MATHEMATICAL NOTES.

Direction-Cosines of the Axes of the Conicoid

\[ ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1. \]

In Mathematical Notes, No. 20 (April 1916), there is a note on the above; I add a form of the equations of these axes which I have not seen in a text-book, and which is perhaps worth recording.

If on transformation of axes

\[ ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = a'x^2 + b'y^2 + c'z^2, \]

then

\[(a - \lambda) x^2 + (b - \lambda) y^2 + (c - \lambda) z^2 + 2fyz + 2gzx + 2hxy = (a' - \lambda) X^2 + (b' - \lambda) Y^2 + (c' - \lambda) Z^2.\]

The right-hand side resolves into factors if \( \lambda = a' \) or \( b' \) or \( c' \),

\[ a', b', c' \]

are the roots of

\[(a - \lambda) (b - \lambda) (c - \lambda) + 2fgh - f^2 (a - \lambda) - g^2 (b - \lambda) - h^2 (c - \lambda) = 0. \]

When \( \lambda = a' \), \( a'X^2 + b'Y^2 + c'Z^2 = 0 \) is the equation of two planes which intersect in the \( X \)-axis, on which the conicoid intercepts a length \( \frac{2}{\sqrt{a'}} \).

\[ \phi(x, y, z) = (a - a') x^2 + (b - a') y^2 + (c - a') z^2 + 2fyz + 2gzx + 2hxy = 0 \]

is the equation of two planes intersecting in the axis of length \( \frac{2}{\sqrt{a'}} \).

The intersection of these planes is given by any two of the equations

\[ \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial z} = 0, \]

i.e. \( (a - a') x + hy + gz = 0, \quad hx + (b - a') y + fz = 0, \quad gx + fy + (c - a') z = 0. \)

From the first and second, any point on the axis satisfies

\[ \frac{x}{hf - bg + ga'} = \frac{y}{gh - af + fa'}, \]

and from the first and third, the point satisfies

\[ \frac{x}{fg - ch + ha'} = \frac{z}{gh - af + fa'}, \]

i.e. \( \frac{x}{G + ga'} = \frac{y}{F + fa'} \) and \( \frac{x}{H + ha'} = \frac{z}{F + fa'} \),

where \( F, G, H \) are the customary minors.

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Therefore the equations of this axis are

\[ x(F + fa') = y(G + ga') = z(H + ha'). \]

Similar equations hold for the other axes, with \( b' \) and \( c' \) instead of \( a' \).

**Lawrence Crawford.**

**Proofs of some Inequalities and Limits.**

In his article in No. 20, Professor Gibson gives proofs of the inequalities

\[ 1 - na < (1 - a)^n < \frac{1}{1 + na}, \]

with certain restrictions as to the values of \( n \) and \( a \). The mode of proof, avoiding as it does the use of the Binomial Theorem, is comparatively short and simple. The following mode of proof is, I think, still simpler, as it does not involve the use of Mathematical Induction.

If \( n \) is a positive integer and \( a \) positive, we have

\[
\frac{(1 + a)^n - 1}{1 + a} = (1 + a)^{n-1} + (1 + a)^{n-2} + (1 + a)^{n-3} + \ldots + (1 + a) + 1, \]

> \( n \),

\[ \therefore (1 + a)^n - 1 > na, \]

\[ \therefore (1 + a)^n > 1 + na. \]

\[ \text{(1)} \]

Again, \( n \) being a positive integer and \( a \) a positive proper fraction, we have

\[
\frac{1 - (1 - a)^n}{1 - (1 - a)} = 1 + (1 - a) + (1 - a)^2 + \ldots + (1 - a)^{n-1}, \]

< \( n \),

\[ \therefore 1 - (1 - a)^n < na, \]

\[ \therefore (1 - a)^n > 1 - na. \]

\[ \text{(2)} \]

Then, since \( (1 - a)(1 + a) = 1 - a^2 \)

< 1,

\[ \therefore 1 - a < \frac{1}{1 + a}, \]

\[ \therefore (1 - a)^n < \frac{1}{(1 + a)^n}, \]

\[ \therefore \text{by (1), } < \frac{1}{1 + na}. \]

\[ \text{(243)} \]