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BILATERAL LAPLACE MULTIPLIERS ON SPACES OF DISTRIBUTIONS

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A bilateral Laplace multiplier theory, based on Rooney's class $\mathscr{A}$, is developed for certain operators defined on the Fréchet spaces $D_{p,\mu}$. The theory is applied to Riesz fractional integrals associated with the one-dimensional wave operator.

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1. Introduction

The $L^p(\mathbb{R}^n)$ Fourier multiplier results, given by the Marcinkiewicz or Mihlin–Hörmander theorems [**8**, p. 109], [**4**, p. 417], serve as a starting point for multiplier theories via other transforms. For example, in [**5**], Rooney developed a Mellin multiplier framework based on the one-dimensional Mihlin–Hörmander theorem together with the relationship between the Fourier and Mellin transforms. Rooney's work was useful in establishing the mapping properties of certain operators, such as the Erdélyi–Kober operators, on weighted $L^p(0,\infty)$ spaces. In a similar manner, we aim to obtain a bilateral Laplace multiplier theory on some other weighted $L^p(\mathbb{R}^n)$ spaces. Moreover, we will extend the results to the corresponding Fréchet spaces and, hence, develop a distributional bilateral Laplace multiplier theory. Our theory will be particularly useful in connection with the Riemann–Liouville and Weyl fractional integrals, which were investigated by means of Fourier multipliers in [**1**], and also for the Riesz fractional integrals associated with the one-dimensional wave operator, which were considered using fractional powers in [**6**].

2. Preliminaries

Throughout, p and q denote real numbers with $1<p,\ q<\infty$ and $1/p+1/q=1$. We use $\mathbb{N}_0^n=\mathbb{N}_0\times\mathbb{N}_0\times\cdots\times\mathbb{N}_0$ to represent the set of all n-dimensional multi-indices and, for $r=(r_1,r_2,\dots,r_n)\in\mathbb{N}_0^n$, $(\partial^r\phi)(\mathbf{x})=(\partial^{r_1,r_2,\dots,r_n}\phi)(\mathbf{x})=(\partial_1^{r_1}\partial_2^{r_2}\dots\partial_n^{r_n}\phi)(\mathbf{x})$ for $\mathbf{x}=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ where $\partial_i=\partial/\partial x_i$.

The main spaces of interest are the exponentially weighted Banach spaces $L_{p,\mu}(\mathbb{R}^n)$

together with the corresponding Fréchet spaces $D_{p,\mu}(\mathbb{R}^n)$. We assume familiarity with $L^p(\mathbb{R}^n)$ and the Schwartz spaces $D_{L^p}(\mathbb{R}^n)$, [7].

Definition 2.1. For $\boldsymbol{\mu}=(\mu_1,\mu_2,\dots,\mu_n)$, $\mathbf{x}=(x_1,x_2,\dots,x_n)\in(\mathbb{R}^n)$,

(i) $(E_\mu\phi)(\mathbf{x})=e^{\boldsymbol{\mu}\cdot\mathbf{x}}\phi(\mathbf{x}), \boldsymbol{\mu}\cdot\mathbf{x}=\sum_{i=1}^n \mu_i x_i$;

(ii) $L_{p,\mu}(\mathbb{R}^n)=\{\phi: E_{-\mu}\phi\in L^p(\mathbb{R}^n)\}$ and is equipped with the norm $\|\phi\|_{p,\mu}=\|E_{-\mu}\phi\|_p$, where $\|\ \|_p$ denotes the usual L^p norm;

(iii) $D_{p,\mu}(\mathbb{R}^n)=\{\phi: E_{-\mu}\phi\in D_{L^p}(\mathbb{R}^n)\}$ and is equipped with the topology generated by the countable multinorm $S=\{v_r^{p,\mu}, r\in\mathbb{N}_0^n\}$ where $v_r^{p,\mu}(\phi)=\|\partial^r(E_{-\mu}\phi)\|_p$.

In addition, we shall mention the spaces $\mathrm{L}_{\mu,p}(\mathbb{R})$ which are polynomially weighted $L^p(0,\infty)$ spaces given by $\mathrm{L}_{\mu,p}(\mathbb{R})=\{\phi: C\phi\in L_{p,\mu}(\mathbb{R})\}$, where $(C\phi)(t)=\phi(e^{-t})$.

Duality theory in the above spaces is defined in the usual way. For example, $L_{q,-\mu}$ denotes the dual space of $L_{p,\mu}$. We define the bilinear form (,) on $L_{q,-\mu}(\mathbb{R}^n)\times L_{p,\mu}(\mathbb{R}^n)$ by

$$(\psi,\phi)=\int_{\mathbb{R}^n}\psi(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x},\quad \phi\in L_{p,\mu},\quad \psi\in L_{q,-\mu}$$

which is well-defined, by Hölder's inequality. Thus, if R is a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu}$ then the formal adjoint, R', is defined as the unique linear operator from $L_{q,-\mu}$ into $L_{q,-\mu}$ such that $(R'\psi,\phi)=(\psi,R\phi)$.

The set $C_0^\infty(\mathbb{R}^n)$ of infinitely differentiable, complex-valued functions on $\mathbb{R}^n$ with compact support is dense in $D_{p,\mu}(\mathbb{R}^n)$. Tensor product spaces [9], of the form $X\otimes X$, are also required. For example $L^p(\mathbb{R})\otimes L^p(\mathbb{R})$ is defined by

$$L^p(\mathbb{R})\otimes L^p(\mathbb{R})=\{\phi:\phi(x_1,x_2)=\sum_{j=1}^k \theta_j(x_1)\psi_j(x_2);\theta_j,\psi_j\in L^p(\mathbb{R})\},\tag{2.1}$$

and is dense in $L^p(\mathbb{R}^2)$.

We shall be particularly concerned with the bilateral Laplace transform, $\mathscr{L}$, defined, for suitable ψ and $\boldsymbol{\mu}\in\mathbb{R}^n$, by

$$(\mathscr{L}\psi)(\mathbf{s})=\int_{\mathbb{R}^n}e^{-\mathbf{s}\cdot\mathbf{x}}\psi(\mathbf{x})\,d\mathbf{x},\quad \mathbf{s}=\boldsymbol{\mu}+i\boldsymbol{\tau},\tag{2.2}$$

but we shall also consider the closely related Fourier transform $\mathscr{F}$, defined for suitable ϕ, by

$$(\mathscr{F}\phi)(\boldsymbol{\tau})=\int_{\mathbb{R}^n}e^{i\boldsymbol{\tau}\cdot\mathbf{x}}\phi(\mathbf{x})\,d\mathbf{x}\tag{2.3}$$

and, for the one-dimensional case, the Mellin transform, $\mathscr{M}$, defined for suitable η and $\mu\in\mathbb{R}$ by

$$(\mathcal{M}\eta)(s) = \int_0^\infty x^{s-1}\eta(x)\,dx, \qquad s = \mu + i\tau. \tag{2.4}$$

It is convenient to introduce the following notation.

