BASIS OF QUADRATIC DIFFERENTIALS FOR RIEMANN SURFACES WITH AUTOMORPHISMS

by GONZALO RIERA*

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Introduction. The uniformization theorem says that any compact Riemann surface S of genus $g \geq 2$ can be represented as the quotient of the upper half plane by the action of a Fuchsian group Λ with a compact fundamental region Δ.

A classical problem going back to Poincaré is to obtain a relation between the group Λ and the algebraic equations defining S in some projective space; the solution to this problem is approached via the construction of a basis of differentials invariant by Λ. Our point of view is that it is interesting and possible to obtain direct relations for special groups, namely those representing surfaces admitting a group of automorphisms. See for example Siegel [**6**] and Streit [**7**].

Wolpert in [**8**] constructed an explicit basis of the space of quadratic differentials $Q(\Lambda)$ in terms of series depending on a partition of the surface by closed geodesics; however, it does not seem to be possible to study the action of a finite group of automorphisms on such a prescribed basis and one therefore does not obtain the equation of the algebraic curve representing S. We propose here a new method to build a basis of quadratic differentials based on the action of a finite group of automorphisms and one therefore does obtain the equation of the algebraic curve representing S. Namely we represent the group on the space of quadratic differentials θ_γ associated to closed geodesics and then find the eigenspaces; these in turn can be seen in the equation of the algebraic curve and we can identify them.

A different construction of a basis for $Q(\Lambda)$ is given by Kra in [**2**] and even though his version holds great promise we will need the analytic formulae established by Wolpert, and we give a sketch of the results needed in the first section.

Since the possible groups acting on Klein surfaces are more varied it is natural also to relate the action of an anti-conformal involution τ of S on the basis $Q(\Lambda)$. We shall determine these for some important examples in low genera leaving a larger classification for a later study.

The importance of explicit basis of quadratic differentials can also be seen in the following theorem of Royden [**5**].

Let $Q_1(\Lambda)$ denote the space $Q(\Lambda)$ with norm L_1. Then any isometry of $Q_1(\Lambda)$ is induced by an automorphism of S.

Thus, if we have an identification $Q(\Lambda) \cong R^d$, $d = 6g - 6$, is is possible to study automorphisms of Riemann surfaces in terms of the geometry of convex sets in R^d. We cannot undertake that study in this paper however.

Preliminaries. Let Λ be a Fuchsian group acting on the upper-half plane with a compact fundamental region Δ. Let C be a hyperbolic element in Λ given by

$$C(z) = \frac{az + b}{cz + d},$$

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where a, b, c, d are real numbers and $ad - bc = 1$. Then the formula

$$w_c(z) = ((a+d)^2 - 4)(cz^2 + (d-a)z - b)^{-2}$$

defines the natural quadratic differential invariant by the cyclic group $\langle C\rangle$. The Petersson series associated to the simple closed geodesic γ in Δ/Λ which C represents is the series

$$\theta_\gamma = \sum_{B \in \langle C\rangle \backslash \Lambda} w_C \circ B \qquad B'^2$$

The sum is taken over the right cosets of $\langle C\rangle$ in Λ. It is proved by Wolpert (c.f. [**8**, Theorem 3.7]) that if α_j, $1 \leq j \leq 3g-3$, is a partition of the surfaces S into pairs of pants, the differentials θ_{α_j} form a basis for $Q(\Lambda)$ over C. We do not have, as we said, any means of representing the action of a finite group of automorphisms in terms of this basis.

The Weil–Petersson metric on $Q(\Lambda)$ is given by the product

$$\langle \phi, \psi\rangle = 1/2 \operatorname{Re} \int_\Delta \phi\bar{\psi}(\operatorname{Im} z)^2$$

and the following fundamental formula holds

$$1/\pi^2(\theta_a, i\theta_\beta) = \sum_{p \in \alpha.\beta} \cos \epsilon_p,$$

where ϵ_p is the angle at each point of intersection measured counter-clockwise from α to β. We denote this skew-symmetric product by the symbol $[\theta_\alpha, \theta_\beta]$. It is a bilinear product over R and therefore the interaction of the real and complex structures on $Q(\Lambda)$ is important. Let then $\{\theta_j\}$, $1 \leq j \leq 6g-6$, be any basis of quadratic differentials over the reals. In terms of this basis, multiplication by i is given by a square matrix R such that $R^2 = -I$, and the bilinear product is given by x^tFy, where F is a skew-symmetric real matrix.

Also, since $x^t(R^tF)y$ represents the product $\langle \phi, \psi\rangle$ it follows that the matrix R^tF is symmetric positive definite.

Let now G be the full group of automorphisms of the Riemann surface, $G \simeq N(\Lambda)/\Lambda$ where $N(\Lambda)$ is the normalizer of Λ. Our method starts with any closed geodesic γ and its associated quadratic differential θ_γ; we act on it by the group and we obtain a vector space $\{\theta_{g\gamma}; g \in G\}$, invariant by G. The dimension of this vector space over R can be computed using the geometric formula for the skew-symmetric product. If this dimension is not yet $6g-6$, we take another geodesic and repeat the process. In the end we arrive to a complete basis of $Q(\Lambda)$ over R and to a formula for the complex structure given by the matrix R.

