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RECONSTRUCTION OF ENTIRE FUNCTIONS FROM IRREGULARLY SPACED SAMPLE POINTS

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ABSTRACT. Let $G(z) := (z-\lambda_0)\prod_{n=1}^{\infty}(1-\frac{z}{\lambda_n})(1-\frac{z}{\lambda_{-n}})$ where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers such that $|\lambda_n - n| \le \Delta$ for some $\Delta > 0$ and all $n \in \mathbb{Z}$. Extending an obvious property of $\sin \pi z$ to which the function G reduces when $\Delta = 0$ we show that $\left|\frac{G^{(k)}(\lambda_n)}{G'(\lambda_n)}\right|$ is bounded by a constant independent of n. The result is then applied to a problem concerning derivative sampling in one and several variables.

1. Introduction.

1.1 *The main result.* Sampling theorems deal with the reconstruction of a function f from its values and possibly those of some of its derivatives at an infinite sequence of points $\{\lambda_n\}_{n\in\mathbb{Z}}$ called nodes. Assuming that the infinite product

$$G(z) := (z-\lambda_0)\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\left(1-\frac{z}{\lambda_{-n}}\right) \tag{1}$$

converges uniformly on all compact subsets of $\mathbb{C}$ the function $L_n(z) := \frac{G(z)}{G'(\lambda_n)(z-\lambda_n)}$ vanishes at all the nodes except λ_n where it takes the value 1. Then $F(z) := \sum_{n=-\infty}^{\infty} f(\lambda_n)L_n(z)$ is defined at least at the points λ_n and $f(\lambda_n) = F(\lambda_n)$ for all n. Clearly, f may not agree with F at any other point, *i.e.*, in general, the formula

$$f(z) = \sum_{n=-\infty}^{\infty} f(\lambda_n)\frac{G(z)}{G'(\lambda_n)(z-\lambda_n)} \tag{2}$$

cannot hold for all $z \in \mathbb{C}$ or even for all $z \in \mathbb{R}$. However, (2) does hold under appropriate conditions on $\{\lambda_n\}_{n\in\mathbb{Z}}$ and on f (see for example [7], [8]). As regards the nodes, they are usually assumed (see [1], [4], [5], [9], [10], [11] in addition to [7], [8]) to satisfy

$$\lambda_n \in \mathbb{R}, \quad |\lambda_n - n| \le \Delta \tag{3}$$

and also $|\lambda_{n+j} - \lambda_n| \ge \delta > 0$ for $n \in \mathbb{Z}, j \in \mathbb{Z}\setminus\{0\}$ whenever $\Delta \ge \frac{1}{2}$. In the special case $\lambda_n = n$ (for all $n \in \mathbb{Z}$) the function G reduces to $\sin \pi z$ which has several characteristic properties. Proofs of (2) and certain other related formulae are based on the observation that these properties are *more or less* preserved when the sequence $\{\lambda_n\}$ is allowed to

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deviate from $\{n\}$ but still remains "close to it" in some sense. It is known ([1], [10], [11], [14]) that if (3) holds, then there exist constants c_1, c_2, *etc.*, such that

$$|G(z)| > c_1 e^{\pi|y|}\left|\frac{y}{z^2}\right|^{2\Delta}, \quad |y| > \Delta; \tag{4}$$

$$|G(z)| > c_2 \frac{|\lambda_n - z| e^{\pi|y|}}{(1+|z-n|)(1+|z|)^{4\Delta}}, \quad \left(n - \frac{1}{2} \le |z| \sec(\arg z) < n + \frac{1}{2}\right); \tag{5}$$

$$|G(z)| < c_2' e^{\pi|y|}(|z|+1)^{4\Delta} \quad \text{for all } z; \tag{6}$$

$$\left|\frac{G(z)}{z-\lambda_n}\right| < c_2'' e^{\pi|y|}(|z|+1)^{4\Delta} \quad \text{for all } z, \tag{7}$$

where the function on the left is assumed to have its singularity at $z = \lambda_n$ removed;

$$|G'(\lambda_n)| > c_3(|\lambda_n|+1)^{-4\Delta} \quad \text{for } \Delta < \frac{1}{2}. \tag{8}$$

Some other interesting estimates for $|G(z)|$ were recently obtained in [9].

In the present paper we shall establish a new property of the function G. It concerns the quantity $\frac{G^{(k)}(\lambda_n)}{G'(\lambda_n)}$ which appears naturally in connection with Hermite interpolation and there are problems in that area to which our result can be applied. Since $G(z)$ reduces to $\sin \pi z$ when $\lambda_n = n$ for all $n \in \mathbb{Z}$ we have (in that special case)

$$\left|\frac{G^{(k)}(\lambda_n)}{G'(\lambda_n)}\right| = \begin{cases} 0 & \text{if } k \text{ is even} \\ 2^{k-1} & \text{if } k \text{ is odd,} \end{cases}$$

and so in particular, $\left|\frac{G^{(k)}(\lambda_n)}{G'(\lambda_n)}\right|$ *is bounded by a constant independent of* n. We show that this property remains true under fairly weak restrictions on the sequence $\{\lambda_n\}$.

THEOREM 1. *If $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence satisfying (3) with $\Delta \le \frac{1}{4}$, then for each $k \ge 2$ there exists a constant $C_{4,k}$ such that*

$$\left|\frac{G^{(k)}(\lambda_n)}{G'(\lambda_n)}\right| < C_{4,k} \tag{9}$$

for all $n \in \mathbb{Z}$.

REMARK 1. From (6) it follows that $|G(z)| < c_2' e^{\pi}(|x|+2)^{4\Delta}$ for $|y| \le 1$ and so, by Cauchy's integral formula for the k-th derivative we have

$$|G^{(k)}(\lambda_n)| < k! c_2' e^{\pi}(|\lambda_n|+2)^{4\Delta}.$$

This, in conjunction with (8), implies that

$$\left|\frac{G^{(k)}(\lambda_n)}{G'(\lambda_n)}\right| < k!\left(\frac{c_2'}{c_3}\right)e^{\pi}\big((|\lambda_n|+2)(|\lambda_n|+1)\big)^{4\Delta}$$

from which the desired estimate follows trivially provided n remains bounded but not otherwise. It is however clear that while proving (9) we may suppose $|n| \ge 1$.

Note that in order to prove Theorem 1 it is enough to establish the following result.

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THEOREM 1′. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $|\lambda_n - n| \le \Delta \le \frac{1}{4}$ and define*

$$G_N(z) := (z-\lambda_0)\prod_{\nu=-N}^{N}{}'\Big(1-\frac{z}{\lambda_\nu}\Big).$$

Then for any given $k \in \mathbb{N}$ we can find a constant C_k such that for each $n \in \{\pm 1, \pm 2, \ldots\}$

$$\Big|\frac{G_N^{(k)}(\lambda_n)}{G_N'(\lambda_n)}\Big| < C_k \tag{10}$$

for all $N \ge N_{n,k}$ where $N_{n,k}$ is a positive integer depending on n and on k.

NOTE. A prime affixed to the summation (product) sign as in $\sum'(\prod')$ indicates that the index of summation (product) does not take the value zero.

REMARK 2. Let $\lambda_n = n$ for all $n \in \mathbb{Z}$ so that $G_N(z) := z\prod'^{N}_{n=-N}\big(1-\frac{z}{n}\big)$. It can be easily seen that $\sup_{-N\le n\le N}\big|\frac{G_N''(\lambda_n)}{G_N'(\lambda_n)}\big|$ is larger than $\ln(2N+1)$ and so goes to $+\infty$ as $N \longrightarrow \infty$. The same example can be used to show that for no given $k \ge 2$ the quantity $\sup_{-N\le n\le N}\big|\frac{G_N^{(k)}(\lambda_n)}{G_N'(\lambda_n)}\big|$ remains bounded as $N \longrightarrow \infty$. In other words, it is not true that for some C_k depending on k but not on N we have

$$\Big|\frac{G_N^{(k)}(z)}{G_N'(z)}\Big| < C_k$$

for all the zeros λ_n of G_N. This makes (9) somewhat more interesting.

