle x, x_i \rangle = 0\}.$$

The number of vertices of Z is then equal to the number of n-dimensional cones into which $\mathbf{R}^n$ is dissected by $\mathscr{H}$. This number has been shown by Winder [**10**] to be $N_e - N_o$ where N_e is the number of *even-degenerate* subsets of $\mathscr{H}$, that is to say, the number of distinct subsets $\mathscr{H}_1 \subseteq \mathscr{H}$ for which

$$\dim(\cap\ \mathscr{H}_1) - n - \operatorname{card}\ \mathscr{H}_1$$

is *even*, and N_o is the number of *odd-degenerate* subsets of $\mathscr{H}$, defined in a similar manner. Now if C is defined as in (4) and $\mathscr{H}_1 = \{H_{\sigma(1)}, \ldots, H_{\sigma(s)}\}$, then by (33),

$$e(C) = \operatorname{card} \mathscr{H}_1 - (n - \dim(\cap \mathscr{H}_1)).$$

Thus N_e and N_o are the numbers of families of cells of Z for which $e(C)$ is even and odd respectively. Thus

$$v(Z) = N_e - N_o = \sum (-1)^{e[C]}$$

as stated.

If Z is not a prism then $i(Z) = v(\bar{Z})$ by (15) and the first part of the theorem shows that $v(Z) = \sum(-1)^{e[\bar{C}]} = \sum(-1)^{d[C]}$ by (34). On the other hand, if Z is a prism with upright S_1, then for each cell C of Z which does *not* contain the component S_1 there exists a cell $C + S_1$ which contains S_1. The corresponding terms in the sum $\sum(-1)^{d[C]}$ cancel out, so that $\sum(-1)^{d[C]} = 0$. Clearly $i(Z) = 0$, and so the third assertion of the theorem is true in this case also.

For the second part of the theorem we temporarily introduce the notation $c(o, o)$ for the number of families of cells for which $d[C]$ and $e[C]$ are odd, $c(e, o)$ for the number of families for which $d[C]$ is even and $e[C]$ is odd, and so on. Then

$$v(Z) = \sum(-1)^{e[C]} = c(e, e) + c(o, e) - c(e, o) - c(o, o)$$
$$i(Z) = \sum(-1)^{d[C]} = c(e, e) - c(o, e) + c(e, o) - c(o, o)$$

and (35) implies

$$c(o, o) + c(e, e) = c(e, o) + c(o, e) = 2^{r-1}.$$

Thus, using (3),

$$b(Z) = 2^r - v(Z) - i(Z) = 4c(o, o) = 2\sum(1 - (-1)^{d[C]e[C]})$$

as required, and the proof of the theorem is completed.

Theorem (37) gives an expression for the number of vertices of Z. This can be easily extended to give an expression for the number of faces of Z in any

given family of cells.

(39) *Let*

$$C_1 = S_{\sigma(1)} + \ldots + S_{\sigma(s)} + x_{\sigma(s+1)} + \ldots + x_{\sigma(r)}$$

be a cell of the zonotope Z. Then the family $[C_1]$ *contains*

$$(-1)^{e(C_1)} \sum (-1)^{e[C]}$$

faces of Z, where the summation is over all those families of cells $[C]$ *of Z that have* $S_{\sigma(1)}, \ldots, S_{\sigma(s)}$ *as components.*

The proof follows the same lines as that of (29) except that we use (37) to determine the number of vertices of $\bar{Z}_1$.

4. The deficiency-excess matrix. The results of the previous section can be conveniently stated in terms of a matrix $A(Z)$ which we shall call the *deficiency-excess matrix* (*d-e* matrix) of Z. The (i, j)th term a_{ij} of $A(Z)$ is defined to be the number of families of cells $[C]$ of Z for which $d[C] = i$ and $e[C] = j$. Since $0 \leq d[C] \leq n$ and $0 \leq e[C] \leq r - n$, we see that all the non-zero terms of $A(Z)$ lie in the leading $(n + 1) \times (r - n + 1)$ submatrix. While it is often convenient to think of $A(Z)$ as consisting of this block of elements, we shall tacitly assume that rows and columns of zeros are to be adjoined when this is necessary to make the matrices compatible with the operations of matrix algebra. In Figures 1, 2 and 3, the *d-e* matrices of the six zonotopes are given.

