FURTHER IDENTITIES AND CONGRUENCES FOR THE COEFFICIENTS OF MODULAR FORMS

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I. If n is a non-negative integer, define $p_r(n)$ by

$$\sum p_r(n)x^n = \prod (1 - x^n)^r;$$

otherwise define $p_r(n)$ as 0. (Here and in what follows all sums will be extended from 0 to ∞ and all products from 1 to ∞ unless otherwise stated.) $p_r(n)$ is thus generated by the powers of $x^{-1/24}\eta(\tau)$, where

$$\eta(\tau) = \exp(\pi i\tau/12) \prod (1 - x^n), x = \exp 2\pi i\tau,$$

is the Dedekind modular form. In **(1)** it was shown that recurrence formulas for these coefficients depending on a parameter p, p a prime, exist for all positive integral r. The number of terms in these recurrence formulas is in general a function of r and p, which is determined in **(1)**. If r is even, $0 < r \leqslant 26$, it was shown in **(2)**, **(3)** that *three* term recurrence formulas exist for these coefficients for p satisfying appropriate congruence conditions with respect to 24 as modulus. These include, for example, Mordell's identity for $\tau(n) = p_{24}(n - 1)$:

$$\tau(np) = \tau(n)\tau(p) - p^{11}\tau(n/p).$$

$p_r(n)$ bears some relation to the function $q_r(n)$, the number of representations of n as a sum of r squares. If

$$n = \sum_{k=1}^{r} \tfrac{1}{2}(3x_k^2 \pm x_k)$$

is a representation of n as a sum of r pentagons, then $p_r(n)$ is the excess of the number of those representations of n in which

$$\sum_{k=1}^{r} x_k$$

is even over those in which it is odd. Since the associated modular form is of fractional dimension when r is odd and of integral dimension when r is even, identities for odd r lie deeper than identities for even r; and indeed quadratic reciprocity symbols appear. A good example is furnished by the identity

$$(1)\qquad q_3(np^2) = \left\{p + 1 - \left(\frac{-n}{p}\right)\right\}q_3(n) - \left\{p - \left(\frac{-n}{p}\right)\right\}q_3\left(\frac{n}{p^2}\right)$$

given by G. Pall in **(7)**.

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In this paper we study the coefficients $p_r(n)$ for r odd, $0 < r < 24$. We shall demonstrate the existence of identities of type (1) for all primes $p > 3$, and for $p = 3$ when r is a multiple of 3. Most of the discussion that follows depends upon **(1)**, and we assume familiarity with the contents of this paper.

After this paper was written the author received from J. H. van Lint a copy of his dissertation, "Hecke Operators and Euler Products" (October 1957, University of Utrecht), which contains a proof of formulas (5) and (11) of the next section. (There are minor inaccuracies in van Lint's expression for formula (5).) van Lint's proof is based upon properties of modular *forms* while the author's is based upon properties of modular *functions*. The methods are quite different and yield different results in general.

II. Let p be a prime. If $g(\tau)$ is a function on $\Gamma_0(p)$, we say that $g(\tau)$ is *entire* if it is regular in the interior of the upper τ half-plane and has polar singularities at most in appropriate uniformizing variables at the two parabolic vertices $\tau = 0, i\infty$ of the fundamental region of $\Gamma_0(p)$. We require the following lemma:

LEMMA 1. *If $g(\tau)$ is a function on $\Gamma_0(p)$, then so is $g(-1/p\tau)$. If in addition $g(\tau)$ is entire, then so is $g(-1/p\tau)$.*

Proof. The second statement is clear, since the substitution $\tau' = -1/p\tau$ permutes the parabolic points $\tau = 0, i\infty$ and takes interior points of the upper τ half-plane into interior points of the upper τ half-plane. To prove the first, let

$$M = \begin{bmatrix} a & b \\ pc & d \end{bmatrix}$$

belong to $\Gamma_0(p)$, and let

$$T_p = \begin{bmatrix} 0 & -1 \\ p & 0 \end{bmatrix}$$

be the matrix of the transformation $\tau' = -1/p\tau$. Then

$$T_p M T_p^{-1} = \begin{bmatrix} d & -c \\ -pb & a \end{bmatrix} = M_0,$$

where M_0 also belongs to $\Gamma_0(p)$.