1.2 *Further results and an application of Theorem 1.* The following result on derivative sampling was recently proved by G. Hinsen (see [9, Theorem 6.1]).

THEOREM A. *Let $m \in \mathbb{N}$, $1 \le p < \infty$ and $\lambda := \{\lambda_n\}_{n\in\mathbb{Z}}$ a sequence satisfying (3) with $\lambda_0 = 0$ and*

$$\Delta < \begin{cases} \frac{1}{4m}, & 1 \le p \le 2 \\ \frac{1}{2pm}, & 2 \le p < \infty. \end{cases}$$

If f is an entire function of exponential type $m\pi$ belonging to $L^p(\mathbb{R})$, then

$$f(z) = \sum_{n=-\infty}^{\infty}\sum_{\mu=0}^{m-1} f^{(\mu)}(\lambda_n)\Psi_{m,n,\mu}(\lambda; z) \tag{11}$$

where

$$\Psi_{m,n,\mu}(\lambda; z) := \sum_{j=0}^{m-1-\mu}\frac{\big(G(z)\big)^m}{(z-\lambda_n)^{j+1}}\frac{\big\{\big((\cdot-\lambda_n)/G(\cdot)\big)^m\big\}^{(m-1-\mu-j)}(\lambda_n)}{\mu!\,(m-1-\mu-j)!}. \tag{12}$$

In addition, he proved

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THEOREM B [9, THEOREM 5.2]. *Let $1 \le p < \infty$ and $\{\lambda_n\}_{n\in\mathbb{Z}}$ a sequence satisfying (3) with $\lambda_0 = 0$ and*

$$\text{(13)} \qquad \begin{array}{ll} \Delta \le \frac{1}{4}, & p = 1 \\ \Delta < \frac{1}{4p}, & 1 < p < \infty. \end{array}$$

If f is an entire function of exponential type π belonging to $L^p(\mathbb{R})$, then

$$\sum_{n=-\infty}^{\infty} \left| f(\lambda_n) \frac{G(z)}{G'(\lambda_n)(z-\lambda_n)} \right|$$

converges uniformly on each bounded subset of $\mathbb{C}$ and (2) holds.

Theorem B is important for certain applications; as an example we mention the generalization of the sampling theorem to more than one dimension obtained by P. L. Butzer and G. Hinsen in [4]. Comparing the two preceding theorems it is natural to ask under what condition does the series in (11) converge absolutely. With the help of (9) we shall prove the following

THEOREM 2. *Let $m \in \mathbb{N}$, $0 < p < \infty$ and $\lambda := \{\lambda_n\}_{n\in\mathbb{Z}}$ a sequence satisfying (3) with*

$$\text{(13')} \qquad \begin{array}{ll} \Delta \le \frac{1}{4m}, & \text{if } \ 0 < p \le 1 \\ \Delta < \frac{1}{4pm}, & \text{if } 1 < p < \infty. \end{array}$$

Further, let

$$\text{(14)} \qquad \Psi_{m,n}(z) := \left(\frac{G(z)}{G'(\lambda_n)(z-\lambda_n)} \right)^m.$$

If f is an entire function of exponential type $m\pi$ belonging to $L^p(\mathbb{R})$ and

$$\text{(15)} \qquad f_{\mu,\lambda}(\lambda_n) := \frac{d^\mu}{dz^\mu} \left\{ \frac{f(z)}{\Psi_{m,n}(z)} \right\} \Big|_{z=\lambda_n} \qquad \text{for } 0 \le \mu \le m-1,$$

then the series

$$\text{(16)} \qquad \mathfrak{h}_{m,\lambda}(f;z) := \sum_{n=-\infty}^{\infty} \left\{ \sum_{\mu=0}^{m-1} \frac{1}{\mu!} f_{\mu,\lambda}(\lambda_n)(z-\lambda_n)^\mu \right\} \Psi_{m,n}(z),$$

which is identical with $\sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} f^{(\mu)}(\lambda_n) \Psi_{m,n,\mu}(\lambda; z)$ (and so represents f, by Theorem A read in conjunction with [3, Theorem 6.7.1]), converges absolutely. The convergence is uniform on each bounded subset of $\mathbb{C}$.

It is interesting to compare conditions (13) and (13′).

REMARK 3. It is not hard to verify that formula (16) can also be written as

$$\text{(17)} \qquad \mathfrak{h}_{m,\lambda}(f;z) = \sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} f^{(\mu)}(\lambda_n) \Phi_{m,n,\mu}(\lambda; z)$$

where

$$\Phi_{m,n,\mu}(\lambda;z) := \frac{1}{\mu!}(z-\lambda_n)^{\mu}\Psi_{m,n}(z)\sum_{j=0}^{m-1-\mu}\frac{1}{j!}\frac{d^j}{dz^j}\left(\frac{1}{\Psi_{m,n}(z)}\right)\bigg|_{z=\lambda_n}(z-\lambda_n)^j. \tag{18}$$

Substituting for $\Psi_{m,n}$ from (14) in the above expression for $\Phi_{m,n,\mu}(\lambda;\cdot)$ we see that it ($\Phi_{m,n,\mu}(\lambda;\cdot)$) is indeed identical with the function $\Psi_{m,n,\mu}(\lambda;\cdot)$ appearing in Theorem A.

1.3 *Nonuniform multidimensional derivative sampling.*

1.3.1. The sum $g(z_1,\ldots,z_n)$ of an everywhere convergent power series in n variables $z_1,\ldots,z_n$ is called an entire function of exponential type $\vec{\tau} := (\tau_1,\ldots,\tau_n)$ if for every $\varepsilon > 0$ there exists a positive number A_ε such that

$$|g(z_1,\ldots,z_n)| \le A_\varepsilon e^{\sum_{\nu=1}^{n}(\tau_\nu+\varepsilon)|z_\nu|}$$

for all $(z_1,\ldots,z_n) \in \mathbb{C}^n$. By definition, the function g belongs to $L^p(\mathbb{R}^n)$, $0 < p < \infty$ if

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|g(x_1,\ldots,x_n)|^p\,dx_1\cdots dx_n < \infty, \quad (x_\nu := \operatorname{Re} z_\nu \text{ for } \nu = 1,\ldots,n).$$

1.3.2. Corresponding to each n-tuple $(j_1,\ldots,j_n)$ where $j_1,\ldots,j_n$ belong to $\mathbb{Z}$ let

$$\vec{\lambda}_{j_1,\ldots,j_n} := \left((\vec{\lambda}_{j_1,\ldots,j_n})_1,\ldots,(\vec{\lambda}_{j_1,\ldots,j_n})_n\right)$$

be a point in $\mathbb{R}^n$ such that

$$\begin{cases} (i) & |(\vec{\lambda}_{j_1,\ldots,j_n})_\nu - j_\nu| \le \Delta_\nu < \frac{1}{2} \text{ for } 1 \le \nu \le n; \\ (ii) & \text{if } j'_\nu = j''_\nu \text{ for } 1 \le \nu \le n_0 \text{ then} \\ & (\vec{\lambda}_{j'_1,\ldots,j'_n})_\nu = (\vec{\lambda}_{j''_1,\ldots,j''_n})_\nu \text{ for } 1 \le \nu \le n_0. \end{cases} \tag{19}$$