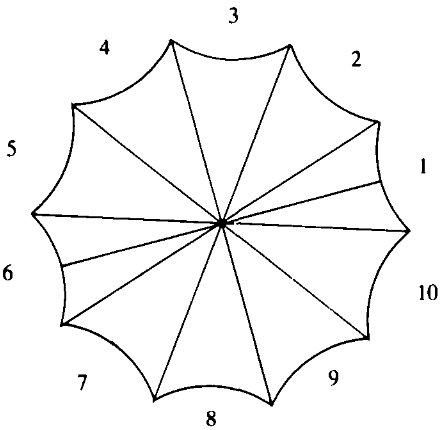
The action of the group G is now completely determined and we can therefore relate this basis of quadratic differentials to the algebraic equations for S.

The curve of genus 2 with a $Z/5$ action. The equation of the curve is

$$y^5 = (x^2 - 1)$$

and the quadratic differentials are

$$\phi_1 = \frac{dx^2}{y(x^2-1)}, \qquad \phi_2 = \frac{dx^2}{y^2(x^2-1)}, \qquad \phi_3 = \frac{dx^2}{y^3(x^2-1)}$$



1–6, 2–7, 3–8, 4–9, 5–10

Figure 1.

They can be characterized in two ways, first if we consider the action of the automorphism

$$\nu(x, y) = (-x, \zeta y) \quad \text{where} \quad \zeta = e^{2\pi i/5}$$

Then

$$\nu^*(\phi_1) = \zeta^4\phi_1, \nu^*(\phi_2) = \zeta^3\phi_2, \qquad \nu^*(\phi_3) = \zeta^2\phi_3$$

so that $\{\phi_1, \phi_2, \phi_3\}$ constitutes a basis over C under which the action of the cyclic group decomposes into eigenvalues.

They can be characterized also in terms of their zeros:

ϕ_1 has double zeros at $-1, 1$

ϕ_2 has simple zeros at $-1, 1$ double zero at ∞

ϕ_3 has a quadruple zero at ∞

We shall now consider a fundamental region for a discrete group acting on the unit disk such that its quotient is isomorphic to the curve above.

The angles at each vertex are equal to $2\pi/5$, the identifications of the sides are shown in the list above and each equilateral triangle has angles $\pi/5$. The rotation ν of order 10 with center 0 takes each triangle to its adjacent one and belongs to the normalizer of the group Λ generated by the side identifications.

Let γ be the geodesic that goes from the middle of side 1 to the middle of side 6. We consider the quadratic differential given by the Petersson series θ_γ.

The automorphism ν acts on θ_γ generating the differentials

$$\nu^*(\theta_\gamma) = \theta_{\nu(\gamma)}, \qquad \nu^*(\theta_{\nu\gamma}) = \theta_{\nu^2(\gamma)}, \qquad \nu^*(\theta_{\nu^2\gamma}) = \theta_{\nu^3\gamma}, \qquad \nu^*(\theta_{\nu^3\gamma}) = \theta_{\nu^4\gamma}$$

and

$$\theta_{\nu^5\gamma} = \theta_\gamma \qquad (\text{for } \theta_\gamma = \theta_{-\gamma}).$$

We would like to consider the dimension of the subspace generated by

$$\langle \theta_\gamma, \theta_{\nu\gamma}, \theta_{\nu^2\gamma}, \theta_{\nu^3\gamma}, \theta_{\nu^4\gamma} \rangle$$

over R. Since v is an automorphism over C and this space is closed under v, its dimension is even, that is it is either 2 or 4.

To prove that it is actually 4 it is enough to prove that three of them are linearly independent and to that effect we compute the skew-symmetric products in this subspace.

$$[\theta_\gamma, \theta_{v\gamma}] = \cos\frac{\pi}{5} = -\cos\frac{4\pi}{5} = \frac{1+\sqrt{5}}{4}$$

$$[\theta_\gamma, \theta_{v^2\gamma}] = \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$$

$$[\theta_\gamma, \theta_{v^3\gamma}] = \cos\frac{3\pi}{5} = -\cos\frac{2\pi}{5} = \frac{1-\sqrt{5}}{4}$$

$$[\theta_\gamma, \theta_{v^4\gamma}] = \cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$$

where we used the formula

$$[\theta_\alpha, \theta_\beta] = \sum_{p\in\alpha\beta} \cos\epsilon_p$$

Suppose we have a linear relation

$$a\theta_\gamma + b\theta_{v\gamma} + c\theta_{v^2\gamma} = 0$$

with real numbers a, b, c. Taking the product with θ_γ, $\theta_{v\gamma}$, $\theta_{v^2\gamma}$ gives

$$\begin{aligned} b\frac{1+\sqrt{5}}{4} + c\frac{-2+\sqrt{5}}{4} &= 0 \\ -a\frac{1+\sqrt{5}}{4} - \qquad\qquad + c\frac{1+\sqrt{5}}{4} &= 0 \\ a\frac{1-\sqrt{5}}{4} - b\frac{1+\sqrt{5}}{4} \qquad\qquad &= 0 \end{aligned}$$

where we used the fact that the product is invariant by the action of v, so that, for instance, $[\theta_{v\gamma}, \theta_{v^2\gamma}] = [\theta_\gamma, \theta_{v\gamma}]$.