Notation 2.2. (i) Ω denotes a region in $\mathbb{R}^n$ of the form $\{\boldsymbol{\mu}: a_j < \mu_j < b_j,\ j = 1, 2, \ldots, n\}$ where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and $a_j, b_j \in \mathbb{R}$.

(ii) For Ω given by (i), $\Omega^\# = \{\mathbf{s} \in \mathbb{C}^n : \operatorname{Re} \mathbf{s} \in \Omega\}$.

(iii) Throughout, we will assume $\mathbf{s} = \boldsymbol{\mu} + i\boldsymbol{\tau}$, $\boldsymbol{\mu}, \boldsymbol{\tau} \in \mathbb{R}^n$, so that $\mathbf{s} \in \Omega^\#$ if and only if $\boldsymbol{\mu} \in \Omega$.

(iv) The functions g, g^* and g_μ are defined by $g(\mathbf{s}) = g(\boldsymbol{\mu} + i\boldsymbol{\tau})$; $g^*(s) = g(\bar{\mathbf{s}}) = g(\boldsymbol{\mu} - i\boldsymbol{\tau})$ and, for fixed $\boldsymbol{\mu} \in \Omega$, $g_\mu(\tau) = g(\boldsymbol{\mu} + i\boldsymbol{\tau})$.

3. One-dimensional theory

In one dimension, a bilateral Laplace multiplier theory on $L_{p,\mu}(\mathbb{R})$ can be obtained from the Mellin multiplier theory on $\mathbf{L}_{\mu,p}(\mathbb{R})$ which was developed by Rooney [**3**, pp. 176–188], [**5**]. Essentially, this relies on the relationship

$$(\mathcal{M}\phi)(s) = (\mathcal{L}(C\phi))(s), \quad C\phi(t) = \phi(e^{-t}), \tag{3.1}$$

which holds initially for $\phi \in \mathbf{L}_{\mu,1}(\mathbb{R}) \cap \mathbf{L}_{\mu,p}(\mathbb{R})$, $1 < p \leqq 2$, but can be extended to all $\mathbf{L}_{\mu,p}(\mathbb{R})$, $1 \leqq p \leqq 2$, by the same process as is used for the extension of the Fourier transform to $L^p(\mathbb{R})$, $1 \leqq p \leqq 2$, in [4, Ch. 6].

Definition 3.1. Let g be a complex-valued function defined on $\Omega^\#$. Then g is an $\mathbf{L}_{\mu,p}(\mathbb{R})$ Mellin multiplier if there exists a unique linear transformation T_g such that

(i) for $1 < p < \infty$ and $\mu \in \Omega$, $T_g \in L(\mathbf{L}_{\mu,p})$;

(ii) for $1 < p \leqq 2$, $\mu \in \Omega$ and $\psi \in \mathbf{L}_{\mu,p}$,

$$(\mathcal{M}(T_g\psi))(s) = g(s)(\mathcal{M}\psi)(s).$$

Definition 3.2. Replacing T_g by R, $\mathbf{L}_{\mu,p}(\mathbb{R})$ by $L_{p,\mu}(\mathbb{R}^n)$, $s \in \mathbb{C}$ by $\mathbf{s} \in \mathbb{C}^n$ and Mellin and $\mathcal{M}$ by bilateral Laplace and $\mathcal{L}$ respectively, Definition 3.1 yields the n-dimensional bilateral Laplace multiplier definition.

In one dimension, it can be shown that the class $\mathcal{A}$ is sufficient for both Mellin and bilateral Laplace multipliers.

Definition 3.3. The function $g \in \mathcal{A}$ if there are extended real numbers $\alpha(g)$ and $\beta(g)$, with $\alpha(g) < \beta(g)$, such that

(i) $g(s)$ is analytic in the strip $\Omega = \operatorname{Re} s : \alpha(g) < \operatorname{Re} s < \beta(g)$;

(ii) in every closed substrip $\alpha_1 \leqq \operatorname{Re} s \leqq \beta_1$, where $\alpha(g) < \alpha_1 \leqq \beta_1 < \beta(g)$, $g(s)$ is bounded;

(iii) for fixed $\mu = \operatorname{Re} s \in (\alpha(g), \beta(g))$, $|g'(\mu + i\tau)| = O(|\tau|^{-1})$, as $\tau \to \infty$.

Lemma 3.4. *Let $g \in \mathscr{A}$ be such that $\mu \in \Omega = (\alpha(g), \beta(g))$. Then g is an $L_{p,\mu}(\mathbb{R})$ bilateral Laplace multiplier.*

Proof. By [3, p. 183, Theorem 4], g is a Mellin multiplier with associated operator $T_g \in L(\mathbf{L}_{\mu,p})$ for $\mu \in \Omega$. Hence, by (3.1) for $\psi \in L_{p,\mu}(\mathbb{R})$, $1 < p \leqq 2$, $(\mathscr{L}(CT_gC^{-1}\psi))(s) = g(s)(\mathscr{L}\psi)(s)$. Moreover, it can be deduced that $R = CT_gC^{-1} \in L(L_{p,\mu}(\mathbb{R}))$, $1 < p < \infty$.

Lemma 3.5. *Let $g \in \mathscr{A}$ and have associated operator R as in Lemma 3.4. Then the restriction of R to $D_{p,\mu}(\mathbb{R})$ defines a continuous linear mapping from $D_{p,\mu}(\mathbb{R})$ into $D_{p,\mu}(\mathbb{R})$.*

Proof. The result follows from the corresponding Fourier multiplier result on $D_{L^p}(\mathbb{R})$ [1, Theorem 3.8], together with the relationship $(\mathscr{F}(E_{-\mu}\psi))(-\tau) = (\mathscr{L}\psi)(s)$, $\psi \in L_{1,\mu}(\mathbb{R}) \cap L_{p,\mu}(\mathbb{R})$, $1 < p \leqq 2$, since $E_{-\mu}: D_{p,\mu} \to D_{L^p}$ is a homeomorphism with inverse E_μ. Alternatively, the proof of [1] can be modified to give the result directly.

In [1], the Riemann–Liouville and Weyl fractional integrals were studied on $D_{p,\mu}(\mathbb{R})$ by means of a Fourier multiplier theory. However, these results are easily restructured in a more natural manner in terms of bilateral Laplace multipliers.