Note that the set of points $\{\vec{\lambda}_{j_1,\ldots,j_n} : j_1,\ldots,j_n \in \mathbb{Z}\}$ has the following structure (see [4, Remark on p. 71]). First we choose points $\{\lambda_{j_1}\}$ on the x_1–axis subject to the condition that $\lambda_{j_1} \in [j_1-\Delta_1, j_1+\Delta_1]$ for all $j_1 \in \mathbb{Z}$. Through the points $\{\lambda_{j_1}\}$ we draw lines parallel to the x_2–axis. On each of these lines we choose points $\{\vec{\lambda}_{j_1,j_2}\}$ satisfying $|(\vec{\lambda}_{j_1,j_2})_2 - j_2| \le \Delta_2$ for all $j_1, j_2 \in \mathbb{Z}$. If $n \ge 3$ we continue by drawing through the points $\{\vec{\lambda}_{j_1,j_2}\}$ lines parallel to the x_3–axis and choose on them points $\{\vec{\lambda}_{j_1,j_2,j_3}\}$ such that $|(\vec{\lambda}_{j_1,j_2,j_3})_3 - j_3| \le \Delta_3$ for all $j_1, j_2, j_3 \in \mathbb{Z}$. This procedure for choosing the coordinates of $\{\vec{\lambda}_{j_1,\ldots,j_n}\}$ one after the other goes on until all of them have been determined. The following alternative notation

$$\vec{\lambda}_{j_1,\ldots,j_n} = \left(\lambda_1(j_1), \lambda_2(j_1,j_2), \ldots, \lambda_n(j_1,\ldots,j_n)\right) \tag{20}$$

would therefore be more indicative as regards the structure of the set $\{\vec{\lambda}_{j_1,\ldots,j_n} : j_1,\ldots,j_n \in \mathbb{Z}\}$.

As an application of the absolute convergence of the series (16) we present:

THEOREM 3. *Let $\vec{m} := (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $p \in (0, \infty)$. If $\{\vec{\lambda}_{j_1,\ldots,j_n} : j_1, \ldots, j_n \in \mathbb{Z}\}$ is a sequence of points in $\mathbb{R}^n$ satisfying conditions (19) with*

$$\begin{aligned} \Delta_\nu &\leq \tfrac{1}{4m_\nu}, \quad \text{if } 0 < p \leq 1 \\ \Delta_\nu &< \tfrac{1}{4pm_\nu}, \quad \text{if } 1 < p < \infty, \end{aligned}$$

for $\nu = 1, 2, \ldots, n$, then for all entire functions f of exponential type $\vec{m}\pi = (m_1\pi, \ldots, m_n\pi)$ belonging to $L^p(\mathbb{R}^n)$ we have

$$f(z_1, \ldots, z_n) = \sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_n=-\infty}^{\infty} \sum_{\mu_1=0}^{m_1-1} \cdots \sum_{\mu_n=0}^{m_n-1} \frac{\partial^{\mu_1} \cdots \partial^{\mu_n} f}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}}\Bigg|_{(x_1,\ldots,x_n)=\vec{\lambda}_{j_1,\ldots,j_n}} \times \Phi_{\vec{m},(j_1,\ldots,j_n),(\mu_1,\ldots,\mu_n)}(z_1, \ldots, z_n) \tag{21}$$

where with $\Phi_{\cdot,\cdot,\cdot}(\cdot;\cdot)$ as in (18) and $\lambda_1(j_1), \ldots \lambda_n(j_1, \ldots, j_n)$ as in (20)

$$\begin{aligned} &\Phi_{\vec{m},(j_1,\ldots,j_n),(\mu_1,\ldots,\mu_n)}(z_1, \ldots, z_n) \\ &\quad = \Phi_{m_1,j_1,\mu_1}\big(\{\lambda_1(j)\}_{j\in\mathbb{Z}}; z_1\big) \times \cdots \times \Phi_{m_n,j_n,\mu_n}\big(\{\lambda_n(j_1, \ldots, j_{n-1}, j)\}_{j\in\mathbb{Z}}; z_n\big). \end{aligned}$$

The series (21) converges absolutely and uniformly on each compact subset of $\mathbb{C}^n$.

2. **Auxiliary Results.**

LEMMA 1. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $|\lambda_n - n| \leq \Delta \leq \frac{1}{4}$. Then for each $n \in \mathbb{Z}$ there exists a positive integer N_n such that for all $N \geq N_n$ we have*

$$\varphi_n(N) := \Bigg| \sum_{\substack{\nu=-N \\ \nu\notin\{-n,0,n\}}}^{N} \frac{1}{\lambda_\nu - \lambda_n} \Bigg| < 10. \tag{22}$$

Further, for all $N \in \mathbb{N}$,

$$\sum_{\substack{\nu=-N \\ \nu\neq n}}^{N} \frac{1}{(\lambda_\nu - \lambda_n)^2} \leq \pi^2, \tag{23}$$

$$\sum_{\substack{\nu=-N \\ \nu\neq n}}^{N} \frac{1}{|\lambda_\nu - \lambda_n|^k} < \pi^2 + 2^{k+1}, \quad (k = 3, 4, \ldots). \tag{24}$$

PROOF. Since the sequence $\{-\lambda_{-n}\}_{n\in\mathbb{Z}}$ also satisfies the condition of the lemma it is enough to prove (22)–(24) for $n \geq 0$. By assumption $\lambda_n = n + \delta_n$ where $|\delta_n| \leq \Delta$ for all $n \in \mathbb{Z}$. Let $n \geq 0$ be arbitrary but fixed. Then for $N \geq n + 1$,

$$\varphi_n(N) = \left| \sum_{\substack{\nu=1 \\ \nu \neq n}}^{N} \frac{2n + 2\delta_n - \delta_\nu - \delta_{-\nu}}{(\nu - n + \delta_\nu - \delta_n)(\nu + n + \delta_n - \delta_{-\nu})} \right|.$$

In particular,

$$\varphi_0(N) < \sum_{\nu=1}^{\infty} \frac{4}{(2\nu - 1)^2} = \frac{\pi^2}{2},$$

$$\varphi_1(N) < \sum_{\nu=2}^{\infty} \frac{3}{\left((\nu - \frac{1}{2}) - 1\right)\left((\nu - \frac{1}{2}) + 1\right)} = 3,$$

$$\varphi_2(N) < \max\left\{ \frac{4 + (\delta_2 - \delta_1) + (\delta_2 - \delta_{-1})}{\left(1 + (\delta_2 - \delta_1)\right)\left(3 + (\delta_2 - \delta_{-1})\right)}, \sum_{\nu=3}^{N} \frac{5}{(\nu - \frac{5}{2})(\nu + \frac{3}{2})} \right\}$$

$$= \max\left\{ \frac{12}{5}, \frac{1126}{312} \right\} = \frac{1126}{312}.$$

For $n \geq 3$, we have $\varphi_n(N) = |B(n) - A(n)|$, where

$$A(n) := \sum_{\nu=1}^{n-1} \frac{2n + 2\delta_n - \delta_\nu - \delta_{-\nu}}{\left(n + \nu + (\delta_n - \delta_{-\nu})\right)\left(n - \nu + (\delta_n - \delta_\nu)\right)},$$

$$B(n) := \sum_{\nu=n+1}^{N} \frac{2n + 2\delta_n - \delta_\nu - \delta_{-\nu}}{\left(\nu + n + (\delta_n - \delta_{-\nu})\right)\left(\nu - n + (\delta_\nu - \delta_n)\right)}.$$