If we take further the product with $\theta_{v^3\gamma}$ we obtain

$$a\frac{1-\sqrt{5}}{4} + b\frac{-1+\sqrt{5}}{4} + c\frac{1+\sqrt{5}}{4} = 0$$

From these equations it follows $a = b = c = 0$. The action of v is then necessarily

$$[v] = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

in terms of the basis $\langle \theta_\gamma, \theta_{\nu\gamma}, \theta_{\nu^2\gamma}, \theta_{\nu^3\gamma}\rangle$, as a vector space over R. (A matrix

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & d \end{pmatrix}$$

of order 5 has the form above). Thus $\theta_{\nu^4\gamma} = -\theta_\gamma - \theta_{\nu\gamma} - \theta_{\nu^2\gamma} - \theta_{\nu^3\gamma}$. The matrix for the skew-symmetric product is:

$$F = \begin{bmatrix} 0 & -\cos\frac{4\pi}{5} & \cos\frac{2\pi}{5} & -\cos\frac{2\pi}{5} \\ \cos\frac{4\pi}{5} & 0 & -\cos\frac{4\pi}{5} & \cos\frac{2\pi}{5} \\ -\cos\frac{2\pi}{5} & \cos\frac{4\pi}{5} & 0 & -\cos\frac{4\pi}{5} \\ \cos\frac{2\pi}{5} & -\cos\frac{2\pi}{5} & \cos\frac{4\pi}{5} & 0 \end{bmatrix}$$

We "diagonalize" the action of $[\nu]$ over R using the matrix

$$\Omega = \begin{bmatrix} \sin\frac{2\pi}{5} & -\cos\frac{2\pi}{5} & \sin\frac{4\pi}{5} & -\cos\frac{4\pi}{5} \\ \sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} & -\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5} & -\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} & -\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5} \\ \sin\frac{2\pi}{5} & 1 + \cos\frac{2\pi}{5} & \sin\frac{4\pi}{5} & 1 + \cos\frac{4\pi}{5} \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

so that

$$\Omega^{-1}\nu\Omega = \begin{bmatrix} \cos\frac{2\pi}{5} & -\sin\frac{2\pi}{5} & 0 & 0 \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} & 0 & 0 \\ 0 & 0 & \cos\frac{4\pi}{5} & -\sin\frac{4\pi}{5} \\ 0 & 0 & \sin\frac{4\pi}{5} & \cos\frac{4\pi}{5} \end{bmatrix}$$

$$\Omega' F \Omega = \begin{bmatrix} 0 & 3{\cdot}44 & 0 & 0 \\ -3{\cdot}44 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0{\cdot}25 \\ 0 & 0 & -0{\cdot}25 & 0 \end{bmatrix}$$

Here Ω' denotes the transpose of the matrix Ω. Therefore, if R is multiplication by i in the original subspace,

$$\Omega^{-1} R \Omega = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

for FR must be negative definite. Let $\langle f_1, f_2, f_3, f_4\rangle$ be the canonical basis for these matrices. In order that the action of $\Omega^{-1}R\Omega$ corresponds to multiplication by i we must make the correspondence

$$f_1 \leftrightarrow (i,0), \qquad f_2 \leftrightarrow (1,0), \qquad f_3 \leftrightarrow (0,i), \qquad f_4 \leftrightarrow (0,1)$$

Then

$$(\Omega^{-1}v\Omega)(f_1) = \cos\frac{2\pi}{5} f_1 + \sin\frac{2\pi}{5} f_2$$

corresponds to

$$i\cos\frac{2\pi}{5} + \sin\frac{2\pi}{5} = ie^{-2\pi i/5}$$

that is

$$(\Omega^{-1}v\Omega)(f_1) = e^{-2\pi i/5} f_1$$

Therefore

$$\begin{aligned}\Omega(f_1) &= \sin\frac{2\pi}{5}\theta_\gamma + \left(\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5}\right)\theta_{v\gamma} + \sin\frac{2\pi}{5}\theta_{v^2\gamma}\\ &= \phi_1\end{aligned}$$

is a quadratic differential such that

$$v^*(\phi_1) = \zeta^4\phi_1$$

and up to a constant multiple, can be identified with the differential $\phi_1 = \dfrac{dx^2}{y(x^2-1)}$.

In the same way

$$\begin{aligned}\Omega(f_3) &= \sin\frac{4\pi}{5}\theta_\gamma + \left(-\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5}\right)\theta_{v\gamma} + \sin\frac{4\pi}{5}\theta_{v^2\gamma}\\ &= \phi_2\end{aligned}$$

satisfies $v^*(\phi_2) = \zeta^3\phi_2$.

To complete the basis over C, let α be the geodesic beginning along side 1 from B to A and ending between the sides 8 and 9.

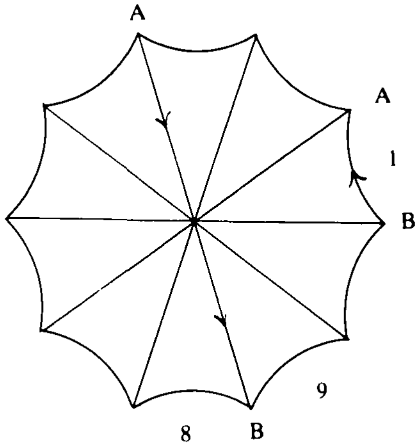


Figure 2.