4. Multidimensional theory

Rooney's Mellin multiplier theory is clearly only valid in one dimension. In order to make progress in higher dimensions, we exploit the relationship between the Fourier and bilateral Laplace transforms, namely

$$(\mathscr{F}(E_{-\mu}\psi))(-\boldsymbol{\tau}) = (\mathscr{L}\psi)(\mathbf{s}), \qquad \psi \in L_{1,\mu}(\mathbb{R}^n) \cap L_{p,\mu}(\mathbb{R}^n),\ 1 < p \leqq 2. \tag{4.1}$$

Definition 4.1. Replacing T_g by P, $\mathbf{L}_{\mu,p}(\mathbb{R})$ by $L^p(\mathbb{R}^n)$, $s \in \mathbb{C}$ by $\mathbf{s} \in \mathbb{C}^n$ and Mellin and $\mathscr{M}$ by Fourier and $\mathscr{F}$ respectively, Definition 3.1 yields the n-dimensional Fourier multiplier definition.

The Fourier multiplier (sufficient) conditions provided by the Mihlin–Hörmander or Marcinkiewicz theorems prove difficult to apply for dimensions greater than one. Instead, we obtain a multidimensional theory via a product of one-dimensional multipliers and, for simplicity, we concentrate on the two-dimensional case. It is natural to ask whether the product of two one-dimensional multipliers defines a two-dimensional multiplier. We answer this with respect to the Marcinkiewicz theorem.

Theorem 4.2. (Marcinkiewicz theorem). *Let h be a bounded function on $\mathbb{R}^n$, defined on each of 2^n octants* [8, p. 108–109]. *Further let h, together with its partial derivatives up to and including order n, be continuous on these octants. Suppose also that*

(a) $|h(\boldsymbol{\tau})| \leqq B$, *for all* $\boldsymbol{\tau} \in \mathbb{R}^n$;

(b) *for each* $k=1,2,\dots,n$,

$$\sup_{\tau_{k+1},\dots,\tau_n} \int_\rho \left|\frac{\partial^k h}{\partial\tau_1\dots\partial\tau_k}\right| d\tau_1\dots d\tau_k \leqq B,$$

as ρ *ranges over dyadic rectangles of* $\mathbb{R}^k$ *where, if* $k=n$, *the* sup *is omitted;*

(c) *the analogue of* (*b*) *holds for each of the* $n!$ *permutations of the variables* $\tau_1,\dots,\tau_n$.

Then h *is an* $L^p(\mathbb{R}^n)$ *Fourier multiplier in the sense of Definition* 4.1.

Proof. [**8**, p. 109–112].

In Theorem 4.2, the dyadic decomposition of $\mathbb{R}^k$ involves considering sets of k-dimensional rectangles with sides parallel to the axes. Thus, it is not a general decomposition and can be contrasted with the alternative decomposition used in the Mihlin–Hörmander theorem [**4**, p. 417]. However, since our approach uses products of one-dimensional multipliers, the dyadic decomposition will prove particularly suitable.

Lemma 4.3. *If* $g\in\mathscr{A}$ *then, for each* $\mu\in\Omega=(\alpha(g),\beta(g)), g_\mu(\tau)$, *defined by Notation* 2.2(*iv*), *satisfies the conditions of the one-dimensional Marcinkiewicz theorem.*

Proof. Since $g\in\mathscr{A}$ it follows that, for each $\mu\in\Omega$, $g_\mu\in L^\infty(\mathbb{R})\cap C^\infty(\mathbb{R})$ and thus, $|g_\mu(\tau)|\leqq M_1$, $\forall\tau\in\mathbb{R}$, where M_1 may depend on μ. Also,

$$\int_{I_j} |g'_\mu(\tau)|d\tau \leqq M_2 \int_{I_j} |\tau|^{-1}\,d\tau = M_2\ln 2,$$

where $I_j=(2^j,2^{j+1})$ or $(-2^{j+1},-2^j)$. Hence, setting $B=\max(M_1,M_2\ln 2)$ gives the required result.

Theorem 4.4. *Let* $h_j\in C^2(\mathbb{R}), j=1,2$, *be one-dimensional Fourier multipliers satisfying the conditions of Theorem* 4.2 *and having associated operators* P_j. *Then* $h(\tau)=h_1(\tau_1)h_2(\tau_2)$ *is a two-dimensional Fourier multiplier satisfying the conditions of Theorem* 4.2 *and having associated operator* $P_1\otimes P_2$, *defined on* $L^p(\mathbb{R})\otimes L^p(\mathbb{R})$ *by* $(P_1\otimes P_2)(\phi)=\sum_{j=1}^k (P_1\theta_j)(P_2\psi_j)$ *where* ϕ *is as in* (2.1), *and extended by continuity and density to* $L^p(\mathbb{R}^2)$.

Proof. By definition h_j are bounded functions on $\mathbb{R}$ defined on the half-lines $(-\infty,0)$ and $(0,\infty)$ and have continuous second order derivatives there. Also, there exist constants B_j such that $|h_j(\tau_j)|\leqq B_j$ and $\int_{\rho_j}|h'_j(\tau_j)|d\tau_j\leqq B_j$, as ρ_j ranges over dyadic intervals of $\mathbb{R}$. Thus, h is a bounded function on $\mathbb{R}^2$, defined on each of the four quadrants, and is continuous together with its derivatives up to and including order two. In addition,

$$|h(\tau_1,\tau_2)|=|h_1(\tau_1)h_2(\tau_2)|\leqq B_1B_2;$$

$$\sup_{\tau_k}\int_{\rho_j}|\partial_j h_j(\tau_j)h_k(\tau_k)|\,d\tau_j, \qquad j\neq k,\ j,k=1,2,$$

$$=\sup_{\tau_k}|h_k(\tau_k)|\int_{\rho_j}|h_j'(\tau_j)|\,d\tau_j$$

$$\leqq B_kB_j=B_1B_2;$$

and

$$\int_{\rho_1\times\rho_2}|\partial_1 h_1(\tau_1)\partial_2 h_2(\tau_2)|\,d\tau_1 d\tau_2=\int_{\rho_1}|h_1'(\tau_1)|\,d\tau_1\int_{\rho_2}|h_2'(\tau_2)|\,d\tau_2$$
$$\leqq B_1B_2;$$

where ρ_1,ρ_2 range over dyadic intervals of $\mathbb{R}$ and $\times$ denotes the Cartesian product. Hence, Theorem 4.2 is satisfied for $n=2$.