It is easily seen that

$$A(n) \geq \sum_{\nu=1}^{n-1} \frac{2n - 1}{(n + \frac{1}{2} + \nu)(n + \frac{1}{2} - \nu)} \geq \sum_{\nu=1}^{n-1} \frac{2n - 1}{(n + 1)^2 - (\nu + \frac{1}{2})^2},$$

$$A(n) \leq \sum_{\nu=1}^{n-1} \frac{2n + 1}{(n - \frac{1}{2})^2 - \nu^2} \leq \sum_{\nu=1}^{n-1} \frac{2n + 1}{(n - 1)^2 - (\nu + \frac{1}{2})^2} + \frac{56}{9},$$

$$B(n) > \sum_{\nu=n}^{N} \frac{2n - 1}{(\nu + \frac{1}{2})^2 - n^2} - 2 \geq \sum_{\nu=n}^{N} \frac{2n - 1}{(\nu + \frac{1}{2})^2 - (n - 1)^2} - 2,$$

$$B(n) \leq \sum_{\nu=n}^{N-1} \frac{2n + 1}{(\nu + \frac{1}{2})^2 - n^2} < \frac{(2n + 1)^2}{(n + \frac{1}{4})(n + \frac{3}{4})} + \sum_{\nu=n}^{N} \frac{2n + 1}{(\nu + \frac{1}{2})^2 - (n + 1)^2}.$$

Hence $\varphi_n(N) \leq \max\{D_{1,N}, D_{2,N}\}$ where

$$D_{1,N} := \left| \sum_{\nu=1}^{n-1} \frac{2n + 1}{(n - 1)^2 - (\nu + \frac{1}{2})^2} - \sum_{\nu=n}^{N} \frac{2n - 1}{(\nu + \frac{1}{2})^2 - (n - 1)^2} + \frac{56}{9} + 2 \right|,$$

$$D_{2,N} := \left| \sum_{\nu=1}^{n-1} \frac{2n - 1}{(n + 1)^2 - (\nu + \frac{1}{2})^2} - \sum_{\nu=n}^{N} \frac{2n + 1}{(\nu + \frac{1}{2})^2 - (n + 1)^2} - \frac{(2n + 1)^2}{(n + \frac{1}{4})(n + \frac{3}{4})} \right|.$$

Now we note that

$$D_{1,N} = \Big| -\sum_{\nu=1}^{N} \frac{2(n-1)}{(\nu+\frac12)^2-(n-1)^2} - \sum_{\nu=1}^{n-1} \frac{3}{(\nu+\frac12)^2-(n-1)^2} - \sum_{\nu=n}^{N} \frac{1}{(\nu+\frac12)^2-(n-1)^2} + \frac{74}{9} \Big|$$

$$= \Big| -\Big(1+\frac{1}{2(n-1)}\Big)\pi \sum_{\nu=0}^{N} \frac{2(n-1)\pi}{\big((\nu+\frac12)\pi\big)^2-\big((n-1)\pi\big)^2} - \frac{2n-1}{(n-1)^2-\frac14} - \sum_{\nu=1}^{n-1} \frac{2}{(\nu+\frac12)^2-(n-1)^2} + \frac{74}{9} \Big|.$$

But

$$\Big| -\frac{2n-1}{(n-1)^2-\frac14} - \sum_{\nu=1}^{n-1} \frac{2}{(\nu+\frac12)^2-(n-1)^2} + \frac{74}{9} \Big|$$
$$< \Big| \frac{74}{9} - \frac{2}{n-\frac34} + (n-2)\Big\{ \frac{1}{(n-1)^2-(\frac32)^2} + \frac{1}{(n-1)^2-(n-\frac32)^2} \Big\} \Big|$$
$$< \frac{74}{9} + 1 = \frac{83}{9},$$

and so

$$D_{1,N} < \Big(1+\frac{1}{2(n-1)}\Big)\pi \Big| \sum_{\nu=0}^{\infty} \frac{2(n-1)\pi}{\big((\nu+\frac12)\pi\big)^2-\big((n-1)\pi\big)^2} - \epsilon_N \Big| + \frac{83}{9}$$

where

$$\epsilon_N := \sum_{\nu=N+1}^{\infty} \frac{2(n-1)\pi}{\big((\nu+\frac12)\pi\big)^2-\big((n-1)\pi\big)^2} \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty.$$

According to Mittag–Leffler's expansion of $\tan \pi z$ [17, p. 113]

$$2(n-1)\pi \sum_{\nu=0}^{\infty} \frac{1}{\big((\nu+\frac12)\pi\big)^2-\big((n-1)\pi\big)^2} = \tan(n-1)\pi = 0.$$

As such $D_{1,N} < 10$ for all large N. Next we estimate $D_{2,N}$. Clearly

$$D_{2,N} = \Big| +\Big(1-\frac{1}{2(n+1)}\Big) \sum_{\nu=0}^{N} \frac{2(n+1)}{(\nu+\frac12)^2-(n+1)^2} + \frac{2n+1}{(n+1)^2-\frac14} + \sum_{\nu=1}^{n-1} \frac{2}{(n+1)^2-(\nu+\frac12)^2} + \frac{(2n+1)^2}{(n+\frac14)(n+\frac34)} \Big|$$
$$< \Big| \sum_{\nu=0}^{\infty} \frac{2(n+1)}{(\nu+\frac12)^2-(n+1)^2} - \epsilon'_N \Big| + \frac49 + \frac23 \frac{n-1}{n+\frac14} + \frac{784}{195}$$

where $\epsilon'_N \longrightarrow 0$ as $N \longrightarrow \infty$. But

$$\sum_{\nu=0}^{\infty} \frac{2(n+1)}{(\nu+\frac12)^2-(n+1)^2} = \pi \tan(n+1)\pi = 0, \qquad \frac{n-1}{n+\frac14} < 1$$

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and so $D_{2,N} < 6$ for all large N. It follows that $\varphi_n(N) < 10$ for each given n provided N is sufficiently large.

In order to prove (23) we note that

$$\sum_{\substack{\nu=-N \\ \nu\neq n}}^{N} \frac{1}{(\lambda_\nu - \lambda_n)^2} \leq \sum_{\nu=n}^{N-1} \frac{1}{(\nu - n + \frac{1}{2})^2} + \sum_{\nu=-N}^{n-1} \frac{1}{(\nu - n + \frac{1}{2})^2}$$
$$< \sum_{\nu=-\infty}^{\infty} \frac{\pi^2}{\big((n-\frac{1}{2})\pi - \nu\pi\big)^2}$$
$$= \pi^2 \operatorname{cosec}^2\Big((n - \frac{1}{2})\pi\Big) = \pi^2.$$

Finally, if $k \geq 3$ then

$$\sum_{\substack{\nu=-N \\ \nu\neq n}}^{N} \frac{1}{|\lambda_\nu - \lambda_n|^k} \leq \frac{1}{|\lambda_{n-1} - \lambda_n|^k} + \frac{1}{|\lambda_{n+1} - \lambda_n|^k} + \sum_{\substack{\nu=-N \\ \nu\notin\{n-1,n,n+1\}}}^{N} \frac{1}{|\lambda_\nu - \lambda_n|^2}$$
$$< 2^k + 2^k + \pi^2$$

and so (24) holds. ■

The next lemma will be needed for the proof of Theorem 2.

LEMMA 2. *Let $\Psi_{m,n}$ be the function defined in (14). Then*

$$\Psi_{m,n}^{(s)}(\lambda_n) = \sum_{\substack{s_1+\cdots+s_m=s \\ 0\leq s_1,\ldots,s_m\leq s}} \frac{s!}{(s_1+1)!\cdots(s_m+1)!} \prod_{j=1}^{m} \frac{G^{(s_j+1)}(\lambda_n)}{G'(\lambda_n)}.$$

PROOF. If $H(z) := \frac{G(z)}{z-\lambda_n}$, then $H^{(s)}(\lambda_n) = \frac{1}{s+1} G^{(s+1)}(\lambda_n)$ and so by the generalized Leibnitz's formula [6, p. 219] for the s-th derivative of the product of m functions we have

$$\Psi_{m,n}^{(s)}(\lambda_n) = \sum_{\substack{s_1+\ldots+s_m=s \\ 0\leq s_1,\ldots,s_m\leq s}} \frac{s!}{s_1!\cdots s_m!} \prod_{j=1}^{m} H^{(s_j)}(\lambda_n)$$
$$= \sum_{\substack{s_1+\cdots+s_m=s \\ 0\leq s_1,\ldots,s_m\leq s}} \frac{s!}{(s_1+1)!\cdots(s_m+1)!} \prod_{j=1}^{m} \frac{G^{(s_j+1)}(\lambda_n)}{G'(\lambda_n)}.$$ ■

REMARK 4. From Theorem 1 it follows that if $C_{4,1} = 1$ and $M_s := \max_{1\leq k\leq s+1} C_{4,k}$ then for all $n \in \mathbb{Z}$

$$|\Psi_{m,n}^{(s)}(\lambda_n)| \leq (M_s)^m \frac{s!\,m^{s+m}}{(s+m)!}.$$

The following formula for the l-th derivative of the reciprocal of a function will also be needed for the proof of Theorem 2.