The skew symmetric form is now given by

$$[\theta_\alpha, \theta_{\nu\alpha}] = \cos\frac{3\pi}{5} + \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{3-\sqrt{5}}{4}$$

$$[\theta_\alpha, \theta_{\nu^2\alpha}] = \cos\frac{\pi}{5} + \cos\frac{2\pi}{5} + \cos\frac{\pi}{5} = \frac{1+3\sqrt{5}}{4}$$

$$[\theta_\alpha, \theta_{\nu^3\alpha}] = \cos\frac{4\pi}{5} + \cos\frac{3\pi}{5} + \cos\frac{4\pi}{5} = \frac{-1-3\sqrt{5}}{4}$$

taking into account the intersections at the points A, O, B.

As before $\langle\theta_\alpha, \theta_{\nu\alpha}, \theta_{\nu^2\alpha}, \theta_{\nu^3\alpha}\rangle$ is a vector space of dimension four and ν acts by the same matrix in it. However we now obtain

$$\Omega' F\Omega = \begin{bmatrix} 0 & 4{\cdot}76 & 0 & 0 \\ -4{\cdot}76 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2{\cdot}38 \\ 0 & 0 & 2{\cdot}38 & 0 \end{bmatrix}$$

In this subspace multiplication by i is now given by R where

$$\Omega^{-1} R\Omega = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and therefore

$$\Omega(f_3) = \sin\frac{4\pi}{5}\,\theta_\alpha + \left(-\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5}\right)\theta_{\nu\alpha} + \sin\frac{4\pi}{5}\,\theta_{\nu^2\alpha} = \phi_3$$

satisfies $\nu^*(\phi_3) = \zeta^2\phi_3$.

The curve of genus 2 with a group G_{24}. The Riemann surface whose algebraic equation is

$$y^2 = x(x+1)(x-1)(x-2)(x-\tfrac{1}{2})s$$

a6admits a group of automorphisms over C of order 24 generated by elements

$$\nu(x,y) = \left(\frac{2x-1}{x+1}, -i\sqrt{27}\frac{y}{(x+1)^3}\right) \text{ of order 6}$$

$$\mu(x,y) = \left(\frac{1}{x}, i\frac{y}{x^3}\right) \text{ of order 4.}$$

A basis of quadratic differentials is

$$\phi_1 = \frac{dx^2}{y^2}, \qquad \phi_2 = \frac{x\,dx^2}{y^2}, \qquad \phi_3 = \frac{x^2\,dx^2}{y^2}$$

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and it is immediate to obtain the action of the group in terms of this basis

$$[\nu] = -\frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -4 \\ 1 & 2 & 4 \end{bmatrix}, \qquad [\nu^6] = I,$$

$$[\mu] = -\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad [\mu^2] = I.$$

The fact that $[\mu]$ has order 2 is due to the fact that $\mu^2(x, y) = (x, -y)$ is the hyperelliptic involution and therefore fixes all quadratic differentials. The group acting on $Q(S)$ is not of order 24 but 12 and is isomorphic to D_6. A check of the character of this representation against all known irreducible representations of D_6 says that there must be a matrix Ω such that

$$\Omega^{-1}[\nu]\Omega = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{bmatrix}, \qquad \zeta = \exp(2\pi i/6)$$

$$\Omega^{-1}[\mu]\Omega = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -\zeta^4 \\ 0 & -\zeta^2 & 0 \end{bmatrix}$$

and indeed $\Omega = \begin{bmatrix} 1 & \zeta^2 & \zeta^4 \\ -1 & 2\zeta^4 & 2\zeta^2 \\ 1 & 1 & 1 \end{bmatrix}$ works.

We shall then look on the upper-half plane for quadratic differentials on which the group acts by these latter matrices.

We uniformize this Riemann surface by a considering a polygon of 12 sides as in Figure 3 in the unit disk, whose sides are identified as shown. There are three cycles at the boundary, proving that if Λ is the Fuchsian group generated by the corresponding Moebius transformations, the quotient Δ/Λ has genus two. The polygon is divided into 48

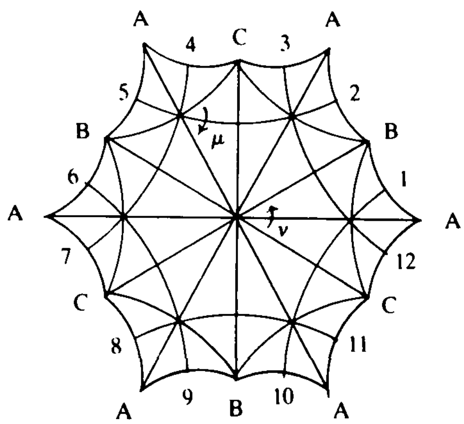

Figure 3.

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triangles with angles $\pi/6$, $\pi/4$, $\pi/2$. A pair of triangles, one shaded and one unshaded, is the fundamental region for a triangle group N such that $[N:\Lambda]=24$; this group is generated by ν, μ as indicated in the figure, of orders 6 and 4 respectively. Thus N/Λ acts on Δ/Λ as a finite group of automorphisms of order 24 and since (it can be shown that) there is only one such surface in genus 2, it is the Riemann surface we are considering.