Next, suppose $\phi\in L^2(\mathbb{R})\otimes L^2(\mathbb{R})$. Clearly, $\mathscr{F}:L^2(\mathbb{R})\otimes L^2(\mathbb{R})\subset L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$. Moreover, for $\phi(x_1,x_2)=\sum_{j=1}^k\theta_j(x_1)\psi_j(x_2)$,

$$(\mathscr{F}(P_1\otimes P_2)\phi)(\tau)=\left(\mathscr{F}\left(\sum_{j=1}^k P_1\theta_j(x_1)P_2\psi_j(x_2)\right)\right)(\tau), \qquad \theta_j,\ \psi_j\in L^2(\mathbb{R}),$$

$$=\sum_{j=1}^k(\mathscr{F}(P_1\theta_j(x_1)P_2\psi_j(x_2)))(\tau), \qquad \text{by linearity of } \mathscr{F},$$

$$=\sum_{j=1}^k(\mathscr{F}^{(1)}(P_1\theta_j(x_1)))(\tau_1)(\mathscr{F}^{(2)}(P_2\psi_j(x_2)))(\tau_2)$$

where $\mathscr{F}^{(i)}$ denotes the one-dimensional Fourier transform with respect to the variable x_i. Thus,

$$(\mathscr{F}(P_1\otimes P_2)\phi)(\tau)=\sum_{j=1}^k h_1(\tau_1)(\mathscr{F}^{(1)}(\theta_j(x_1)))(\tau_1)h_2(\tau_2)(\mathscr{F}^{(2)}(\psi_j(x_2)))(\tau_2)$$

$$=h_1(\tau_1)h_2(\tau_2)\left(\mathscr{F}\left(\sum_{j=1}^k\theta_j(x_1)\psi_j(x_2)\right)\right)(\tau), \qquad \text{by linearity,}$$

$$=h(\tau)(\mathscr{F}\phi)(\tau).$$

Finally, since h is a two-dimensional Fourier multiplier, there exists a mapping

$P \in L(L^p(\mathbb{R}^2))$ such that $P = P_1 \otimes P_2$ on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. By density arguments we can extend $P_1 \otimes P_2$ to $P \in L(L^p(\mathbb{R}^2))$ as required. This completes the proof.

Corollary 4.5. *For $j = 1, 2$, let g^j be of class $\mathscr{A}$ and have associated operators P_{μ_j} and let $\mu_j \in \Omega_j = (\alpha(g^j), \beta(g^j))$. Then $g_{\mu}(\tau) = g^1_{\mu_1}(\tau_1) g^2_{\mu_2}(\tau_2)$ is a two-dimensional Fourier multiplier for $\mu = (\mu_1, \mu_2) \in \Omega = \Omega_1 \times \Omega_2$, with associated operator $P_{\mu_1} \otimes P_{\mu_2}$.*

Proof. This is immediate in view of Lemma 4.3 and Theorem 4.4.

The corresponding result for $L_{p,\mu}(\mathbb{R}^2)$ bilateral Laplace multipliers can now be deduced.

Corollary 4.6. *For $j = 1, 2$, let g^j be of class $\mathscr{A}$ and let g^* be defined, for $\mu \in \Omega = \Omega_1 \times \Omega_2 = (\alpha(g^1), \beta(g^1)) \times (\alpha(g^2), \beta(g^2))$ by $g^*(\mu + i\tau) = g^1(\mu_1 - i\tau_1) g^2(\mu_2 - i\tau_2)$. Then g^* is a two-dimensional bilateral Laplace multiplier with associated operator $R = E_{\mu}(P_{\mu_1} \otimes P_{\mu_2}) E_{-\mu}$ for $\mu \in \Omega$.*

Proof. By Corollary 4.5, for $\phi \in L^p(\mathbb{R}^2)$, $1 < p \leqq 2$, $(\mathscr{F}(P_{\mu_1} \otimes P_{\mu_2}\phi))(\tau) = g_{\mu}(\tau)(\mathscr{F}\phi)(\tau)$. Hence, by (4.1), $(\mathscr{L}(E_{\mu}(P_{\mu_1} \otimes P_{\mu_2})\phi))(\mu - i\tau) = g_{\mu}(\tau)(\mathscr{L}(E_{\mu}\phi))(\mu - i\tau)$ from which the required result follows.

Remark 4.7. (i) The operator R in Corollary 4.6 can be expressed as $(E_{\mu_1} P_{\mu_1} E_{-\mu_1}) \otimes (E_{\mu_2} P_{\mu_2} E_{-\mu_2})$. With this form, Corollary 4.6 may be obtained by working in the tensor product space $L_{p,\mu_1}(\mathbb{R}) \otimes L_{p,\mu_2}(\mathbb{R})$ directly.

(ii) There would seem to be no difficulties in extending Theorem 4.4 – Corollary 4.6 to higher dimensions.

Next, we extend the above results to a $D_{p,\mu}(\mathbb{R}^2)$ setting. On replacing $L_{p,\mu}$ by $D_{p,\mu}$ in Definition 3.2 we get a $D_{p,\mu}(\mathbb{R}^n)$ bilateral Laplace multiplier definition. Applying the approach of [2; II, Theorem 3.3], we shall prove that every $L_{p,\mu}(\mathbb{R}^2)$ multiplier is a $D_{p,\mu}(\mathbb{R}^2)$ multiplier. This requires the following preliminary results.

Definition 4.8. The operators I_i^{α} and K_i^{α}, $i = 1, 2, \ldots, n$ are defined, for suitable functions ϕ and $\operatorname{Re}\alpha > 0$, by

$$I_i^{\alpha}\phi(\mathbf{x}) = [\Gamma(\alpha)]^{-1} \int_{-\infty}^{x_i} (x_i - y)^{\alpha - 1} \phi(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)\, dy, \quad \mathbf{x} \in \mathbb{R}^n,$$

and

$$K_i^{\alpha}\phi(\mathbf{x}) = [\Gamma(\alpha)]^{-1} \int_{x_i}^{\infty} (y - x_i)^{\alpha - 1} \phi(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)\, dy, \quad \mathbf{x} \in \mathbb{R}^n,$$

We denote by I_i and K_i the operators corresponding to $\alpha = 1$.