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LEMMA 3. *If ψ is l times differentiable at ξ and $\psi(\xi) \neq 0$ then*

$$\left(\frac{d^l}{dx^l}\left(\frac{1}{\psi(x)}\right)\right)\Big|_{x=\xi} = -\left(\frac{-1}{\psi(\xi)}\right)^{l+1} \begin{vmatrix} \binom{l}{1}\psi'(\xi) & \binom{l}{2}\psi''(\xi) & \cdots & \binom{l}{l}\psi^{(l)}(\xi) \\ \psi(\xi) & \binom{l-1}{1}\psi'(\xi) & \cdots & \binom{l-1}{l-1}\psi^{(l-1)}(\xi) \\ 0 & \psi(\xi) & \cdots & \binom{l-2}{l-2}\psi^{(l-2)}(\xi) \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \psi(\xi) & \psi'(\xi) \end{vmatrix}$$

PROOF. The formula may be obtained by solving (for $y^{(l)}(\xi)$) the system

$$\sum_{j=0}^{k} \binom{k}{j} y^{(j)}(\xi)\psi^{(k-j)}(\xi) = 0 \quad \text{for } k = l, \quad l-1, \ldots, 1 \quad \text{and} \quad y(\xi)\psi(\xi) = 1. \qquad \blacksquare$$

REMARK 5. Applying Lemma 3 to the function $\Psi_{m,n}(x)$ at the point λ_n we conclude that the quantities $\left(\frac{d^l}{dx^l}\left(\frac{1}{\Psi_{m,n}(x)}\right)\right)_{|x=\lambda_n}$ are *particular* polynomials in $\Psi'_{m,n}(\lambda_n), \ldots, \Psi^{(l)}_{m,n}(\lambda_n)$. Hence by Remark 4 they are bounded by a constant C_5 depending only on Δ and m.

The next two lemmas contain certain facts about entire functions of exponential type belonging to $L^p(\mathbb{R})$ which we will use.

LEMMA 4 [3, THEOREM 11.3.3; 15]. *If f is an entire function of exponential type τ and if $f \in L^p(\mathbb{R}), p > 0$, then*

$$\int_{-\infty}^{\infty} |f'(x)|^p \, dx \leq \tau^p \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

LEMMA 5 [13, P. 126; 3, THEOREM 6.7.15]. *If f is as in Lemma 4 then for any real increasing sequence $\{\lambda_n\}$ such that $\lambda_{n+1} - \lambda_n \geq \delta > 0$*

$$\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^p \leq \frac{8}{\pi\delta^2} \frac{e^{p\tau\frac{\delta}{2}} - 1}{p\tau} \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

The following lemmas will be needed for the proof of Theorem 3.

LEMMA 6 [13, P. 147; 12, THEOREM 3.4.2]. *Let $f(z_1, \ldots, z_n)$ be an entire function of exponential type $\vec{\tau} = (\tau_1, \ldots, \tau_n)$. If some of the variables $z_1, \ldots, z_n$ are fixed, then the resulting function is entire and of exponential type in the remaining variables. If $f \in L^p(\mathbb{R}^n)$ where $1 \leq p < \infty$ then for $1 \leq m < n$ we have*

$$\left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)|^p \, dx_1 \cdots dx_m\right)^{1/p}$$
$$\leq 2^{n-m} \left(\prod_{\nu=m+1}^{n} \tau_\nu\right)^{1/p} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1, \ldots, x_n)|^p \, dx_1 \cdots dx_n\right)^{1/p}.$$

LEMMA 7 [13, P. 160; 12, P. 116]. *If $f(z_1,\ldots,z_n)$ is an entire function of exponential type $\vec{\tau}=(\tau_1,\ldots,\tau_n)$ belonging to $L^p(\mathbb{R}^n)$ for some $p\in(0,\infty)$, then $\frac{\partial f}{\partial z_\nu}$, $\nu=1,\ldots,n$ are entire functions of exponential type $\vec{\tau}$ belonging to $L^p(\mathbb{R}^n)$.*

LEMMA 8 [13, P. 146]. *Let $f(z_1,\ldots,z_n)$ be an entire function of exponential type $\vec{\tau}=(\tau_1,\ldots,\tau_n)$ belonging to $L^p(\mathbb{R}^n)$ for some $p>0$. If x_ν, y_ν are the real and imaginary parts of z_ν for $\nu=1,\ldots,n$, then*

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|f(x_1+iy_1,\ldots,x_n+iy_n)|^p\,dx_1\cdots dx_n \\ \leq e^{p(\tau_1|y_1|+\cdots+\tau_n|y_n|)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|f(x_1,\ldots,x_n)|^p\,dx_1\cdots dx_n.$$

LEMMA 9. *If $f(z_1,\ldots,z_n)$ is an entire function of exponential type $\vec{\tau}=(\tau_1,\ldots,\tau_n)$ belonging to $L^p(\mathbb{R}^n)$ for some $p\in(0,\infty)$ and $\{(x_1^{(j)},\ldots,x_n^{(j)});j=1,2,3,\ldots\}$ is a sequence of points in $\mathbb{R}^n$ such that the euclidian distance between any two of them is at least $\delta(>0)$, then there exists a constant C depending only on p, $\vec{\tau}$, n and δ such that*

$$\sum_{j=1}^{\infty}|f(x_1^{(j)},\ldots,x_n^{(j)})|^p\leq C\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|f(x_1,\ldots,x_n)|^p\,dx_1\cdots dx_n. \tag{25}$$

PROOF. As it is well known (see [16, p. 74]), the function $|f(z_1,\ldots,z_n)|^p$ is plurisubharmonic in $\mathbb{C}^n$ and (see (2.1.1), (2.1.2) in [16]) for $\delta_1>0,\ldots,\delta_n>0$ we have

$$|f(z_1,\ldots,z_n)|^p \\ \leq\frac{2^n}{(2\pi)^n\delta_1^2\cdots\delta_n^2}\int_0^{\delta_1}\cdots\int_0^{\delta_n}\int_0^{2\pi}\cdots\int_0^{2\pi}r_1\cdots r_n|f(z_1+r_1e^{i\theta_1},\ldots,z_n+r_ne^{i\theta_n})|^p \\ dr_1\cdots dr_n\,d\theta_1\cdots d\theta_n \\ =\frac{1}{\pi^n\delta_1^2\cdots\delta_n^2}\underset{\substack{s_\nu^2+t_\nu^2\leq\delta_\nu^2\\ \nu=1,\ldots,n}}{\int\cdots\int}|f(z_1+s_1+it_1,\ldots,z_n+s_n+it_n)|^p\,ds_1\,dt_1,\ldots,ds_n\,dt_n.$$

If we now follow the method used by Plancherel and Pólya [13, p. 126] to prove Lemma 5 we will obtain (25) with

$$C=\frac{2^{2n}n^n2^n}{\pi^n\delta^{2n}p^n\tau_1\ldots\tau_n}\prod_{\nu=1}^{n}\left(e^{p\tau_\nu\frac{\delta}{2\sqrt{n}}}-1\right).$$ ∎