Suppose now that $g(\tau)$ is a function on $\Gamma_0(p)$, and put $f(\tau) = g(-1/p\tau) = g(T_p\tau)$. Then $f(M\tau) = g(T_pM\tau) = g(M_0T_p\tau) = g(T_p\tau) = f(\tau)$, so that $f(\tau)$ is also a function on $\Gamma_0(p)$. The lemma is therefore proved.

As in **(1)** we write $T_pg(\tau) = g(T_p\tau)$.

Following the notation of **(1)**, let p be a prime > 3, and Q a power of p. Define

$$\epsilon = \begin{cases} p & Q \text{ a square} \\ 1 & \text{otherwise,} \end{cases}$$

and set

$$h(\tau) = \frac{\eta(pQ\tau)}{\eta(\epsilon\tau)}.$$

Let

$$R_n = \begin{bmatrix} 1 & 0 \\ -np & 1 \end{bmatrix}.$$

Then if r is an integer, it is shown in **(1)** that the function

$$F(r, p, Q; \tau) = \sum_{n=0}^{Q-1} h^r(R_n\tau)$$

is an entire modular function on $\Gamma_0(p)$. Define

$$G(r, p, Q; \tau) = T_p F(r, p, Q; \tau).$$

By Lemma 1, $G(r, p, Q; \tau)$ is also an entire modular function on $\Gamma_0(p)$. It is shown in **(1)** that

$$G(r, p, Q; \tau) = \left(\frac{pQ}{\epsilon}\right)^{-r/2} \eta^{-r}\left(\frac{p\tau}{\epsilon}\right) \sum_{n=0}^{Q-1} \eta^r\left(\frac{\tau + 24n}{Q}\right).$$

We write $n{:}Q$ in a summation to indicate that n runs over a reduced set of residues mod Q. We shall prove the following lemma:

LEMMA 2. *Suppose that Q is a square, and put $Q' = Q/p$. Then*

$$F(r, p, Q; \tau) + G(r, p, Q'; \tau) = F(r, p, Q'; p\tau) + G(r, p, Q; p\tau).$$

Proof. Put

$$g_n = h^r(R_n\tau) = \left\{\frac{\eta(pQR_n\tau)}{\eta(pR_n\tau\,)}\right\}^r.$$

Then

$$F(r, p, Q; \tau) = \sum_{n=0}^{Q-1} g_n = \sum_{n:Q} g_n + \sum_{n=0}^{Q'-1} g_{np}.$$

Now

$$\frac{\eta(pQR_{np}\tau)}{\eta(pR_{np}\tau)} = \frac{\eta(pQ'R_n p\tau)}{\eta(R_n p\tau)},$$

which implies that

$$F(r, p, Q; \tau) = \sum_{n:Q} g_n + F(r, p, Q'; p\tau).$$

Thus we need only consider $\sum_{n:Q} g_n$. This sum is treated in **(1)**, where it is shown by means of the transformation formula for the Dedekind η-function that

$$\sum_{n:Q} g_n = Q^{-r/2}\eta^{-r}(p\tau) \sum_{n:Q} \eta^r\left(\frac{p\tau + 24n}{Q}\right).$$

Transforming this sum by means of the identity

$$\sum_{n:Q} f(n) = \sum_{n=0}^{Q-1} f(n) - \sum_{n=0}^{Q'-1} f(np)$$

we find easily that

$$\sum_{n:Q} g_n = G(r, p, Q; p\tau) - G(r, p, Q'; \tau).$$

The lemma is thus proved.

The functions so defined are also entire modular functions on $\Gamma_0(p)$ when $p = 3$, if r is a multiple of 3. We assume from now on that r is odd, $0 < r < 24$; and that p is a prime such that $p > 3$ when $(r, 3) = 1$ and $p > 2$ when $3|r$. We put

$$\nu = \frac{(p^2 - 1)}{24}, \mu = \left[\frac{r\nu}{p}\right], \delta = r\nu - p\mu;$$

and define

$$\alpha_p = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ i & p \equiv 3 \pmod 4 \end{cases}.$$

LEMMA 3. *The function*

$$f = F(r, p, p^2; \tau) + G(r, p, p; \tau)$$

is constant.