Lemma 4.9. *Let $n=2$ in Definition 4.8 and $\boldsymbol{\mu}=(\mu_1,\mu_2)\in\mathbb{R}^2$.*

(i) *$I_i(K_i)$ is a continuous linear mapping from $L_{p,\mu}(\mathbb{R}^2)$ into $L_{p,\mu}(\mathbb{R}^2)$ for $\mu_i>0(\mu_i<0)$.*

(ii) *∂_i is a homeomorphism from $D_{p,\mu}(\mathbb{R}^2)$ onto $D_{p,\mu}(\mathbb{R}^2)$ provided $\mu_i\neq 0$ and has inverse $\partial_i^{-1}=I_i$ for $\mu_i>0$ and $\partial_i^{-1}=K_i$ for $\mu_i<0$.*

(iii) *The operator pairs I_1,I_2; K_1,K_2; I_1,K_2 and K_1,I_2 commute on $L_{p,\mu}(\mathbb{R}^2)$ for $\boldsymbol{\mu}>\mathbf{0}$; $\boldsymbol{\mu}<\mathbf{0}$; $\mu_1>0$, $\mu_2<0$ and $\mu_1<0$, $\mu_2>0$ respectively.*

(iv) *For $1<p\leqq 2$, $\psi\in D_{p,\mu}(\mathbb{R}^2)$ and $\mathbf{s}=\boldsymbol{\mu}+i\boldsymbol{\tau}$, $(\mathscr{L}(\partial_i\psi))(\mathbf{s})=s_i(\mathscr{L}\psi)(\mathbf{s})$.*

Proof. Parts (i) and (ii) are given in [6, Lemma 2.4, Theorem 5.1]. In (iii), commutativity of each pair is proved similarly. For example, for $\psi\in L_{p,\mu}(\mathbb{R}^2)$ and $\boldsymbol{\mu}>\mathbf{0}$, $I_1I_2\psi=I_2I_1\psi$ since, by Hölder's inequality

$$I_1I_2|\psi(x_1,x_2)|=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}|\psi(\tau_1,\tau_2)|d\tau$$

$$\leqq\|\psi\|_{p,\mu}\left[\frac{e^{q\boldsymbol{\mu}\cdot\mathbf{x}}}{q^2\mu_1\mu_2}\right]^{1/q}<\infty.$$

Hence, Fubini's theorem can be applied to justify inverting the order of integration in the double integral. To prove (iv), we assume initially that $\psi\in C_0^\infty(\mathbb{R}^2)$. Then

$$(\mathscr{L}(\partial_i\psi))(\mathbf{s})=\int_{\mathbb{R}^2}e^{-\mathbf{s}\cdot\mathbf{x}}\partial_i\psi(\mathbf{x})\,d\mathbf{x}=s_i(\mathscr{L}\psi)(\mathbf{s}),\qquad \mathbf{s}=(s_1,s_2),$$

on integrating by parts. By standard continuity and density arguments, the required result now holds.

We can now prove the main result.

Theorem 4.10. *Every $L_{p,\mu}(\mathbb{R}^2)$ multiplier is a $D_{p,\mu}(\mathbb{R}^2)$ multiplier.*

Proof. Let $g(\mathbf{s})=g(s_1,s_2)$ be an $L_{p,\mu}(\mathbb{R}^2)$ multiplier for $\boldsymbol{\mu}\in\Omega$ with associated operator $Q\in L(L_{p,\mu})$ and let $\phi\in C_0^\infty(\mathbb{R}^2)$ be regarded as an element of $L_{2,\mu}(\mathbb{R}^2)$ with $\mu_i\neq 0$, $i=1,2$. Consider the case $\boldsymbol{\mu}>0$. (The other cases can be treated similarly). By Lemma 4.9, $\partial_1\partial_2\in L_{2,\mu}(\mathbb{R}^2)$ and

$$(\mathscr{L}(Q\partial_1\partial_2\phi))(\mathbf{s})=g(\mathbf{s})(\mathscr{L}(\partial_1\partial_2\phi))(\mathbf{s})=s_1s_2g(\mathbf{s})(\mathscr{L}\phi)(\mathbf{s}).$$

Hence,

$$(\mathscr{L}(Q\phi))(\mathbf{s})=(s_1s_2)^{-1}(\mathscr{L}(Q\partial_1\partial_2\phi))(\mathbf{s})=(\mathscr{L}(\partial_1^{-1}\partial_2^{-1}Q\partial_1\partial_2\phi))(\mathbf{s})$$

so that $Q\phi=I_1I_2Q\partial_1\partial_2\phi$ almost everywhere on $\mathbb{R}^2$, since $\mathscr{L}$ is one–one on $L_{2,\mu}$. We require that this relationship holds everywhere. To prove this, let $R: C_0^\infty(\mathbb{R}^2) \to L_{2,\mu}(\mathbb{R}^2)$ be defined by $R\phi(\mathbf{x}) = I_1 I_2 Q \partial_1 \partial_2 \phi(\mathbf{x})$. The operator R is well-defined for all $\mathbf{x} \in \mathbb{R}^2$, by an argument similar to that used in Lemma 4.9 (iii). Moreover, a straightforward argument establishes that $R \in C(\mathbb{R}^2)$. Next, we prove that $R\phi \in C^\infty(\mathbb{R}^2)$ and that $\partial^{k,l} R\phi = R\partial^{k,l}\phi$, for all $k, l \in \mathbb{N}_0$. Since $(\mathscr{L}(Q\partial_1\phi))(\mathbf{s}) = s_1 g(\mathbf{s})(\mathscr{L}\phi)(\mathbf{s})$, arguing as in the first part of the proof it can be shown that $Q\phi = I_1 I_2 Q \partial_1 \partial_2 \phi = I_1 Q \partial_1 \phi$ almost everywhere on $\mathbb{R}^2$. Thus,

$$R\phi(\mathbf{x}) = \int_{-\infty}^{x_1} (I_2 Q \partial_2 \partial_1 \phi)(\tau_1, x_2)\, d\tau_1$$

$$= \int_{-\infty}^{x_1} (Q\partial_1\phi)(\tau_1, x_2)\, d\tau_1$$

$$= \int_{-\infty}^{x_1} (R\partial_1\phi)(\tau_1, x_2)\, d\tau_1,$$

since $Q\psi = R\psi$ almost everywhere for $\psi \in C_0^\infty(\mathbb{R}^2)$. Then, by the mean value theorem for integrals (see [2; II, p. 138]), $(\partial_1 R\phi)(\mathbf{x}) = (R\partial_1\phi)(\mathbf{x})$ because $R\partial_1\phi$ is continuous. Similarly, $(\partial_2 R\phi)(\mathbf{x}) = (R\partial_2\phi)(\mathbf{x})$ and successive applications of the two results yield $\partial^{k,l} R\phi = R\partial^{k,1}\phi$ as required.