Let α be the geodesic from A to A passing through the fixed points of ν, μ. The skew-symmetric product, computed from the geometry of this fundamental polygon, gives the values

$$[\theta_\alpha, \theta_{\nu\alpha}] = \cos\frac{\pi}{3} + \cos\frac{\pi}{3} = 1,$$

$$[\theta_\alpha, \theta_{\mu\alpha}] = \cos\frac{\pi}{2} + \cos\frac{\pi}{2} = 0,$$

$$[\theta_\alpha, \theta_{\nu\mu\alpha}] = 0, \qquad [\theta_{\mu\alpha}, \theta_{\nu\mu\alpha}] = \cos\frac{2\pi}{3} + \cos\frac{2\pi}{3} = -1,$$

$$[\theta_{\nu\alpha}, \theta_{\mu\alpha}] = 0,$$

$$[\theta_{\nu\alpha}, \theta_{\nu\mu\alpha}] = 0.$$

It follows that the vector space over R

$$\langle \theta_\alpha, \theta_{\nu\alpha}, \theta_{\mu\alpha}, \theta_{\nu\mu\alpha} \rangle$$

is of dimension 4 and that the skew-symmetric form on it is

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The action of the generators of the automorphism group obtained is

$$[\nu] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$[\mu] = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

With the change of variables given by

$$\Omega = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}, \qquad D = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 0 & 1 \end{bmatrix},$$

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we obtain

$$\Delta^{-1}[\nu]\Delta = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 & 0 \\ \sin\frac{2\pi}{3} & \cos\frac{3\pi}{3} & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ 0 & 0 & \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}$$

$$\Delta^{-1}[\mu]\Delta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\Delta' F \Delta = \begin{bmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3}/2 \\ 0 & 0 & +\sqrt{3}/2 & 0 \end{bmatrix}$$

The only real matrix $\Delta^{-1}R\Delta$ commuting with these and satisfying RF negative definite is

$$\Delta^{-1}R\Delta = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

whence the complex structure is determined. It follows that $\{\theta_\alpha, \theta_{\mu\alpha}\}$ is a vector space of dimension two over C over which the group acts by the required matrices.

Thus, up to a constant factor,

$$\theta_\alpha = (-1 + 2\zeta^4 x + 2\zeta^2 x^3)\frac{dx^2}{y^2},$$
$$\theta_{\mu\alpha} = (1 + x + x^2)\frac{dx^2}{y^2}.$$

We have yet to complete this basis to a three dimensional complex space, or, what is the same, the first space to a six dimensional real space. Let β be the geodesic from sides 1 to 10. The sixth geodesic is harder to find and is as follows.

It is a simple closed geodesic and the skew symmetric form is now

$$F^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 1/\sqrt{2} & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 & 2\cos\theta \\ 0 & 0 & 0 & 0 & -2\cos\theta & 0 \end{bmatrix}$$

Here $0 < \theta < \frac{\pi}{4}$ is the angle that β makes with ε at the points of intersection. Hence $\{\theta_\alpha, \theta_{\nu\alpha}, \theta_{\mu\alpha}, \theta_{\nu\mu\alpha}, \theta_\beta, \theta_\varepsilon\}$ is a vector space of dimension 6 over R and it can be shown that the group acts via $-I$ in the last two factors. Thus $\{\theta_\alpha, \theta_{\nu\alpha}, \theta_\beta\}$ is of dimension 3 over

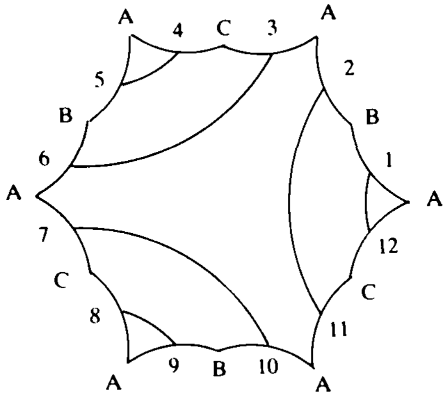


Figure 4.

C and, up to a constant,

$$\theta_B = (1 + \zeta^2 x + \zeta^4 x^2)\frac{dx^2}{y^2}.$$

The Riemann surface with 168 automorphisms. Let us consider the group $PSL(2, Z)$ of integer matrices with determinant 1, modulo ±id, acting on the upper half plane H in C via the formula

$$z \to (Az + b)(cz + d)^{-1}.$$

We use the notation $\Gamma(p)$ to denote the set of all matrices congruent to the identity modulo p, p prime, so that we have the exact sequence

$$1 \to \Gamma(p) \to \Gamma(1) \to PSL(2, F_p) \to 1,$$

where F_p denotes the field with p elements. The group $\Gamma(p)$ acts discontinuously on H and the quotient S can be compactified by adding the parabolic punctures; hence we have a branched covering map

$$C \to \overline{H/\Gamma(p)} \to \overline{H/\Gamma(1)} = P_1$$

Figure 5 modelled on Klein [**1**] (where $-$ denotes compactification) making C into a Riemann surface with Galois group $PSL(2, F_p) = G$ over P_1.