Proof. From **(3)**, formula (2.5.2) and **(1)**, page 354 we have

(2) $$F(r, p, p; \tau) = x^{r\nu} \prod (1 - x^{np^2})^r (1 - x^n)^{-r} +$$
$$a p^{(1-r)/2} \prod (1 - x^n)^{-r} \sum \left(\frac{r\nu - n}{p}\right) p_r(n) x^n,$$

where $a = \alpha_p \exp\{-i\pi r(p-1)/4\}$, and

$$\left(\frac{r\nu - n}{p}\right)$$

is the Legendre-Jacobi symbol of quadratic reciprocity; and

(3) $$G(r, p, p; \tau) = p^{1-r} x^{-\mu} \prod (1 - x^{np})^{-r} \sum p_r(np + \delta) x^n.$$

Similarly, from **(1**, p. 354**)** we have (since $r\nu < p^2$)

(4) $$G(r, p, p^2; \tau) = p^{2-r} \prod (1 - x^n)^{-r} \sum p_r(np^2 + r\nu) x^n.$$

(We take this opportunity to correct an error in the second displayed formula for $T_p F$ on page 354 of **(1)**. The coefficient should be $Q(pQ/\epsilon)^{-r/2}$ instead of $p(pQ/\epsilon)^{-r/2}$.)

From Lemma 2 with $Q = p^2$ we have that

$$f = F(r, p, p; p\tau) + G(r, p, p^2; p\tau),$$

which is regular at $\tau = i\infty$ by formulas (2) and (4). In addition,

$$T_p f = F(r, p, p; \tau) + G(r, p, p^2; \tau)$$

so that (2) and (4) imply that f is regular at $\tau = 0$ as well. Since f is an entire modular function on $\Gamma_0(p)$, this implies that f is constant, proving the lemma.

If we consider the expansion of $T_p f$ in powers of x as in **(1)** we obtain our principal result, by comparing coefficients of like powers of x:

THEOREM 1. *For all integral n,*

$$p_r(np^2 + r\nu) - \gamma_n p_r(n) + p^{r-2} p_r\left(\frac{n - r\nu}{p^2}\right) = 0, \tag{5}$$

where

$$\gamma_n = c - \left(\frac{r\nu - n}{p}\right) p^{(r-3)/2} a \quad \textit{and} \quad c = p_r(r\nu) + \left(\frac{r\nu}{p}\right) p^{(r-3)/2} a.$$

If in this identity n is replaced by $np + \delta = np + r\nu - p\mu$,

$$\left(\frac{r\nu - n}{p}\right)$$

vanishes since $p | r\nu - n$ and we obtain

COROLLARY 1. *Put $\Delta = p^2\delta + r\nu$. Then for all integral n,*

$$p_r(np^3 + \Delta) - c p_r(np + \delta) + p^{r-2} p_r\left(\frac{n - \mu}{p}\right) = 0. \tag{6}$$

This identity is equivalent to the statement that the functions 1, $F(r, p, p; \tau)$, $F(r, p, p^3; \tau)$ are linearly dependent. Another expression for c, obtained by choosing $n = 0$ in (6), is

$$c = \frac{p_r(\Delta)}{p_r(\delta)}.$$

We also have

COROLLARY 2. *If $n - r\nu$ is not divisible by p^2 then*

$$p_r(np^2 + r\nu) = \gamma_n p_r(n).$$

We go on now to some applications of Theorem 1. Suppose that $r \geqslant 5$. Then $\gamma_n \equiv c \equiv p_r(r\nu) \pmod p$, so that

$$p_r(np^2 + r\nu) \equiv p_r(r\nu) p_r(n) \pmod p, \qquad r \geqslant 5. \tag{7}$$

We choose $r = 11$, $p = 13$ in (7) as a significant example. Then from **(4)**, $p_r(r\nu) = p_{11}(77) = -16257 \equiv 6 \pmod{13}$, so that

$$p_{11}(13^2 n + 77) \equiv 6 p_{11}(n) \pmod{13}. \tag{8}$$

It is known **(5; 8)** that

$$p(13n + 6) \equiv 11 p_{11}(n) \pmod{13}. \tag{9}$$

Combining (8) and (9), we obtain the following congruence for the partition function mod 13, already given in **(5)**:

COROLLARY 3. If $n \equiv 6 \pmod{13}$, then

$$p(13^2 n - 7) \equiv 6p(n) \pmod{13}.$$

We can also obtain a general congruence mod p from (7), similar to those given in (**5**; **6**).