Finally, we must show that R defines a continuous mapping from $D_{p,\mu}$ into $D_{p,\mu}$ whenever $\mu \in \Omega$. This follows since

$$v_\alpha^{p,\mu}(R\phi) = \|\partial^\alpha(E_{-\mu} R\phi)\|_p, \qquad \phi \in C_0^\infty(\mathbb{R}^2),$$

$$= \left\| \sum_{\beta \leqq \alpha} C_{\alpha\beta} \partial^{\alpha-\beta} e^{-\mu\cdot \mathbf{x}} \partial^\beta R\phi(\mathbf{x}) \right\|_p, \qquad \text{by Leibniz' rule,}$$

$$\leqq \sum_{\beta \leqq \alpha} C_{\alpha\beta} |-\mu|^{\alpha-\beta} \|\partial^\beta R\phi\|_{p,\mu}$$

$$= \sum_{\beta \leqq \alpha} C_{\alpha\beta} |-\mu|^{\alpha-\beta} \|R\partial^\beta\phi\|_{p,\mu}, \qquad \text{from above,}$$

$$\leqq \sum_{\beta \leqq \alpha} C_{\alpha\beta} |-\mu|^{\alpha-\beta} \|Q\| \, \|\partial^\beta\phi\|_{p,\mu} \qquad \text{since } R\partial^\beta\phi = Q\partial^\beta\phi \quad \text{a.e.,}$$

$$\leqq \sum_{\beta \leqq \alpha} C_{\alpha\beta} |-\mu|^{\alpha-\beta} \|Q\| \left(\sum_{\gamma \leqq \beta} D_{\beta\gamma} \|\partial^\gamma(E_{-\mu}\phi)\|_p \right),$$

by an induction argument. The constants $C_{\alpha\beta}$ and $D_{\beta\gamma}$ are independent of ϕ and $\|Q\|$ is the operator norm of $Q \in L(L_{p,\mu})$. We have

$$v_\alpha^{p,\mu}(R\phi) \leqq \sum_{\beta \leqq \alpha} C_{\alpha\beta} |-\mu|^{\alpha-\beta} \|Q\| \sum_{\gamma \leqq \beta} D_{\beta\gamma} v_\gamma^{p,\mu}(\phi)$$

so that arguing as in [**2**; II, Theorem 3.3] we obtain the required result. This completes the proof.

From Theorem 4.10, we can deduce that every $L^p(\mathbb{R}^2)$ (Fourier) multiplier is a $D_{L^p}(\mathbb{R}^2)$ (Fourier) multiplier.

The class of $D_{p,\mu}$ multipliers is strictly larger than that of $L_{p,\mu}$ multipliers. For example, multipliers associated with differential operators are now included.

Theorem 4.11. *Let g be an $L_{p,\mu}(\mathbb{R}^2)$ multiplier with associated operator R and let $P(\mathbf{s}) = P(s_1, s_2)$ be a polynomial in the variables s_1 and s_2. Then the function $P(\mathbf{s})g(\mathbf{s})$ is a $D_{p,\mu}(\mathbb{R}^2)$ multiplier with associated operator $P(\partial_1, \partial_2)R = RP(\partial_1, \partial_2)$.*

Proof. This is a straightforward modification of [**2**; II, Theorem 3.6].

A distributional multiplier theory on the dual spaces of $D_{p,\mu}(\mathbb{R}^2)$, denoted by $D'_{p,\mu}(\mathbb{R}^2)$, see [**6**], can now be derived. The key to this lies in the following two results, the first of which is a modified version of Parseval's theorem, [**10**, p. 154].

Lemma 4.12. *For any functions ϕ_1 and ϕ_2 in $L_{2,\mu}(\mathbb{R}^n)$ and $L_{2,-\mu}(\mathbb{R}^n)$ respectively*

$$\int_{\mathbb{R}^n} (\mathscr{L}\phi_1)(\mu + i\tau)(\mathscr{L}\phi_2)(-\mu - i\tau)\, d\tau = (2\pi)^n \int_{\mathbb{R}^n} \phi_1(\mathbf{x})\phi_2(\mathbf{x})\, d\mathbf{x}. \tag{4.2}$$

Proof. By [**10**, p. 154], for $f_1, f_2 \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (\mathscr{F} f_1)(\mathbf{u})(\mathscr{F} f_2)(-\mathbf{u})\, d\mathbf{u} = (2\pi)^n \int_{\mathbb{R}^n} f_1(\mathbf{x}) f_2(\mathbf{x})\, d\mathbf{x}.$$

Hence, by (4.1), the identity (4.2) can be deduced.

Lemma 4.13. *Let g be an $L_{p,\mu}$ multiplier for $\mu \in \Omega$. Then g^-, defined by $g^-(\mathbf{s}) = g(-\mathbf{s})$, is an $L_{p,\mu}$ multiplier for $-\mu \in \Omega$. Moreover, if g has associated operator R then g^- has associated operator R', the $L_{p,\mu}$ adjoint of R.*

Proof. As in [**2**; I, Theorem 4.5] we show first that g^- generates an operator $\hat{R} \in L(L_{p,\mu})$ for $-\mu \in \Omega$ and then verify that $\hat{R}\psi = R'\psi$ for all $\psi \in C_0^\infty(\mathbb{R}^2)$. We define $\hat{R}$ by $\hat{R} = URU$ where U is given by $U\phi(\tau) = \phi(-\tau)$ and is a homeomorphism from $L_{p,\mu}$ onto $L_{p,-\mu}$ for all $\mu \in \mathbb{R}^2$ with inverse $U^{-1} = U$. Thus, $\hat{R} \in L(L_{p,\mu})$ for $-\mu \in \Omega$. Moreover, for $\psi \in L_{p,\mu}$, $1 < p \leqq 2$ and $-\mu \in \Omega$,

$$(\mathscr{L}(\hat{R}\psi))(\mathbf{s}) = (\mathscr{L}(URU\psi))(\mathbf{s}) = (\mathscr{L}(RU\psi))(-\mathbf{s}) = g(-\mathbf{s})(\mathscr{L}\psi)(\mathbf{s})$$

since g is an $L_{p,-\mu}$ multiplier for $-\mu \in \Omega$ with associated operator R. For $-\mu \in \Omega$, both

R' and $\hat{R}$ belong to $L(L_{p,\mu})$ so that $\hat{R}=R'$ provided $\hat{R}\phi=R'\phi$ for all $\phi\in C_0^\infty(\mathbb{R}^2)$. For the case $p=q=2$, apply (4.2) to each side of the identity

$$\int_{\mathbb{R}^2} R\psi(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x}=\int_{\mathbb{R}^2}\psi(\mathbf{x})R'\phi(\mathbf{x})\,d\mathbf{x},\qquad \phi\in L_{2,\mu},\ \psi\in L_{2,-\mu}$$

to give

$$\int_{\mathbb{R}^2}(\mathscr{L}\phi)(\mu+i\tau)g(-\mathbf{s})(\mathscr{L}\psi)(-\mu-i\tau)\,d\tau=\int_{\mathbb{R}^2}(\mathscr{L}R'\phi)(\mu+i\tau)(\mathscr{L}\psi)(-\mu-i\tau)\,d\tau.$$

Since the range of $\mathscr{L}$ is the whole of $L^2(\mathbb{R}^2)$, it follows that $(\mathscr{L}R'\phi)(\mathbf{s})=g(-\mathbf{s})(\mathscr{L}\phi)(\mathbf{s})$, for $-\mu\in\Omega$ and $\phi\in\mathscr{L}_{2,\mu}$ and, because R' coincides with $\hat{R}$ on $C_0^\infty(\mathbb{R}^2)$ we obtain the required result. This completes the proof.

