CLOSURE OPERATIONS FOR SCHUNCK CLASSES

Dedicated to the memory of Hanna Neumann

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In his Canberra lectures on finite soluble groups, [3], Gaschütz observed that a Schunck class (sometimes called a saturated homomorph) is \( \{Q, E_\phi, D_0\} \)-closed but not necessarily \( R_0 \)-closed(*). In Problem 7.8 of the notes he then asks whether every \( \{Q, E_\phi, D_0\} \)-closed class is a Schunck class. We show below with an example† that this is not the case, and then we construct a closure operation \( R_0 \) satisfying \( D_0 < R_0 < R_0 \) such that \( \mathcal{X} \) is a Schunck class if and only if \( \mathcal{X} = \{Q, E_\phi, R_0\} \mathcal{X} \).

In what follows the class of finite soluble groups is universal. Let \( \mathcal{T} \) denote the class of primitive groups. We recall that a Schunck class \( \mathcal{X} \) is one which satisfies:

(a) \( \mathcal{X} = Q \mathcal{X} \), and

(b) \( \mathcal{X} \) contains all groups \( G \) such that \( Q(G) \cap \mathcal{T} \subseteq \mathcal{X} \).

**Example.** Let \( \mathcal{Y} \) denote the class comprising groups of order 1, groups of order 2 and non-Abelian groups of order 6. Set \( \mathcal{X} = E_\phi D_0 \mathcal{Y} \). Since \( E_\phi D_0 \) is a closure operation (see [4]), \( \mathcal{X} \) is \( \{E_\phi, D_0\} \)-closed. It is not difficult to see that \( D_0 \mathcal{Y} \) is \( Q \)-closed, and since \( E_\phi Q \) is a closure operation (again see [4]), it follows that \( Q \mathcal{X} = Q E_\phi D_0 \mathcal{Y} \subseteq E_\phi Q D_0 \mathcal{Y} = E_\phi D_0 \mathcal{Y} = \mathcal{X} \), and hence that \( \mathcal{X} \) is \( Q \)-closed.

Let \( G \) denote the extension of an elementary Abelian group of order 9 by an inverting involution. Then clearly \( G \notin \mathcal{X} \) and every primitive epimorphic image of \( G \) does belong to \( \mathcal{X} \), even to \( \mathcal{T} \). Therefore \( \mathcal{X} = \{Q, E_\phi, D_0\} \mathcal{X} \) but \( \mathcal{X} \) is not a Schunck class.

In order to formulate the closure operation \( R_0 \) we need the concept of a crown, due to Gaschütz [2]. Let \( H/K \) be a complemented chief factor of a group \( G \) and \( M \) one of the maximal subgroups of \( G \) complementing it. Writing \( C = \text{Core}(H/K) \) it is well known that \( \text{Core}(M) = M \cap C \) and that \( C/C \cap M \) is a chief factor

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* The closure operation of taking finite direct products is denoted by \( D_0 \); the other closure operations mentioned are defined in [1]. A more detailed analysis of their properties appears in [4].

† I am pleased to acknowledge a similar example constructed by John Cossey of which I was unaware when I submitted this note.
$G$-isomorphic with $H/K$. Let $R$ be the intersection of all normal subgroups $T$ of $G$ such that $C/T \cong G \cong H/K$. $C/R$ is called the crown of $H/K$. A crown of $G$ is a normal factor $C/R$ associated in this way with some complemented chief factor. The following lemma shows that a normal subgroup of $G$ either covers a crown $C/R$ or is properly contained in $C$.

**Lemma.** Let $C/R$ be a crown of $G$ and $N \triangleleft G$. Then the following statements are equivalent:

1. The image of $C/R$ under the natural homomorphism $G \twoheadrightarrow G/N$ is a crown of $G/N$;
2. $N$ does not cover the factor $C/R$;
3. $RN < C$.

**Proof.** Since a crown is by definition a non-trivial normal factor it is clear that (1) implies (2). Assume (2) holds, and set $L = C \cap NR \leq C \cap L \leq C$. Now $[N, C] \leq C \cap L \leq C$, and since $C$ is the centralizer in $G$ of any non-trivial normal factor of $G$ between $C$ and $R$, we have $N \leq C_G(C/L) = C$. Therefore $NR = NR \cap C = L < C$, and (3) is true. Finally assume condition (3) is satisfied. If $C/K$ is a chief factor of $G$ with $K \geq R$, then $C/K$ is complemented and $C/R$ is the crown associated with it. By (3) we may choose such a $K$ containing $RN$. Let $(C/N)/(T/N)$ be a chief factor of $G/N$ isomorphic with $(C/N)/(K/N)$; then $C/T \cong G \cong C/K$, so $R \leq T$. Since $C/RN$ is a semi-simple (in fact, homogeneous) $G$-module, $RN$ is the intersection of such $T$. Hence $(C/N)/(RN/N)$ is the crown of $G/N$ associated with $(C/N)/(K/N)$.

**Definition.** If $\mathcal{X}$ is a class of groups, define $\bar{R}_0\mathcal{X}$ as follows:

- $G \in \bar{R}_0\mathcal{X}$ if and only if $G$ has a set $\{N_i\}_{i=1}^t$ of normal subgroups $N_i$ satisfying
  - $(a)$ $G/N_i \in \mathcal{X}$ for $i = 1, \ldots, t$,
  - $(b)$ $\bigcap_{i=1}^t N_i = 1$, and
  - $(c)$ for each crown $C/R$ of $G$, there exists $i \in \{1, \ldots, t\}$ such that $N_i$ does not cover $C/R$.