Statements (34) and (35) yield the identities

(40) $A(Z) = (A(\bar{Z}))^T$

and

(41) $\displaystyle\sum_{i,j} (-1)^{i+j} a_{ij} = 0.$

From (37) we obtain

(42) $v(Z) = \displaystyle\sum_{i,j} (-1)^j a_{ij}, \quad i(Z) = \sum_{i,j} (-1)^i a_{ij}, \quad b(Z) = 4 \sum a_{ij}$

where, in this latter case only, summation is over those terms for which both i and j are odd. Also from (36)

(43) $\displaystyle\sum_{j-i=p} a_{ij} = \binom{r}{p+n}.$

Our next result enables us to construct $A(Z)$ inductively from the *d-e* matrices of zonotopes with fewer components. As usual Z is taken in the form (1) and we write, for $k = 1, \ldots, r,$

(44) $Z(k) = S_1 + \ldots + S_{k-1} + S_{k+1} + \ldots + S_r,$
       $\bar{Z}(k) = \bar{S}_1 + \ldots + \bar{S}_{k-1} + \bar{S}_{k+1} + \ldots + \bar{S}_r.$

(45)   THEOREM. *For each $k = 1, \ldots, r$, if Z is not a prism with upright S_k, then*

(46) $A(Z) = A(Z(k)) + (A(\bar{Z}(k)))^T.$

If, on the other hand, Z is a prism with upright S_k, then

(47) $A(Z) = \Lambda A(Z(k))$

where Λ is the matrix defined by

$$\lambda_{ii} = \lambda_{i+1,i} = 1 \quad \text{for all } i \geqq 0,$$
$$\lambda_{ij} = 0 \qquad \qquad \text{otherwise.}$$

Consider any family $[C]$ of cells of Z, where C is defined as in (4). To begin with let us suppose that S_k is not parallel to any other component of Z. If S_k is not a component of C then there is a uniquely defined family of cells $[C_1]$ of $Z(k)$ corresponding to $[C]$, namely that with components $S_{\sigma(1)}, \ldots, S_{\sigma(s)}$. If Z is not a prism with upright S_k then dim $Z(k) = n$ and so

(48) $d[C] = d[C_1]$ and $e[C] = e[C_1].$

On the other hand, if S_k is a component of C (say $k = \sigma(s)$) then write π_k for parallel projection in the direction S_k on to a complementary hyperplane. To $[C]$ there corresponds a uniquely defined family of cells $[C_2]$ of $\pi_k(Z)$ namely that with components $\pi_k(S_{\sigma(1)}), \ldots, \pi_k(S_{\sigma(s-1)})$. Since dim $\pi_k(Z) = n - 1$ we deduce that

(49) $d[C] = d[C_2]$ and $e[C] = e[C_2].$

Statements (48) and (49) together imply that

$$A(Z) = A(Z(k)) + A(\pi_k(Z))$$

and statements (30) and (40) together imply that

$$A(\pi_k(Z)) = (A(\bar{Z}(k)))^T,$$

from which (46) follows immediately. If we do not make the restriction that S_k is parallel to no other component of Z, then the above argument has to be modified slightly but is completely straightforward.

Secondly, let us suppose that Z is a prism with upright S_k, then, continuing with the same notation as above, we see that dim $Z(k) = n - 1$ and so instead of (48) we have

(50) $d[C] = d[C_1] + 1$ and $e[C] = e[C_1].$

Also we may take $\pi_k(Z) = Z(k)$ and hence

(51) $d[C] = d[C_2]$ and $e[C] = e[C_2].$

Together, (50) and (51) imply relation (47) between the matrices $A(Z)$ and $A(Z(k))$ and hence the proof of the theorem is completed.

If we apply Theorem (45) $r - n$ times to an n-dimensional zonotope with r components we see that we can express the matrix $A(Z)$ in terms of the d-e matrices of cubes. These take a particularly simple form since for every family of cubes $e[C] = 0$ and then (43) shows that

$$a_{i0} = \binom{n}{i},$$

$$a_{ij} = 0 \qquad \text{(all } j \geqq 1\text{)}.$$

(52) *Let Z be any zonotope. If, for some i, j the element a_{ij} of $A(Z)$ is zero, then $a_{kl} = 0$ for all $k \geqq i$ and $l \geqq j$.*

We notice that this assertion is true for the d-e matrices of cubes, and remains true if we combine these matrices as in (46) and (47). Hence (52) is true generally.