We can now deduce the desired result for $D'_{p,\mu}(\mathbb{R}^2)$.

Theorem 4.14. *Let g be an $L_{p,\mu}(\mathbb{R}^2)$ multiplier for $\mu\in\Omega$ with associated operator R. Then the extension, $\tilde{R}$, of R from $L_{p,-\mu}(\mathbb{R}^2)$ to $D'_{p,\mu}(\mathbb{R}^2)$ exists in $L(D'_{p,\mu}(\mathbb{R}^2))$ for each $-\mu\in\Omega$.*

Proof. In the notation of [**6**], we note that $\langle\tilde{R}f,\phi\rangle=\langle f,R'\phi\rangle$ for $\phi\in D_{p,\mu}$ and $f\in D'_{p,\mu}$. By Lemma 4.13, $R'\in L(L_{p,\mu})$ for $-\mu\in\Omega$ and, from Theorem 4.10, we deduce that $R'\in L(D_{p,\mu})$ for $-\mu\in\Omega$. The required result follows.

If g is a $D_{p,\mu}$ bilateral Laplace multiplier, but not an $L_{p,\mu}$ multiplier, it is still possible to obtain a distributional extension for the associated operator by proceeding along the lines of [**2**; II, Lemma 4.16]. We omit the details.

5. An application to the Riesz fractional integrals $I^{2\alpha}$ and $K^{2\alpha}$

For suitable functions ϕ and $\operatorname{Re}\alpha>0$, we define $I^{2\alpha}$, the Riesz fractional integral associated with the one-dimensional wave operator $\square=\partial^2/\partial t^2-\partial^2/\partial x^2$, by

$$I^{2\alpha}\phi(x;t)=[2^{2\alpha-1}\Gamma^2(\alpha)]^{-1}\int_{V(P)}R^{\alpha-1}\phi(\xi;\tau)\,d\xi\,d\tau, \tag{5.1}$$

where $R=(t-\tau)^2-(x-\xi)^2$ and $V(P)=\{(\xi;\tau):R\geqq 0,\tau<t\}$. The operator $I^{2\alpha}$ forms an adjoint pair with $K^{2\alpha}$ which is defined by (5.1) on replacing $V(P)$ by $V'(P)$ where $V'(P)=\{(\xi;\tau):R\geqq 0,\ \tau>t\}$. These fractional integrals are used in the solution of the wave equation by Riesz's method; see, for example, [**6**] where a distributional version of $I^{2\alpha}$ is applied to solve the generalized half-space Cauchy problem for the one-dimensional wave equation. We show how properties required for the application of Riesz's method may be derived using bilateral Laplace multiplier theory.

We require decompositions of $I^{2\alpha}$ and $K^{2\alpha}$ into the forms [**6**],

$$I^{2\alpha}\phi=2^{-\alpha}T_Q I_1^\alpha I_2^\alpha T_Q^{-1}\phi, \tag{5.2}$$

$$K^{2\alpha}\phi = 2^{-\alpha} T_Q K_1^\alpha K_2^\alpha T_Q^{-1}\phi \tag{5.3}$$

for $\operatorname{Re}\alpha > 0$ and suitable ϕ, where I_i^α and K_i^α are given by Definition 4.8 with $n=2$ and $T_Q\phi(\mathbf{x}) = \phi(Q\mathbf{x})$ with

$$Q = 2^{-1/2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and $T_Q^{-1} = T_{Q'}$, where Q' denotes the transpose of Q. We also require a preliminary lemma concerning the action of $\mathscr{L}$ on the operators T_Q and T_Q^{-1}.

Lemma 5.1. *Let g be an $L_{p,\mu}(\mathbb{R}^2)$ bilateral Laplace multiplier, for $\mu \in \Omega$, with associated operator R and let h be defined on $Q^{-1}\Omega^{\#} = \{\mathbf{s} : Q\mathbf{s} \in \Omega^{\#}\}$ by $h(s) = g(Q\mathbf{s})$. Then h is an $L_{p,\mu}$ bilateral Laplace multiplier for each $\mu \in Q^{-1}\Omega$ and has associated operator $R_1 = T_Q R T_Q^{-1}$.*

Proof. By [6, Lemma 2.3], T_Q is a homeomorphism from $L_{p,\omega}(\mathbb{R}^2)$ onto $L_{p,\mu}(\mathbb{R}^2)$ with inverse T_Q^{-1}, where $\omega = Q\mu$. Hence, R_1 is a continuous operator on $L_{p,\mu}$ for each $p \in (1,\infty)$ and $\mu \in Q^{-1}\Omega$. Moreover, for $\psi \in L_{1,\mu} \cap L_{p,\mu}$, $1 < p \leqq 2$, where $\mu \in Q^{-1}\Omega$, we have

$$\begin{aligned}
(\mathscr{L}(R_1\psi))(\mathbf{s}) &= \int_{\mathbb{R}^2} e^{-\mathbf{s}\cdot\mathbf{x}}(T_Q R T_Q^{-1}\psi)(\mathbf{x})\,d\mathbf{x}, \qquad \mathbf{s} \in Q^{-1}\Omega^{\#}, \\
&= |\det Q|^{-1} \int_{\mathbb{R}^2} e^{-\mathbf{s}\cdot Q^{-1}\mu}(R\chi)(\mu)\,d\mu, \qquad \chi(\mathbf{y}) = \psi(Q^{-1}\mathbf{y}), \\
&= |\det Q|^{-1} \int_{\mathbb{R}^2} e^{-Q\mathbf{s}\cdot\mu}(R\chi)(\mu)\,d\mu, \qquad \text{since } Q^{-1} = Q', \\
&= |\det Q|^{-1} g(Q\mathbf{s})(\mathscr{L}\chi)(Q\mathbf{s}) \\
&= g(Q\mathbf{s})(\mathscr{L}\psi)(\mathbf{s}).
\end{aligned}$$

Using standard arguments, this can be extended to hold for $\psi \in L_{p,\mu}$, for all $p : 1 < p \leqq 2$. This completes the proof.