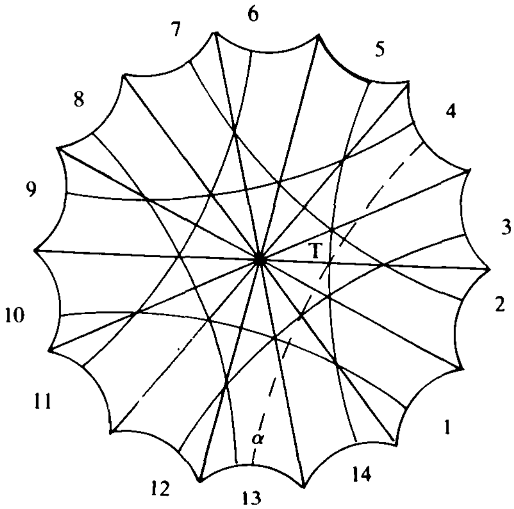
Felix Klein considered this construction for $p = 7$ in [*c.f.* [**1**]] where he also built a universal cover in the unit disk as a subgroup of a triangular group of type [**2, 3, 7**] whose fundamental region we shall use in Figure 5.

The curve C has genus three and it is the only such curve admitting a maximal group of automorphisms of order 168. Algebraically, the equation of the curve is

$$x^3 y + y^3 z + z^3 x = 0$$

in complex projective space, and the generators of the group G represented as linear automorphisms are

$$S = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^4 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \qquad \zeta = \exp(2\pi i/7),$$



11–2, 1–6, 3–8, 5–10, 7–12, 9–14, 13–4

Figure 5.

of order seven and

$$T=\begin{pmatrix} a & b & c\\ b & c & a\\ c & a & b\end{pmatrix}$$

with $a=(\zeta^2-\zeta^{-2})/\sqrt{-7}$, $b=(\zeta^4-\zeta^{-4})/\sqrt{-7}$, $c=(\zeta-\zeta^{-1})/\sqrt{-7}$, of order two. See Riera [**3**].

We determine first the action of the generators of the group on a basis of quadratic differentials; these are

$$\varphi_1=x^2\Omega^2,\qquad \varphi_2=xy\Omega^2,\qquad \varphi_3=y^2\Omega^2,$$
$$\varphi_4=xz\Omega^2,\qquad \varphi_5=yz\Omega^2,\qquad \varphi_6=z^2\Omega^2,$$

with $\Omega=(x\,dy-y\,dx)/(y^3+3z^2x)$.

It is easy to see that S acts via the diagonal matrix $[\zeta^2,\zeta^5,\zeta,\zeta^3,\zeta^6,\zeta^4]$ with different eigenvalues and that T acts via the symmetric product of T with itself.

Let α be the closed geodesic from side 4 to side 13 through the fixed points of order two. We identify S with the rotation of order seven at the origin, and denote by Δ the acute angle between α and $S(\alpha)$. The skew-symmetric product is given then on the real vector space $V=\langle\theta_\alpha,\theta_{S\alpha},\theta_{S^2\alpha},\theta_{S^3\alpha},\theta_{S^4\alpha},\theta_{S^5\alpha}\rangle$ by

$$\theta_\alpha\,.\,\theta_{S\alpha}=\cos\Delta,$$
$$\theta_\alpha\,.\,\theta_{S^2\alpha}=\cos(\pi-\Delta)=-\cos\Delta,$$
$$\theta_\alpha\,.\,\theta_{S^3\alpha}=0,$$
$$\theta_\alpha\,.\,\theta_{S^4\alpha}=0,$$
$$\theta_\alpha\,.\,\theta_{S^5\alpha}=\cos\Delta.$$

The matrix of this product on V is

$$C=\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}. \cos\Delta,$$

which shows that $\dim_R V = 6$. If $V \neq iV$ there would be a repeated eigenvalue under the decomposition of S; therefore $V = iV(=RV)$ and the matrix on V of S is

$$S=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Let now $\beta = TS^3T(\alpha)$, where we mean by S the rotation of order seven at the center of the fundamental polygon and T the elliptic element of order two with fixed point at the nearby point of the triangulation in Figure 4. The matrix of the intersection product on $W = \langle \theta_\beta, \theta_{S\beta}, \theta_{S^3\beta}, \theta_{S^4\beta}, \theta_{S^5\beta} \rangle$ is now

$$D=\begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{pmatrix} \cos\Delta$$

so that $\dim_R W = 6$ and also $W = iW$. S acts on W by the same matrix as in V.