3. **Proofs of the theorems.**

PROOF OF THEOREM 1′. As indicated earlier (see Remark 1) we may suppose $|n|\geq 1$. Applying standard rules of differentiation we obtain

$$G_N'(z)=\prod_{\nu=-N}^{N}{}'\left(1-\frac{z}{\lambda_n}\right)+(z-\lambda_0)\sum_{j_1=-N}^{N}{}'\frac{(-1)}{\lambda_{j_1}}\prod_{\substack{\nu=-N\\ \nu\neq j_1}}^{N}{}'\left(1-\frac{z}{\lambda_\nu}\right),$$

and for $k \geq 2$

$$G_N^{(k)}(z) = k \sum_{j_1=-N}^{N}{}' \frac{(-1)}{\lambda_{j_1}} \cdots \sum_{\substack{j_{k-1}=-N \\ j_{k-1}\notin\{j_1,\dots,j_{k-2}\}}}^{N}{}' \frac{(-1)}{\lambda_{j_{k-1}}} \prod_{\substack{\nu=-N \\ \nu\notin\{j_1,\dots,j_{k-1}\}}}^{N}{}' \left(1 - \frac{z}{\lambda_\nu}\right)$$

$$+ (z - \lambda_0) \sum_{j_1=-N}^{N}{}' \frac{(-1)}{\lambda_{j_1}} \cdots \sum_{\substack{j_k=-N \\ j_k\notin\{j_1,\dots,j_{k-1}\}}}^{N}{}' \frac{(-1)}{\lambda_{j_k}} \prod_{\substack{\nu=-N \\ \nu\notin\{j_1,\dots,j_k\}}}^{N}{}' \left(1 - \frac{z}{\lambda_\nu}\right),$$

where $\{j_1, \dots, j_r\} = \emptyset$ if $r < 1$. Hence, if $n \in \{\pm1, \pm2, \dots, \pm N\}$, then

$$G_N'(\lambda_n) = -\frac{\lambda_n - \lambda_0}{\lambda_n} \prod_{\substack{\nu=-N \\ \nu\neq n}}^{N}{}' \left(1 - \frac{\lambda_n}{\lambda_\nu}\right),$$

$$G_N''(\lambda_n) = 2\left\{\frac{(-1)}{\lambda_n} \prod_{\substack{\nu=-N \\ \nu\neq n}}^{N}{}' \left(1 - \frac{\lambda_n}{\lambda_\nu}\right) + \frac{\lambda_n - \lambda_0}{\lambda_n} \sum_{\substack{j_1=-N \\ j_1\neq n}}^{N}{}' \frac{1}{\lambda_{j_1}} \prod_{\substack{\nu=-N \\ \nu\notin\{n,j_1\}}}^{N}{}' \left(1 - \frac{\lambda_n}{\lambda_\nu}\right)\right\},$$

$$G_N'''(\lambda_n) = 6 \sum_{\substack{j_1=-N \\ j_1\neq n}}^{N}{}' \frac{1}{\lambda_n\lambda_{j_1}} \prod_{\substack{\nu=-N \\ \nu\notin\{n,j_1\}}}^{N}{}' \left(1 - \frac{\lambda_n}{\lambda_\nu}\right)$$

$$- 3(\lambda_n - \lambda_0) \sum_{\substack{j_1=-N \\ j_1\neq n}}^{N}{}' \sum_{\substack{j_2=-N \\ j_2\notin\{n,j_1\}}}^{N}{}' \frac{1}{\lambda_n\lambda_{j_1}\lambda_{j_2}} \prod_{\substack{\nu=-N \\ \nu\notin\{n,j_1,j_2\}}}^{N}{}' \left(1 - \frac{\lambda_n}{\lambda_\nu}\right)$$

and for $k \geq 4$

$$G_N^{(k)}(\lambda_n) = k(k-1) \sum_{\substack{j_1=-N \\ j_1\neq n}}^{N}{}' \cdots \sum_{\substack{j_{k-2}=-N \\ j_{k-2}\notin\{n,j_1,\dots,j_{k-3}\}}}^{N}{}' \frac{(-1)^{k-1}}{\lambda_n\lambda_{j_1}\cdots\lambda_{j_{k-2}}} \prod_{\substack{\nu=-N \\ \nu\notin\{n,j_1,\dots,j_{k-2}\}}}^{N}{}' \left(1 - \frac{\lambda_n}{\lambda_\nu}\right)$$

$$+ k(\lambda_n - \lambda_0) \sum_{\substack{j_1=-N \\ j_1\neq n}}^{N}{}' \cdots \sum_{\substack{j_{k-1}=-N \\ j_{k-1}\notin\{n,j_1,\dots,j_{k-2}\}}}^{N}{}' \frac{(-1)^k}{\lambda_n\lambda_{j_1}\cdots\lambda_{j_{k-1}}} \prod_{\substack{\nu=-N \\ \nu\notin\{n,j_1,\dots,j_{k-1}\}}}^{N}{}' \left(1 - \frac{\lambda_n}{\lambda_\nu}\right).$$

Now, we are ready to prove (10). First let $k = 2$. For any given $n \in \{\pm1, \pm2, \dots\}$ and $N \geq |n|$ we clearly have

$$\frac{G_N''(\lambda_n)}{G_N'(\lambda_n)} = 2\left(\frac{1}{\lambda_n - \lambda_0} - \frac{1}{\lambda_{-n} - \lambda_n} - \sum_{\substack{j_1=-N \\ j_1\notin\{-n,n\}}}^{N}{}' \frac{1}{\lambda_{j_1} - \lambda_n}\right).$$

Since $0 \leq \Delta \leq \frac{1}{4}$,

$$\left|\frac{1}{\lambda_n - \lambda_0} - \frac{1}{\lambda_{-n} - \lambda_n}\right| \leq 2 + \frac{2}{3}$$

and so by Lemma 1

$$\left|\frac{G_N''(\lambda_n)}{G_N'(\lambda_n)}\right| < \frac{76}{3} \quad \text{for all } N \geq N_n.$$

Thus (10) holds with $C_2 = \frac{76}{3}$ and $N_{n,2} = N_n$.

Next we shall prove (10) for $k = 3$. For $n \in \{\pm 1, \pm 2, \dots\}$ and $N \geq |n|$ we have

$$\left|\frac{G_N'''(\lambda_n)}{G_N'(\lambda_n)}\right| = \left| -6\frac{1}{\lambda_n - \lambda_0} \sum_{\substack{j_1=-N \\ j_1 \neq n}}^{N}{}' \frac{1}{\lambda_{j_1} - \lambda_n} + 3 \sum_{\substack{j_1=-N \\ j_1 \neq n}}^{N}{}' \sum_{\substack{j_2=-N \\ j_2 \notin \{n, j_1\}}}^{N}{}' \frac{1}{(\lambda_{j_1} - \lambda_n)(\lambda_{j_2} - \lambda_n)} \right|$$

$$< 12\left(10 + \frac{2}{3}\right) + 3 \left| \sum_{\substack{j_1=-N \\ j_1 \neq n}}^{N}{}' \frac{1}{\lambda_{j_1} - \lambda_n} \left(\sum_{\substack{j_2=-N \\ j_2 \neq n}}^{N}{}' \frac{1}{\lambda_{j_2} - \lambda_n} - \frac{1}{\lambda_{j_1} - \lambda_n} \right) \right|$$

$$\leq 128 + 3 \max\left\{ \left(\sum_{\substack{j_1=-N \\ j_1 \neq n}}^{N}{}' \frac{1}{\lambda_{j_1} - \lambda_n} \right)^2, \sum_{\substack{j_1=-N \\ j_1 \neq n}}^{N}{}' \frac{1}{(\lambda_{j_1} - \lambda_n)^2} \right\}.$$

Hence by Lemma 1 there exists a constant C_3 such that

$$\left|\frac{G_N'''(\lambda_n)}{G_N'(\lambda_n)}\right| < C_3$$

for all large N, say for $N \geq N_{n,3}$.