Instead of considering I_i^α and K_i^α directly we use the related direct products $R^\alpha \otimes R^\alpha$ and $W^\alpha \otimes W^\alpha$ which are defined on the tensor product spaces $L_{p,\mu} \otimes L_{p,\mu}$. The operators R^α and W^α denote the Riemann–Liouville and Weyl fractional integrals defined, for $\operatorname{Re}\alpha > 0$, by

$$(R^\alpha\phi)(t) = [\Gamma(\alpha)]^{-1} \int_{-\infty}^{t} (t-\tau)^{\alpha-1}\phi(\tau)\,d\tau$$

and

$$(W^\alpha\phi)(t) = [\Gamma(\alpha)]^{-1} \int_{t}^{\infty} (\tau-t)^{\alpha-1}\phi(\tau)\,d\tau.$$

By [1] and the identity

$$(\mathscr{F}(E_{-\mu}R^{\alpha}\psi))(-\tau)=(\mathscr{L}(R^{\alpha}\psi))(s)=g(s)(\mathscr{L}\psi)(s)=g_{\mu}(\tau)(\mathscr{F}E_{-\mu}\psi))(-\tau), g(s)=s^{-\alpha} \quad (5.4)$$

we deduce that for $\mu>0$ and $\operatorname{Re}\alpha>0$, $g(s)$ is an $L_{p,\mu}$ bilateral Laplace multiplier with associated operator R^{α}. Hence, for $\mu<0$ and $\operatorname{Re}\alpha>0$, $g^{-}(s)=g(-s)$ is an $L_{p,\mu}$ bilateral Laplace multiplier with associated operator W^{α}. It follows, by Theorem 4.10, that g and g^{-} also define $D_{p,\mu}$ bilateral Laplace multipliers under the stated restrictions on μ and α.

By [**1**, Theorem 4.3], $g\in\mathscr{A}$ for $\operatorname{Re}\alpha>0$ with $\Omega=(0,\infty)$. Hence, for $\boldsymbol{\mu}>\mathbf{0}$, by Corollary 4.6, G_{α} defined by $G_{\alpha}(\mathbf{s})=g(s_1)g(s_2)=s_1^{-\alpha}s_2^{-\alpha}$, is a two-dimensional bilateral Laplace multiplier with associated operator $E_{\mu}(E_{-\mu_1}R^{\alpha}E_{\mu_1})\otimes(E_{-\mu_2}R^{\alpha}E_{\mu_2})E_{-\mu}=R^{\alpha}\otimes R^{\alpha}$. Similarly, $(-s_1)^{-\alpha}(-s_2)^{-\alpha}$ is a two-dimensional bilateral Laplace multiplier for $\boldsymbol{\mu}<\mathbf{0}$ and $\operatorname{Re}\alpha>0$ with associated operator $W^{\alpha}\otimes W^{\alpha}$. Moreover, it is easily verified that the extensions of $R^{\alpha}\otimes R^{\alpha}$ and $W^{\alpha}\otimes W^{\alpha}$ to the whole of $L_{p,\mu}(\mathbb{R}^2)$ coincide with the operators $I_1^{\alpha}I_2^{\alpha}$ and $K_1^{\alpha}K_2^{\alpha}$ respectively.

In view of Lemma 5.1, we conclude that $I^{2\alpha}$ and $K^{2\alpha}$ belong to $L(L_{p,\mathbf{v}}(\mathbb{R}^2))$ for $\mathbf{v}>\mathbf{0}$ and $\mathbf{v}<\mathbf{0}$ respectively where $\mathbf{v}=Q\boldsymbol{\mu}=2^{-1/2}(\mu_1+\mu_2,\mu_2-\mu_1)$. By choosing $\mu_1=-2^{-1/2}\mu$ and $\mu_2=2^{-1/2}\mu$ we find that $I^{2\alpha}$ and $K^{2\alpha}$ define continuous linear mappings on $L_{p,\mu}(\mathbb{R}^2)\equiv L_{p,(0,\mu)}(\mathbb{R}^2)$ for $\mu>0$ and $\mu<0$ respectively.

The index laws for $I^{2\alpha}$ and $K^{2\alpha}$ on $L_{p,\mu}(\mathbb{R}^2)$ are now readily established. For example, $I^{2\alpha}$ has multiplier $h_{\alpha}(\mathbf{s})=2^{-\alpha}G_{\alpha}(Q\mathbf{s})=(s_2^2-s_1^2)^{-\alpha}$ and clearly, $h_{\alpha}(\mathbf{s})h_{\beta}(\mathbf{s})=h_{\alpha+\beta}(\mathbf{s})$ so that $I^{2\alpha}I^{2\beta}=I^{2\alpha+2\beta}$ as operators on $L_{p,\mu}$, for all $\mu>0$. By Theorem 4.10, it is clear that h_{α} and h_{α}^{-} are $D_{p,\mu}(\mathbb{R}^2)$ $(\equiv D_{p,(0,\mu)}(\mathbb{R}^2))$ multipliers for $\mu>0$ and $\mu<0$ respectively. In the $D_{p,\mu}$ setting, the associated operators $I^{2\alpha}$ and $K^{2\alpha}$ can be extended to negative values of α by means of Theorem 4.11. For example, choosing N: $\operatorname{Re}(\alpha+N)>0$, $I^{2\alpha}$ is defined by $\square^N I^{2(\alpha+N)}=I^{2(\alpha+N)}\square^N$ and has $D_{p,\mu}(\mathbb{R}^2)$ bilateral Laplace multiplier given by

$$P(\mathbf{s})h_{\alpha+N}(\mathbf{s})=2^{-(\alpha+N)}(s_2^2-s_1^2)^N G_{\alpha+N}(Q\mathbf{s})$$

$$=(s_2^2-s_1^2)^N(s_2^2-s_1^2)^{-(\alpha+N)}, \qquad \mu_1=0,\ \mu_2=\mu,\ \mu>0.$$

Finally, we can obtain properties of the extended operators $\tilde{I}^{2\alpha}$ and $\tilde{K}^{2\alpha}$ on $D'_{p,\mu}$, for $\mu<0$ and $\mu>0$ respectively, by using Theorem 4.14. In particular, $h_{\alpha}(-\mathbf{s})=(s_2^2-s_1^2)^{-\alpha}$, $\mu_1=0$, $\mu_2=\mu$, is a bilateral Laplace multiplier for $\mu<0$ with associated operator $K^{2\alpha}$. Hence, $\tilde{I}^{2\alpha}$, defined by $\langle\tilde{I}^{2\alpha}f,\phi\rangle=\langle f,K^{2\alpha}\phi\rangle$, $\phi\in D_{p,\mu}$, $f\in D'_{p,\mu}$ is in $L(D'_{p,\mu}(\mathbb{R}^2))$ for $\mu<0$.

In higher dimensions there is no equivalent decomposition for the Riesz fractional integrals so that the above techniques cannot be applied and a more general multidimensional multiplier theory is required.

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