THEOREM 2. *Suppose that $r \geqslant 5$. Let q be an arbitrary integer, and set $R = qp^2 + r$. Then for all integral n,*

$$p_R(np^2 + r\nu) \equiv p_r(r\nu)p_{q+r}(n) \pmod{p}. \tag{10}$$

Proof. We have

$$\sum p_R(n)x^n = \prod (1 - x^n)^{qp^2+r}$$
$$\equiv \prod (1 - x^{np^2})^q(1 - x^n)^r \pmod{p}.$$

Thus

$$p_R(n) \equiv \sum_{0 \leqslant k \leqslant n/p^2} p_q(k)p_r(n - p^2 k) \pmod{p}.$$

Replace n by $np^2 + r\nu$. Since $r\nu < p^2$, we obtain

$$p_R(np^2 + r\nu) \equiv \sum_{k=0}^{n} p_q(k)p_r((n - k)p^2 + r\nu) \pmod{p}.$$

Formula (7) now implies that

$$p_R(np^2 + r\nu) \equiv p_r(r\nu) \sum_{k=0}^{n} p_q(k)p_r(n - k) \pmod{p},$$

so that $p_R(np^2 + r\nu) \equiv p_r(r\nu)p_{q+r}(n) \pmod{p}$, which is just (10).

As another application we prove

THEOREM 3. *For all odd n,*

$$p_{15}\left(53n^2 + \frac{5}{8}(n^2 - 1)\right) = 0. \tag{11}$$

Proof. The proof is by induction on the total number of prime factors of n. For $n = 1$, (11) states that $p_{15}(53) = 0$, which is actually the case (**4**). Suppose (11) proved for all integers with not more than t prime factors. Let p be an odd prime. Then if n has precisely t prime factors, it will suffice to prove (11) for pn. Put

$$a_n = 53n^2 + \frac{5}{8}(n^2 - 1).$$

Then

$$a_{pn} = p^2 a_n + \frac{5}{8}(p^2 - 1),$$

and Theorem 1 implies (with $r = 15$) that $p_{15}(a_{pn})$ is linear in $p_{15}(a_n)$ and $p_{15}(a_{n/p})$. Now $p_{15}(a_n)$ vanishes by the induction hypothesis, and so does

$p_{15}(a_{n/p})$ if $p|n$. If $p \nmid n$, however, $a_{n/p}$ is not an integer (since 429 is square-free) and so $p_{15}(a_{n/p})$ vanishes in this instance as well. Thus $p_{15}(a_{pn}) = 0$ and the proof is complete.

We now prove

THEOREM 4. *Suppose that a is such that for the mod m,* $p_r(a) \equiv 0 \pmod{m}$. *Suppose further that* $24a + r$ *is square-free. Then*

$$p_r\left(an^2 + \frac{r}{24}(n^2 - 1)\right) \equiv 0 \pmod{m}, \tag{12}$$

where $(n, 2) = 1$ *if* $3|r$ *and* $(n, 6) = 1$ *otherwise.*

Proof. As in Theorem 3, the proof is by induction on the total number of prime factors of n. If $n = 1$, (12) states that $p_r(a) \equiv 0 \pmod{m}$, which is true by hypothesis. Suppose (12) proved for all integers with not more than t prime factors. Let p be a prime such that $p > 3$ when $(r, 3) = 1$ and $p > 2$ otherwise. Then if n has precisely t prime factors, it will suffice to prove (12) for pn. Put

$$\lambda_n = an^2 + \frac{r}{24}(n^2 - 1).$$

Then

$$\lambda_{pn} = p^2\lambda_n + \frac{r}{24}(p^2 - 1),$$

and Theorem 1 implies that $p_r(\lambda_{pn})$ is linear in $p_r(\lambda_n)$ and $p_r(\lambda_{n/p})$. Now $p_r(\lambda_n) \equiv 0 \pmod{m}$ by hypothesis, and the same is true for $p_r(\lambda_{n/p})$ if $p|n$. If $p \nmid n$ however, $\lambda_{n/p}$ is not at integer since $24a + r$ is square-free, and so $p_r(\lambda_{n/p})$ vanishes. Thus $p_r(\lambda_{pn}) \equiv 0 \pmod{m}$ in either case, and the proof of Theorem 4 is complete.

Theorem 4 can be strengthened slightly by discarding the condition that $24a + r$ be square-free and restricting n to be divisible only by primes p such that $p > 2$ when $3|r$, $p > 3$ when $(r, 3) = 1$, and $p^2 \nmid 24a + r$.