In order to prove (10) for $k \geq 4$ we first observe that

$$\frac{G_N^{(k)}(\lambda_n)}{G_N'(\lambda_n)} = k(k-1)\frac{(-1)^k}{\lambda_n - \lambda_0} \sum_{\substack{j_1=-N \\ j_1 \neq n}}^{N}{}' \cdots \sum_{\substack{j_{k-2}=-N \\ j_{k-2} \notin \{n, j_1, \dots, j_{k-3}\}}}^{N}{}' \frac{1}{(\lambda_{j_1} - \lambda_n) \cdots (\lambda_{j_{k-2}} - \lambda_n)}$$

$$+ (-1)^{k+1} k \sum_{\substack{j_1=-N \\ j_1 \neq n}}^{N}{}' \sum_{\substack{j_2=-N \\ j_2 \notin \{n, j_1\}}}^{N}{}' \cdots$$

$$\sum_{\substack{j_{k-1}=-N \\ j_{k-1} \notin \{n, j_1, \dots, j_{k-2}\}}}^{N}{}' \frac{1}{(\lambda_{j_1} - \lambda_n)(\lambda_{j_2} - \lambda_n) \dots (\lambda_{j_{k-1}} - \lambda_n)}.$$

Next we note that

$$Q := \sum_{\substack{j_1=-N \\ j_1 \neq n}}^{N}{}' \sum_{\substack{j_2=-N \\ j_2 \notin \{n, j_1\}}}^{N}{}' \cdots \sum_{\substack{j_l=-N \\ j_l \notin \{n, j_1, \dots, j_{l-1}\}}}^{N}{}' \frac{1}{(\lambda_{j_1} - \lambda_n)(\lambda_{j_2} - \lambda_n) \cdots (\lambda_{j_l} - \lambda_n)}$$

can be written as a polynomial of degree at most $l - 1$ in the quantities

(26) $$\sum_{\substack{\nu=-N \\ \nu \neq n}}^{N}{}' \frac{1}{\lambda_\nu - \lambda_n}, \quad \sum_{\substack{\nu=-N \\ \nu \neq n}}^{N}{}' \frac{1}{(\lambda_\nu - \lambda_n)^2}, \dots, \sum_{\substack{\nu=-N \\ \nu \neq n}}^{N}{}' \frac{1}{(\lambda_\nu - \lambda_n)^{l-1}}.$$

It follows that $\frac{G_N^{(k)}(\lambda_n)}{G_N'(\lambda_n)}$ is also a polynomial of degree $k - 1$ in the quantities (26) and we may apply Lemma 1 to complete the proof of Theorem 1′. ∎

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REMARK 6. We wish to point out that in Theorem 1 the restriction on Δ can be considerably relaxed. Inequality (9) remains true if Δ is arbitrary but fixed and

$$|\lambda_{n+j} - \lambda_n| \geq \delta > 0 \quad \text{for} \quad n \in \mathbb{Z}, \quad j \in \mathbb{Z} \setminus \{0\}.$$

For the proof of Theorem 1 we used (6), (8) and Lemma 1. The estimate (6) holds under the weaker assumption on $\{\lambda_n\}$ whereas (8) gets replaced [2, Lemma 1] by

$$(8') \qquad |G'(\lambda_n)| \geq c_3(|\lambda_n| + 1)^{-4\Delta-1}$$

which is good enough. The remaining details are left to the reader.

PROOF OF THEOREM 2. Let E be any given compact subset of $\mathbb{C}$ and let n_0 be the smallest positive integer such that $|z| \leq n_0$ for all $z \in E$. It is easily seen that for each $\nu \in \mathbb{Z}$ the series

$$\sum_{n=-\infty}^{\infty} \left(\sum_{\mu=0}^{m-1} \frac{1}{\mu!} |f_{\mu,\lambda}(\lambda_n)(z-\lambda_n)^{\mu}| \right) |\Psi_{n,m}(z)|$$

reduces to $|f(\lambda_\nu)|$ at $z = \lambda_\nu$. Therefore, in order to prove the absolute and uniform convergence of $\mathfrak{h}_{m,\lambda}(f;z)$ on E we may assume $z \neq \lambda_\nu$ for $\nu \in \mathbb{Z}$. Now if $p > 1$ and $q := \frac{p}{p-1}$, then for all large $N \in \mathbb{N}$ we have

$$(27) \qquad \sum_{n=-N}^{N} \left(\sum_{\mu=0}^{m-1} \frac{1}{\mu!} |f_{\mu,\lambda}(\lambda_n)| |z-\lambda_n|^{\mu} \right) |\Psi_{m,n}(z)| \leq \sum_{\mu=0}^{m-1} A_\mu(N) B_\mu(N)$$

where

$$A_\mu(N) := \frac{1}{\mu!} \left(\sum_{n=-N}^{N} |f_{\mu,\lambda}(\lambda_n)|^p \right)^{1/p},$$

$$B_\mu(N) := \left(\sum_{n=-N}^{N} |(z-\lambda_n)^{\mu} \Psi_{m,n}(z)|^q \right)^{1/q}.$$

If $N \geq n_0 + 2$, then clearly

$$\begin{aligned}\big(B_\mu(N)\big)^q &= \left(\sum_{|n| \leq n_0+1} + \sum_{n_0+2 \leq |n| \leq N} \right) \left| \frac{\big(G(z)\big)^m}{(z-\lambda_n)^{m-\mu} \big(G'(\lambda_n)\big)^m} \right|^q \\ &\leq \sum_{|n| \leq n_0+1} |G(z)|^{\mu q} \left| \frac{G(z)}{z-\lambda_n} \right|^{(m-\mu)q} \frac{1}{|G'(\lambda_n)|^{mq}} \\ &\quad + \sum_{n_0+2 \leq |n| \leq N} \left| \frac{\big(G(z)\big)^m}{(z-\lambda_n)\big(G'(\lambda_n)\big)^m} \right|^q.\end{aligned}$$

We note that for $|n| \geq n_0 + 2$,

$$\frac{|\lambda_n| + 1}{|z-\lambda_n|} \leq \frac{|\lambda_n| + 1}{|\lambda_n| - |\operatorname{Re} z|} \leq \frac{n_0 + 3 - \Delta}{2 - \Delta}$$

and so from (6), (7) and (8) it follows that

$$\begin{aligned}
\big(B_\mu(N)\big)^q &\leq \sum_{|n|\leq n_0+1}\Big\{\Big(\frac{c_2'}{c_2''}\Big)^\mu\Big(\frac{c_2''}{c_3}\Big)^m e^{\pi n_0 m}(n_0+1)^{4\Delta m}(n_0+2+\Delta)^{4\Delta m}\Big\}^q\\
&\quad+\sum_{n_0+2\leq|n|\leq N}\Big\{\Big(\frac{c_2'}{c_3}\Big)^m e^{\pi n_0 m}(n_0+1)^{4\Delta m}\Big(\frac{n_0+3-\Delta}{2-\Delta}\Big)\Big\}^q\frac{1}{(|\lambda_n|+1)^{q(1-4\Delta m)}}\\
&< (2n_0+3)\Big\{\Big(\frac{c_2'}{c_2''}\Big)^\mu\Big(\frac{c_2''}{c_3}\Big)^m e^{\pi n_0 m}(n_0+1)^{4\Delta m}(n_0+2+\Delta)^{4\Delta m}\Big\}^q\\
&\quad+\Big\{\Big(\frac{c_2'}{c_3}\Big)^m e^{\pi n_0 m}(n_0+1)^{4\Delta m}\Big(\frac{n_0+3-\Delta}{2-\Delta}\Big)\Big\}^q\sum_{n=-\infty}^{\infty}\frac{1}{(|\lambda_n|+1)^{q(1-4\Delta m)}}.
\end{aligned}$$