If Ω is the matrix that diagonalizes S over R, that is

$$\Omega^{-1}S\Omega = \begin{bmatrix} \cos\frac{2\pi}{7} & -\sin\frac{2\pi}{7} \\ \sin\frac{2\pi}{7} & \cos\frac{2\pi}{7} \end{bmatrix} \oplus \begin{bmatrix} \cos\frac{4\pi}{7} & -\sin\frac{4\pi}{7} \\ \sin\frac{4\pi}{7} & \cos\frac{4\pi}{7} \end{bmatrix}$$
$$\oplus \begin{bmatrix} \cos\frac{6\pi}{7} & -\sin\frac{6\pi}{7} \\ \sin\frac{6\pi}{7} & \cos\frac{6\pi}{7} \end{bmatrix}$$

we obtain

$$\Omega' C\Omega = \begin{bmatrix} 0 & -1{\cdot}8 \\ 1{\cdot}8 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2{\cdot}2 \\ -2{\cdot}2 & 0 \end{bmatrix} (\cos \Delta),$$

$$\Omega' D\Omega = \begin{bmatrix} 0 & -11{\cdot}3 \\ 11{\cdot}3 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -0{\cdot}6 \\ 0{\cdot}6 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -2{\cdot}6 \\ 2{\cdot}6 & 0 \end{bmatrix} (\cos \Delta).$$

It follows that we can obtain theta series that are eigenfunctions for the eigenvalues ζ, ζ^2, ζ^{-2}, ζ^3, ζ^{-3}; we have to find ζ^{-1}.

To this end, let γ be the geodesic joining the vertex between sides 10 and 11 through the center of the polygon to the opposite vertex between sides 3 and 4 and continuing as side 1.

We compute in this case

$$[\theta_\gamma, \theta_{S\gamma}] = \cos\frac{2\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{3\pi}{7},$$

$$[\theta_\gamma, \theta_{S^2\gamma}] = \cos\frac{4\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{2\pi}{7},$$

$$[\theta_\gamma, \theta_{S^3\gamma}] = \cos\frac{6\pi}{7} + \cos\frac{3\pi}{7} - \cos\frac{5\pi}{7}.$$

The intersection matrix E is determined and we obtain

$$\Omega' E\Omega = \begin{bmatrix} 0 & 3{\cdot}0 \\ -3{\cdot}0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 7{\cdot}4 \\ -7{\cdot}4 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -2{\cdot}1 \\ 2{\cdot}1 & 0 \end{bmatrix},$$

where we have in the first summand the eigenvalue ζ^{-1}.

In all we obtain

$$\theta_\alpha + \left(1 + \cos\frac{2\pi}{7}\right)\theta_{S\alpha} + \left(1 + \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7}\right)\theta_{S^2\alpha} - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right)\theta_{S^3\alpha} - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7}\right)\theta_{S^4\alpha} - \cos\frac{2\pi}{7}\,\theta_{S^5\alpha}$$

as eigenfunction for S with eigenvalue ζ

$$\theta_\beta + \left(1 + \cos\frac{4\pi}{7}\right)\theta_{S\beta} + \left(1 + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right)\theta_{S^2\beta} - \left(\cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{12\pi}{7}\right)\theta_{S^3\beta} - \left(\cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right)\theta_{S^4\beta} - \cos\frac{4\pi}{7}\,\theta_{S^5\beta}$$

for ζ^2,

$$\theta_\beta + \left(1 + \cos\frac{6\pi}{7}\right)\theta_{S\beta} + \left(1 + \cos\frac{6\pi}{7} + \cos\frac{12\pi}{7}\right)\theta_{S^2\beta} - \left(\cos\frac{6\pi}{7} + \cos\frac{18\pi}{7}\right)\theta_{S^3\beta} - \left(\cos\frac{6\pi}{7}\right)\theta_{S^4\beta} - \cos\frac{6\pi}{7}\,\theta_{S^5\beta}$$

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for ζ^3,

$$\theta_\alpha + \left(1+\cos\frac{6\pi}{7}\right)\theta_{S\alpha} + \left(1+\cos\frac{6\pi}{7}+\cos\frac{12\pi}{7}\right)\theta_{S^2\alpha} - \left(\cos\frac{6\pi}{7}+\cos\frac{12\pi}{7}+\cos\frac{18\pi}{7}\right)\theta_{S^3\alpha} - \left(\cos\frac{6\pi}{7}+\cos\frac{12\pi}{7}\right)\theta_{S^4\alpha} - \cos\frac{6\pi}{7}\theta_{S^5\alpha}$$

for ζ^4,

$$\theta_\alpha + \left(1+\cos\frac{4\pi}{7}\right)\theta_{S\alpha} + \left(1+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)\theta_{S^2\alpha} - \left(\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{12\pi}{7}\right)\theta_{S^3\alpha} - \left(\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)\theta_{S^4\alpha} - \cos\frac{4\pi}{7}\theta_{S^5\alpha}$$

for ζ^5, and

$$\theta_\gamma + \left(1+\cos\frac{2\pi}{7}\right)\theta_{S\gamma} + \left(1+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}\right)\theta_{S^2\gamma} - \left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)\theta_{S^3\gamma} - \left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}\right)\theta_{S^4\gamma} - \cos\frac{2\pi}{7}\theta_{S^5\gamma}$$

for ζ^6.

The action of Klein's involution. Let $\tau: S \to S$ be an anti-conformal involution. This involution acts naturally on Petersson quadratic differentials by the formula

$$\tau(\theta_\alpha) = \theta_{\tau(\alpha)}.$$

If (θ_{α_j}), $1 \le j \le 6g-6$, is a basis of $Q(S)$ over R, the action of τ is given by a real square matrix B satisfying $B^2 = I$. We will establish the main relations of this matrix with F, the matrix of the skew-symmetric product, and with R, the matrix of multiplication by i in $Q(S)$.

