*Proceedings of the Edinburgh Mathematical Society* (1981) **24**, 231–232 ©

# BOOK REVIEWS

SLEEMAN, B. D. *Multiparameter spectral theory in Hilbert space* (Research Notes in Mathematics 22, Pitman, 1978), 118 p. £6·00.

This is only the second book to have been written which is devoted entirely to topics in the theory of multiparameter problems. As the title indicates this particular book is concerned more with spectral theory than with general multiparameter problems; nevertheless it makes a significant contribution to an area of research which over recent years has seen a considerable renewal of interest. Clearly written, in a readable and persuasive manner, it collects together most of the recent Hilbert space developments which have occurred in the subject.

The book is divided into nine chapters. The first gives an introduction to multiparameter problems and provides some of the motivation for subsequent chapters. The second introduces certain basic notions and techniques which are necessary for the theoretical developments given in Chapters three to eight. The final chapter is devoted to a consideration of open problems.

The introductory chapter demonstrates how multiparameter problems arise and how they are linked with other areas of mathematics, in particular with the study of higher special functions. Much of the motivation for the work in subsequent chapters is provided here in a discussion of a multiparameter eigenvalue problem, for a second order, ordinary differential equation. Formulating this problem as a generalisation of the familiar one parameter counterpart it is pointed out that a natural setting for multiparameter problems is in some tensor product of Hilbert spaces. Furthermore, indications are given of the way in which certain definiteness conditions, familiar in the one parameter case, generalise to a multiparameter setting. The chapter concludes with a survey of contributions to the field.

Chapter Two sets out the basic concepts of tensor products of Hilbert spaces and of operators acting between them. The chapter is mainly concerned with spectral properties of linear operators in such spaces and gathers together those results which will be needed later. Because most of the theorems are simply quoted, their proofs being adequately referenced, a clear insight is easily and quickly obtained into the strategy to be adopted in the theoretical development of the subject. This quality is particularly true of the sections dealing with functions of several commuting self-adjoint operators and solvability of a linear operator system.

Chapter Three, dealing with multiparameter spectral theory for bounded operators, begins by introducing a definiteness condition appropriate for multiparameter problems. If this definiteness condition is available for a given multiparameter problem then the associated tensor product Hilbert space can be renormed in a manner similar to that which obtains in one parameter theory by introducing a so called energy norm. The consequent symmetry which this procedure imparts to the problem is then used to develop an appropriate Parseval equality and eigenvector expansion. The chapter ends with a discussion of the eigenvalue spectrum of the system of operators defining the multiparameter problem.

The next two chapters are concerned with extending the results of Chapter Three to unbounded operators. In effecting such an extension fundamental roles are played by appropriate generalisations to multiparameter eigenvalue problems of the concepts of right and left definiteness used in one parameter spectral theory. In Chapter Four the right definite case is treated. Following the pattern as in Chapter Three a Parseval equality and eigenvector expansion are also established in this case followed as before by a discussion of the eigenvalue spectrum. In the latter the compact case of Chapter Three is replaced by the requirement that certain operators should have compact resolvents. An application to a problem in ordinary differential equations is given. Chapter Five is concerned with the left definite case. Again an application is given to ordinary differential equations. The chapter ends with a comparison of the two definiteness conditions.

In one parameter problems for differential equations it is frequently of a decided advantage to replace the given problem by an equivalent integral equation since the associated bounded operators are somewhat easier to handle than their differential counterparts. In Chapter Six a generalisation of this approach to multiparameter problems is considered. Applications to one and many parameter problems for ordinary differential equations are given.

So far in this book multiparameter problems have been represented by systems of equations coupled together by spectral parameters only; so called weakly coupled systems. If, in addition, coupling is effected through the unknowns of the system then such a system is said to be completely coupled. In Chapter Seven it is shown, by introducing the notion of operators acting between certain direct sums of Hilbert spaces, that a completely coupled system can be reduced to an equivalent weakly coupled system. This reduction lays the foundation from which a spectral theory for completely coupled systems is constructed.

Representations of multiparameter problems also occur in the form of one operator equation containing many parameters. Of particular interest among such representations is that in which all the parameters appear as integral powers of some one parameter thus giving rise to so called operator bundle equations. The purpose of Chapter Eight is to establish completeness and expansion theorems for the frequently occurring quadradic bundles. These results are obtained as a consequence of reformulating the problem as a two parameter system and then using the theory described in Chapters Three to Five.

The final chapter discusses some open problems in multiparameter theory and indicates possible lines of further research.

For both the newcomer to the subject and those engaged in research this is an excellent book.

G. F. ROACH

MURPHY, IAN S., *Basic Mathematical Analysis* (Arklay Publishers, 1980), 245 pp., £4·95.

This book is intended as an aid for a student embarking on a first course in analysis. It assumes that he will have some knowledge of calculus, and this familiarity is exploited; indeed the author says that his aim has been to provide a clear overall picture of the development and properties of the exponential, trigonometric and hyperbolic functions. With one exception (the introduction of metric spaces and compactness) the ground covered is substantially what one would expect, namely the upper bound axiom, sequences, series, continuity and the Riemann integral.

As is perhaps implied by its full title, *Basic Mathematical Analysis*: *The Facts* is in some ways more like a set of lecture notes than a conventional textbook. However this may well be an advantage, since it is known that students have difficulty in finding their way through expansive books on analysis. Moreover everything is carefully explained, if not always motivated, and there are plenty of worked examples, as well as over three hundred for the reader to tackle with twenty pages of hints and answers. The style is informal, the level well judged, and the treatment generally good. I would criticise the inclusion of metric spaces. These are unnecessary in a text which does not even venture into $R^n$, and their introduction as early as page 62 must make the book harder for the average student. There are other details one might question—it seems to me that heavy weather is made of estimates such as $(n+5)/(n^2-2n+3)<K/n$ through dwelling on them before proving that $n(n+5)/(n^2-2n+3)\to 1$—but certainly the author is to be congratulated on covering so much ground in so few pages without seeming to hurry. After introducing the exponential function via its series, with the logarithm as its inverse, he finds time to point out the alternative route starting from the integral of $1/x$, for example, and Riemann–Stieltjes integrals, Dirichlet's test, the gamma function and the $O$-notation all appear briefly in later chapters.

This, then, is a useful book for first year mathematicians in an Honours stream. The text has been reproduced from typescript, and is clear and pleasing. Two small criticisms are the straight commas (which make $a_n$, look like $a_{n'}$) and the translation of $n$ to $N$ in headings such as "LIMITS AS $N$ TENDS TO INFINITY". I did not find many misprints, though there are unfortunate ones on page 112, where $(\sin n)/n^2$ appears twice as $(\sin n)/n$, and on page 82, where $n\to\infty$ has replaced $x\to a$ in the proof that $D(x^n)=nx^{n-1}$.

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