The hypothesis $\Delta < \frac{1}{4pm}$ implies that $q(1-4\Delta m) > 1$ and therefore

$$\sum_{n=-\infty}^{\infty}\frac{1}{(|\lambda_n|+1)^{q(1-4\Delta m)}} < 1+2\sum_{n=1}^{\infty}\Big(\frac{1}{n}\Big)^{q(1-4\Delta m)} < \infty.$$

It follows that if $\Delta < \frac{1}{4pm}$ then for all $z \in E$ and all $N \geq n_0+2$ the quantity $B_\mu(N)$, where $0 \leq \mu \leq m-1$, is bounded by a constant C_{m,p,Δ,n_0} depending only on m, p, Δ and n_0. The same can be said about $A_\mu(N)$. Indeed by Leibnitz's formula and Remark 5 we have

$$|f_{\mu,\lambda}(\lambda_n)| \leq C_5\sum_{l=0}^{\mu}\binom{\mu}{l}|f^{(\mu-l)}(\lambda_n)|. \tag{28}$$

Since $|w_0+\cdots+w_\mu|^p \leq (\mu+1)^{p-1}(|w_0|^p+\cdots+|w_\mu|^p)$ we obtain

$$\begin{aligned}
A_\mu(N) &\leq \frac{1}{\mu!}C_5\bigg(\sum_{n=-N}^{N}\Big(\sum_{l=0}^{\mu}\binom{\mu}{l}|f^{(\mu-l)}(\lambda_n)|\Big)^p\bigg)^{1/p}\\
&\leq \frac{1}{\mu!}C_5(\mu+1)^{1/q}\bigg(\sum_{l=0}^{\mu}\Big(\binom{\mu}{l}\Big)^p\sum_{n=-\infty}^{\infty}|f^{(\mu-l)}(\lambda_n)|^p\bigg)^{1/p}
\end{aligned}$$

and so the desired property of $A_\mu(N)$ follows from Lemmas 4 and 5.

From (27) we conclude that if $p > 1$ then

$$\sum_{n=-\infty}^{\infty}\Big(\sum_{\mu=0}^{m-1}\frac{1}{\mu!}|f_{\mu,\lambda}(\lambda_n)(z-\lambda_n)^\mu|\Big)|\Psi_{m,n}(z)|$$

is uniformly convergent on E.

Now let $0 < p \leq 1$. If $f \in L^p(\mathbb{R})$ then ([13], [3, Theorem 6.7.1]) it belongs to $L^1(\mathbb{R})$. From Lemmas 4 and 5 it follows that $\sum_{n=-\infty}^{\infty}|f^{(\mu)}(\lambda_n)| < \infty$ for $\mu = 0, 1, \ldots$. Hence in view of (28) it suffices to check that if $\Delta \leq \frac{1}{4m}$ then for all $z \in E$ the quantity $|z-\lambda_n|^\mu|\Psi_{m,n}(z)|$ is bounded by a constant not depending on n. This can be done with the help of (6), (7) and (8) in a manner analogous to the above estimation of $B_\mu(N)$. ■

REMARK 7. At this stage we wish to mention that for $q > 1$ and all $z \in E$ (as above)

$$\sum_{n=-N}^{N}\sum_{\mu=0}^{m-1}|\Phi_{m,n,\mu}(\lambda;z)|^q \le \sum_{n=-N}^{N}\sum_{\mu=0}^{m-1}\frac{(m-\mu)^{q-1}}{(\mu!)^q}\sum_{j=0}^{m-1-\mu}\Big(\frac{1}{j!}C_5|z-\lambda_n|^{\mu+j}|\Psi_{m,n}(z)|\Big)^q$$
$$\le \sum_{\mu=0}^{m-1}\frac{(m-\mu)^{q-1}}{(\mu!)^q}C_5^q\sum_{n=-N}^{N}\sum_{j=0}^{m-1-\mu}\big(|z-\lambda_n|^{\mu+j}|\Psi_{m,n}(z)|\big)^q$$
$$\le (C_5C_{m,p,\Delta,n_0})^q\sum_{\mu=0}^{m-1}\Big(\frac{m-\mu}{\mu!}\Big)^q$$

and so the series $\sum_{n=-\infty}^{\infty}\sum_{\mu=0}^{m-1}|\Phi_{m,n,\mu}(\lambda;z)|^q$ converges uniformly on E.

In addition, it can be seen that if $\Delta \le \frac{1}{4m}$ then like $|z-\lambda_n|^{\mu}|\Psi_{m,n}(z)|$ the quantity $|\Phi_{m,n,\mu}(\lambda;z)|$ is bounded on E by a constant not depending on n.

PROOF OF THEOREM 3. By Lemma 6, $f(\cdot,z_2,\dots,z_n)$ is an entire function of exponential type $m_1\tau_1$ in the variable z_1 and belongs to $L^p(\mathbb{R})$. Hence Theorem 2 may be applied to conclude (see Remark 3) that

$$f(z_1,z_2,\dots,z_n) = \sum_{j_1=-\infty}^{\infty}\sum_{\mu_1=0}^{m_1-1}\frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}}f\big(\lambda_1(j_1),z_2,\dots,z_n\big)\Phi_{m_1,j_1,\mu_1}\big(\{\lambda_1(j)\}_{j\in\mathbf{Z}};z_1\big) \tag{29}$$

the series being absolutely and uniformly convergent on compact subsets of the z_1–plane. From Lemmas 7 and 6 it follows that $\frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}}f\big(\lambda_1(j_1),\cdot,z_3,\dots,z_n\big)$ is an entire function of exponential type $m_2\tau_2$ in z_2 and belongs to $L^p(\mathbb{R})$. By Theorem 2

$$\frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}}f\big(\lambda_1(j_1),z_2,\dots,z_n\big) = \sum_{j_2=-\infty}^{\infty}\sum_{\mu_2=0}^{m_2-1}\frac{\partial^{\mu_2}}{\partial x_2^{\mu_2}}\Big(\frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}}f\big(\lambda_1(j_1),\lambda_2(j_1,j_2),\dots,z_n\big)\Big)\Phi_{m_2,j_2,\mu_2}\big(\{\lambda_2(j_1,j)\}_{j\in\mathbf{Z}};z_2\big). \tag{30}$$

Again here the series converges absolutely and uniformly on compact subsets of the z_2–plane. Substituting the right hand side of (30) for $\frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}}f\big(\lambda_1(j_1),z_2,\dots,z_n\big)$ in (29) and using the absolute convergence of the two series we obtain

$$f(z_1,z_2,\dots,z_n) = \sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty}\sum_{\mu_1=0}^{m_1-1}\sum_{\mu_2=0}^{m_2-1}\frac{\partial^{\mu_1}\partial^{\mu_2}}{\partial x_1^{\mu_1}\partial x_2^{\mu_2}}f\big(\lambda_1(j_1),\lambda_2(j_1,j_2),\dots,z_n\big)$$
$$\times\,\Phi_{m_1,j_1,\mu_1}\big(\{\lambda_1(j)\}_{j\in\mathbf{Z}};z_1\big)\times\Phi_{m_2,j_2,\mu_2}\big(\{\lambda_2(j_1,j)\}_{j\in\mathbf{Z}};z_2\big).$$

This latter series not only converges absolutely but also uniformly on compact subsets of (z_1–plane) × (z_2–plane) by Lemma 9 and Remark 7.

Repeatedly applying Theorem 2 followed by Lemma 9 and Remark 7 as above we obtain the desired representation (21). ∎

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