(53) *Z is a cubical zonotope if and only if $A(Z)$ has zero elements everywhere except in the first row and in the first column.*

This follows from (52) since clearly $a_{11} = 0$. As in the case of cubes, (53) implies that the non-zero entries in $A(Z)$ are binomial coefficients; see, for example, the d-e matrices corresponding to the cubical zonotopes of Figure 1. Equalities (42) enable us to calculate the number of vertices in this case, thus extending statement (28). In fact the number of k-faces of Z ($0 \leqq k \leqq n - 1$) can also be determined by noticing that it is equal to twice the number of $(n - k - 1)$-dimensional regions in the $(n - 1)$-arrangement of hyperplanes in which $\mathscr{H}$ (defined in (38)) cuts the hyperplane at infinity. (This is Coxeter's second projective diagram [**3**, § 5].) As Z is cubical, this arrangement is simple, and the required numerical values can be determined immediately from [**4**, Theorem 18.1.2]. For further information concerning the connection between zonotopes and arrangements, see [**1**] and [**6**].

We note that, for any zonotope Z the $(r - n)$th column and nth row of $A(Z)$ are of the form $(1, 0, 0, \ldots)$. This follows from (52) and the obvious equalities

$$a_{0,r-n} = a_{n,0} = 1 \quad \text{and} \quad a_{1,r-n} = a_{n,1} = 0.$$

5. Cubical dissections of zonotopes. The element a_{00} of $A(Z)$ has an interesting geometrical interpretation. By a *cubical dissection* of Z we mean a cell complex $\mathscr{C}(Z)$ such that
    (i) set $\mathscr{C}(Z) = Z$,
    (ii) every cell of $\mathscr{C}(Z)$ is a cube, and
    (iii) every cell of $\mathscr{C}(Z)$ is a cell of Z.
By the *order* of a cubical dissection $\mathscr{C}(Z)$ we mean the number of n-cells in $\mathscr{C}(Z)$, where $n = \dim Z$, and we shall denote this quantity by ord $\mathscr{C}(Z)$. In Figure 5 we indicate cubical dissections of the zonotopes of Figure 2; each of
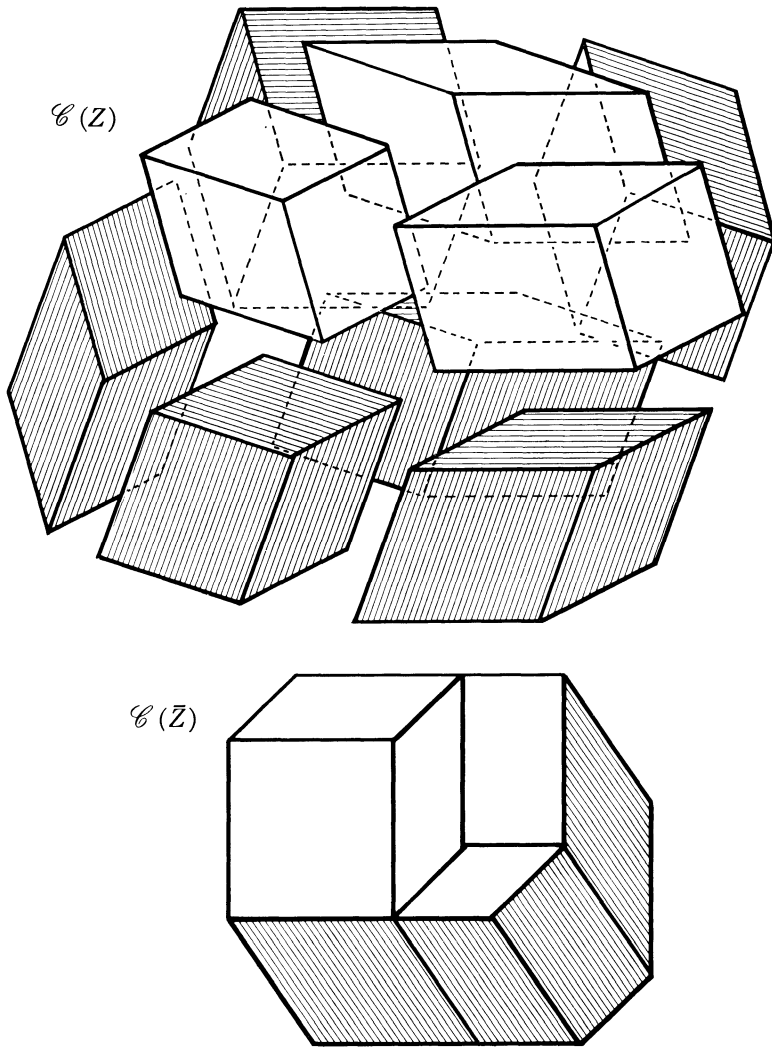
$\mathscr{C}(Z)$

$\mathscr{C}(\bar{Z})$

FIGURE 5

these dissections is of order nine. In the upper figure the 3-dimensional cubes have been 'exploded' to show the internal structure of $\mathscr{C}(Z)$.