If we choose $r = 11$, $m = 13$ and $a = 6$ we find from **(4)** that $p_r(a) = p_{11}(6) = -143 \equiv 0 \pmod{13}$, while $24a + r = 155$ is square-free. Theorem 4 applies and we have

$$p_{11}\left(6n^2 + \frac{11}{24}(n^2 - 1)\right) \equiv 0 \pmod{13}, \qquad (n, 6) = 1. \tag{13}$$

Using formula (9) once again, we obtain the following interesting congruence for the partition function mod 13:

$$p\left(84n^2 - \frac{1}{24}(n^2 - 1)\right) \equiv 0 \pmod{13}, \qquad (n, 6) = 1. \tag{14}$$

Formula (14) is a Ramanujan congruence for the partition function, with the difference that the terms form a quadratic, rather than an arithmetic, progression.

More generally, we have

THEOREM 5. *Suppose that* $p_{11}(a) \equiv 0 \pmod{13}$, *and that* $24a + 11$ *is square-free. Then*

$$p_{11}\left(an^2 + \frac{11}{24}(n^2 - 1)\right) \equiv 0 \pmod{13}, \qquad (n, 6) = 1, \tag{15}$$

$$p\left((13a + 6)n^2 - \frac{1}{24}(n^2 - 1)\right) \equiv 0 \pmod{13}, \qquad (n, 6) = 1. \tag{16}$$

The first few admissible a's are 6, 10, 17, 18, 24, 27, 57, 68, 69, 74, 90, 95. (This information is extracted from **(4)**.) It is of interest to note that two progressions

$$\left\{a_1 n^2 + \frac{11}{24}(n^2 - 1)\right\}, \qquad \left\{a_2 n^2 + \frac{11}{24}(n^2 - 1)\right\}$$

or

$$\left\{(13a_1 + 6)n^2 - \frac{1}{24}(n^2 - 1)\right\}, \qquad \left\{(13a_2 + 6)n^2 - \frac{1}{24}(n^2 - 1)\right\}$$

have no integers in common, since $24a_1 + 11$ and $24a_2 + 11$ are square-free.

III. In this section Table I gives $p_r(r\nu)$ for r odd, $5 \leqslant r \leqslant 23$ and for $3 \leqslant p \leqslant 23$. We exclude $r = 1, 3$ from the table since $p_1(n)$, $p_3(n)$ are known explicitly. For $p = 3$ there is no entry unless r is a multiple of 3. Using Table 1 we can construct Table II of values of c, and we do so for r odd, $5 \leqslant r \leqslant 23$ and for $p = 3, 5, 7$. The values of $p_r(r\nu)$ were extracted from **(4)** and some

TABLE II

$r \backslash p$	3	5	7
5		−6	16
7		66	−176
9	−12	−210	−1016
11		−2694	3544
13		11730	50008
15	1836	3990	4 33432
17		1 14810	30 34528
19		−6 45150	−39 74432
21	53028	−55 56930	444 96424
23		232 45050	13229 77768

unpublished tables in the author's possession giving the first 1000 coefficients of $p_r(n)$ for r odd, $5 \leqslant r \leqslant 23$. These were computed by means of a double precision program on the IBM 704 of the National Bureau of Standards in Washington, D.C.

TABLE I

r \ p	3	5	7	11	13	17	19	23
5		−6	9	1	51	109	−1	191
7		41	−176	−181	−489	125	−1317	71
9	−12	−85	−673	−2423	−815	19619	46799	−80879
11		−2069	1143	29580	−16257	18413	7 64763	−13 92841
13		8605	33201	−3 70369	13 32566	65 29535	−53 77503	241 60657
15	1107	3990	3 15783	1 51789	−60 51657	847 06867	2901 77621	4580 60567
17		36685	22 10985	−839 64529	−1671 14351	−9976 89762	−40401 43705	1 17297 45647
19		−2 54525	17 90369	−7144 27549	−46099 30593	1 24427 50399	5 38898 77060	50 05958 51255
21	33345	−36 03805	444 96424	−39627 27241	−2 25204 73725	−60 37673 75677	98 99329 74081	−157 86000 80689
23		134 79425	10405 02519	−8 32370 19227	33 04666 24117	−467 07919 73011	3758 17200 49059	−21686 12339 64744

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