Evidently $\mathcal{X} \subseteq \bar{R}_0\mathcal{X}$ and, if $X \subseteq \mathcal{Y}$, then $\bar{R}_0\mathcal{X} \subseteq \bar{R}_0\mathcal{Y}$. To prove that $\bar{R}_0$ is a closure operation, it remains to show it is idempotent. Let $G \in \bar{R}_0^2\mathcal{X} = \bar{R}_0(\bar{R}_0\mathcal{X})$. Then $G$ has normal subgroups $\{N_i\}_{i=1}^t$ with $G/N_i \in \bar{R}_0\mathcal{X}$ satisfying conditions $(b)$ and $(c)$ above. Thus each $G/N_i$ has normal subgroups $\{N_{ij}\}_{j=1}^t$ such that

- $(a)$ $G/N_{ij} \in \mathcal{X}$ for $j = 1, \ldots, t$,
- $(b)$ $\bigcap_{j=1}^t N_{ij} = N_i$, and
- $(c)$ each crown of $G/N_i$ is nor covered by at least one $N_{ij}/N_i$.

The full set $\{N_{ij} | j = 1, \ldots, t_i, i = 1, \ldots, t\}$ of normal subgroups of $G$ clearly satisfies conditions $(a)$ and $(b)$ of the Definition. Let $C/R$ be a crown of $G$. There exists an $i \in \{1, \ldots, t\}$ such that $N_i$ does not cover $C/R$. By the Lemma $(C/N_i)/(RN_i/N_i)$ is a crown of $G/N_i$ and by condition $(c)$ above there exists a
j \in \{1, \ldots, t\} \) such that \( N_{ij} \) does not cover it. Again by the Lemma we have \( RN_{ij} < C \) and therefore the set \( \{N_{ij}\} \) also satisfies \((\gamma)\). Thus \( G \in \bar{R}_0\mathcal{X} \). It follows that \( \bar{R}_0^2 = \bar{R}_0 \) and that \( \bar{R}_0 \) is a closure operation.

It is obvious that \( \bar{R}_0 \leq R_0 \). To see that \( D_0 \leq \bar{R}_0 \), let \( G = G_1 \times \cdots \times G_t \) with \( 1 \neq G_i \in \mathcal{X} \) for \( i = 1, \ldots, t \). Set \( N_i = \prod_{j \neq i} G_j \). Then the normal subgroups \( \{N_{ij}\}_{i=1}^t \) clearly satisfy conditions \((\alpha)\) and \((\beta)\). It follows easily from the properties of a direct product that each chief factor is centralized by at least one \( N_i \) and that a factor of the form \( N_i/R \) is never a crown. Hence \((\gamma)\) is also satisfied and we have \( G \in \bar{R}_0\mathcal{X} \). It remains to prove the following

**Theorem.** The condition \( \mathcal{X} = \{Q, E_\phi, \bar{R}_0\}\mathcal{X} \) is both necessary and sufficient for \( \mathcal{X} \) to be a Schunck class.

**Proof.** Let \( \mathcal{X} \) be a Schunck class and let \( G \in \bar{R}_0\mathcal{X} \). \( G \) has a family \( \{N_{ij}\}_{i=1}^t \) of normal subgroups satisfying conditions \((\alpha)\), \((\beta)\) and \((\gamma)\). Let \( G/K \) be a primitive epimorphic image of \( G \). Let \( C/K \) denote the monolith of \( G/K \) and \( C/R \) the crown of \( G \) associated with \( C/K \). It follows from the hypothesis and the Lemma that there is an \( i \in \{1, \ldots, t\} \) such that \( N_i R < C \). Let \( C/T \) be a chief factor of \( G \) with \( T \geq N_i R \). Then \( G/K \cong G/T \in Q(G/N_i K) \cong \bar{Q}\mathcal{X} = \mathcal{X} \). Thus \( Q(G) \cap \mathcal{Y} \subseteq \mathcal{X} \), and so \( G \in \mathcal{X} \). This shows that \( \mathcal{X} = \bar{R}_0\mathcal{X} \). Since Schunck classes are \( Q \)-closed and \( E_\phi \)-closed, the necessity of the condition is established.

We prove the sufficiency arguing by contradiction. Suppose there exists a \( \{Q, E_\phi, \bar{R}_0\}\)-closed class \( \mathcal{X} \) which is not a Schunck class. Let \( G \) be a group of minimal order subject to satisfying \( Q(G) \cap \mathcal{Y} \subseteq \mathcal{X} \) and \( G \notin \mathcal{X} \). If \( 1 \neq N \triangleleft G \), then \( Q(G/N) \cap \mathcal{Y} \subseteq \mathcal{X} \), and by minimality \( G/N \notin \mathcal{X} \). Hence by the \( E_\phi \)-closure of \( \mathcal{X} \) we have \( \Phi(G) = 1 \). Let \( S \) denote the set of all minimal normal subgroups of \( G \). If \( |S| = 1 \), \( G \in \mathcal{Y} \) and so \( G \in \mathcal{X} \), a contradiction. If \( |S| > 1 \), \( \bigcap \{N \mid N \in S\} = 1 \). If \( C/R \) is a crown of \( G \), either \( R > 1 \), or \( R = 1 \) and \( C \notin S \). In either case, there is an \( N \in S \) such that \( RN < C \). Thus the set \( S \) of all minimal normal subgroups of \( G \) satisfies conditions \((\alpha)\), \((\beta)\) and \((\gamma)\) of the Definition. Hence \( G \in \bar{R}_0\mathcal{X} = \mathcal{X} \). This final contradiction completes the proof.

**References**


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