(54) THEOREM. *Every zonotope Z admits a cubical dissection, and every such dissection has order a_{00}.*

Unless it is a cube every zonotope admits more than one cubical dissection. Theorem (54), along with (40), shows that every cubical dissection of Z has the same order as every cubical dissection of Z. Thus we obtain another curious and surprising geometrical relationship between associated zonotopes.

Both assertions of (54) are proved by induction on the dimension n of Z and on the number r of its components. The induction starts by noticing that both statements are true if either Z is a cube (when the dissection is trivial and $a_{00} = 1$) or when Z is one-dimensional (when Z can be dissected into $a_{00} = r$ one-dimensional cubes or line segments). In the general case, let us suppose that Z, $Z(k)$ and π_k are defined as in (44) and the proof of (45). Then we can construct a cubical dissection of Z in the following way. We first partition Z into two pieces, namely $Z(K) - x_k$ and $\mathrm{cl}(Z \backslash (Z(K) - x_k))$. The first of these admits a cubical dissection by the inductive hypothesis, and we shall now show that the second does so also. Write B_k for the union of those facets F of Z such that $\langle u, x_k \rangle > 0$, where u is the outward normal to the hyperplane that supports Z in F. Then $\pi_k(B_k) = \pi_k(Z)$. But $\pi_k(Z)$ admits a cubical dissection by the inductive hypothesis, and this implies that a cubical dissection $\mathscr{C}(B_k)$ also exists. (Each cell of $\mathscr{C}(B_k)$ is the inverse image under π_k of a cell in a cubical dissection of $\pi_k(Z)$.) Consider the set of n-dimensional cubes $\{C + S_k\}$ where C runs through the set of $(n - 1)$-dimensional cubes of $\mathscr{C}(B_k)$. These clearly form a partition of $\mathrm{cl}(Z \backslash (Z(k) - x_k))$ and hence lead to a cubical dissection of this set. The first assertion of the theorem is therefore proved. We note also that for this dissection $\mathscr{C}(Z)$ we have

(55) $\mathrm{ord}\ \mathscr{C}(Z) = \mathrm{ord}\ \mathscr{C}(Z(k)) + \mathrm{ord}\ \mathscr{C}(\pi_k(Z)),$

from which, using (46), we obtain

$$\mathrm{ord}\ \mathscr{C}(Z) = a_{00}$$

as required.

The set of n-dimensional cubes in $\mathrm{cl}(Z \backslash (Z(k) - x_k))$ may be called a *cup* of cubes which 'holds' $Z(k)$. In Figure 4 we have indicated by shading in each case, a cup of cubes. It may be conjectured that every cubical dissection of a zonotope contains at least one cup of cubes,† and if this could be established then the above argument would complete the proof of the theorem. In the absence of a proof of this conjecture, it is necessary to modify the procedure slightly.

Let $\mathscr{C}(Z)$ be given and any component S_k be chosen. The n-dimensional cubes of $\mathscr{C}(Z)$ may be partitioned into two classes:

   (i) those that have the component S_k, and
   (ii) those that do not.

Cubes of the first class project by π_k into the $(n - 1)$-dimensional cubes of some dissection $\mathscr{C}(\pi_k(Z))$, and hence their number is $\mathrm{ord}\ \mathscr{C}(\pi_k(Z))$. Cubes of the second class, after suitable translation through $+x_k$ or $-x_k$ are the cubes of some cubical dissection of $Z(k)$ and so their number is $\mathrm{ord}\ \mathscr{C}(Z(k))$.

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†This conjecture is now known to be incorrect. Consider the projection of a regular triacontahedron, parallel to one of its 5-fold axes of symmetry, on to a plane. The images of its 2-faces form a cubical dissection of a regular 10-gon with no cup of cubes.

Hence we are led again to (55) and the second statement of the theorem is true generally.

A refinement of the above argument leads to the following slightly stronger statement.

(56) *Let $\mathscr{C}(Z)$ be any given cubical dissection of Z. Then precisely one n-dimensional cube of $\mathscr{C}(Z)$ belongs to each of the families of cells $[C]$ of Z for which $d[C] = e[C] = 0$.*

From (56) we immediately obtain the following general expression for the volume $V(Z)$ of a zonotope Z.

(57) $$V(Z) = 2^n \sum |\det(x_{i_1}, \ldots, x_{i_n})|$$

where the summation is over all n-membered subsets $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, r\}$.

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