Notes on an Orthocentric Triangle.

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Figure 18.

1. In the accompanying figure DEF is the pedal triangle of ABC and P, Q, R are the orthocentres of AFE, BDF, CED.

AP, BQ, CR evidently meet in the circumcentre, O, of ABC, which is the orthocentre of PQR,

2. Now the circumradius of $AFE(\rho_a) = R\cos A$,

$$\therefore\ EP = 2\rho_a\cos B = 2R\cos A\cos B$$
$$= FH = DQ:$$

hence the sides of PQR are equal and parallel to the sides of DEF, *i.e.*, the triangles are congruent, and their centre of perspective, L, bisects DP, EQ, FR.

The hexagon DQFPER = twice the pedal triangle.

3. The coordinates of P, D are respectively

$$R[\cos A - \cos 2A\cos(B-C)],\quad 2R\cos^2 A\cos B,\quad 2R\cos^2 A\cos C;$$
$$0\quad,\quad R\sin 2C\sin B\ ,\quad R\sin 2B\sin C\ ;$$

hence the coordinates of L are as

$$\cos A - \cos 2A\cos(B-C),\quad \cos B - \cos 2B\cos(C-A),$$
$$\cos C - \cos 2C\cos(A-B),\quad \text{or}\quad \sin A.\Pi\sin A - \cos A.\Pi\cos A,\ \ldots,\ \ldots,^*$$

4. The perpendicular from P on $EF = 2R\cos A\cos B\cos C$ = the perpendicular from D on QR; and so for the other vertices and sides.

5. The equation to the circle PQR is

$$\sin A\sin B\sin C.\Sigma\alpha\beta\gamma = \Sigma a\alpha.\Sigma\alpha\sin A\cos^2 A(1+2\cos 2B\cos 2C)$$

and the α – coordinate of the centre is

$$R[\cos C\sin(2A-C) + \cos B\sin(2A-B)]/2\sin A.$$

The equation to PQ is

$$\alpha\sin 2A\cos 2B\cos A + \beta\cos 2A\sin 2B\cos B + \gamma\sin C(1+\cos 2A\cos 2B) = 0.$$

* See *Proceedings*, London Mathematical Society, Vol. xv., Appendix.)

https://doi.org/10.1017/S0013091500001784 Published online by Cambridge University Press

6. If v is the incentre of PQR then

$$Pv = DH = 2R\cos B\cos C,$$

and the α – ordinate of v is $2R\cos A(\cos^2 B + \cos^2 C)$:
hence vLH is a straight line, as is also readily seen from the symmetry of the figure, L being the mid point of vH.

7. The α – ordinate of H , the orthocentre of DEF, is (*l.c.*),

$$-R\cos 2A\cos(B - C),$$

hence OLH′ is a straight line, in fact it is the circum-Brocard axis of ABC.

8. If OP, OQ, OR cut EF, FD, DE in X, Y, Z,

then since $$\frac{EX}{XF} = \frac{\tan C}{\tan B},$$

it is seen that DX, EY, FZ cointersect in a point, viz., in v.

9. If a side, as PR, is cut by EF, ED in x, y, then

$$Px : xy : yR = \sin 2A : \sin 2B : \sin 2C.$$

[Since the above Note was written, vol. i. of the Society's *Proceedings* has been published. I must refer readers to Figs. 58, 59, from which it will be seen that some of my points were noted in that admirable piece of Geometry by Dr Mackay.]