First, since τ reverses orientation we have as a direct consequence of the formula in terms of the angles of intersection the relation

$$[\theta_\alpha, \theta_\beta] = -[\theta_{\tau(\alpha)}, \theta_{\tau(\beta)}]$$

or, in terms of real matrices $B'FB = -F$. Consider as before a fixed point free Fuchsian group Γ acting on the upper half plane H such that $H/\Gamma = S$. The involution τ lifts to an anti-conformal involution in H and we may assume, modulo conjugation, that we have $\tau(z) = -\bar{z}$. Then we have

$$\theta_c(\tau(z)) = \overline{\theta_{\tau(c)}(z)}. \tag{1}$$

Proof. Let us write $\tau B r = B^*$ for B in Γ.

$$\theta_c(\tau(z)) = \sum_{B \in \langle c\rangle\backslash\Gamma} w_c(B(\tau(z))) \qquad B'(\tau(z))^2$$
$$= \sum_{B^* \in \langle c^*\rangle\backslash\Gamma} w_c(\tau(B^*(z))) \qquad \overline{B^{*\prime}(z)^2}.$$

But it is easy to see that

$$w_c(\tau(z)) = \overline{w_{c^*}(z)} = \overline{w_{\tau(c)}(z)}.$$

Therefore

$$\theta_c(\tau(z)) = \sum_{B^* \in \langle\tau(c)\rangle\backslash\Gamma} \overline{w_{\tau(c)}(B^*(z))} \qquad \overline{B^{*\prime}(z)^2}$$
$$= \overline{\theta_{\tau(c)}(z)}.$$

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We now prove that

$$[\theta_{\tau(\alpha)}, i\theta_\beta] = -[\theta_{\tau(\beta)}, i\theta_\alpha]. \tag{2}$$

Proof.

$$\begin{aligned}[\theta_{\tau(\alpha)}, i\theta_\beta] &= -\frac{1}{2\pi^2}\operatorname{Re}\int_\Delta \theta_{\tau(\alpha)}(z)\overline{\theta_\beta(z)}\,(\operatorname{Im} z^2)\\ &= -\frac{1}{2\pi^2}\operatorname{Re}\int_\Delta \overline{\theta_\alpha(\tau(z))}\,\overline{\theta_\beta(z)}\,(\operatorname{Im} z)^2\\ &= \frac{1}{2\pi^2}\operatorname{Re}\int_\Delta \overline{\theta_\alpha(\tau(z)}\,\overline{\theta_\beta(\tau(z))}\,(\operatorname{Im} z)^2\\ &= \frac{1}{2\pi^2}\operatorname{Re}\int_\Delta \overline{\theta_\alpha(z)}\,\overline{\theta_{\tau(\beta)}(z)}\,(\operatorname{Im} z^2)\\ &= -[\theta_{\tau(\beta)}, i\theta_\alpha],\end{aligned}$$

where we used twice the formula (1) and the fact that $dx\,dy$ changes sign under $\tau(z) = -\bar{z}$.

Since in terms of a basis over R we have

$$B(\theta_{\alpha_j}) = \theta_{\tau(\alpha_j)} = \sum_{k=1}^{6g-6} b_{kj}\theta_{\alpha_k},$$

$$R(\theta_{\alpha_j}) = i\theta_{\alpha_j} = \sum_{k=1}^{6g-6} r_{kj}\theta_{\alpha_k},$$

the formula (2) gives the identity

$$B^tFR = -(F^R)^tB = -R^tFB.$$

But R^tF is symmetric so that $-FR = R^tF$. Also, since $B^tFB = -F$, we have

$$\begin{aligned}(B^tFB)BR &= +FRB,\\ -FBR &= +FRB,\end{aligned}$$

Hence $BR = -RB$.

This proves the following

THEOREM 1. *In terms of a basis of $Q(\Gamma)$ over R the matrices of multiplication by i, of the skew-symmetric product and of Klein's involution are given by a triple (R, F, B) such that $R^2 = -I$, $F^t = -F$, $B^tFB = -F$ and $BR = -RB$.*

It is to be observed that in terms of abelian varieties a similar construction of a Jacobian of a Klein surface is obtained in Riera [**4**]. The difference is that in this case F need not be an integral matrix.

An automorphism of a Klein surface is given by a conformal mapping $\sigma: S \to S$ commuting with τ. In terms of a basis over R of $Q(S)$ it is given by a matrix A such that

$$AR = RA, \qquad A^tFA = F, \qquad AB = BA.$$

We find these in the examples we already have.

1. In the case of genus two with an action of $Z/5$, the reflection τ is the reflection on the geodesic from the middle of side 1 to the middle of side 6.

The matrix of τ in the first subspace of dimension four spanned by θ_γ, $\theta_{\nu\gamma}$, $\theta_{\nu^2\gamma}$, $\theta_{\nu^3\gamma}$

$$B=\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & -1 & 0 & 1\\ 0 & -1 & 1 & 0\end{bmatrix}.$$

Under conjugation by Ω we obtain

$$\Omega^{-1}B\Omega=\begin{bmatrix}\cos\frac{4\pi}{5} & -\sin\frac{4\pi}{5} & 0 & 0\\ -\sin\frac{4\pi}{5} & -\cos\frac{4\pi}{5} & 0 & 0\\ 0 & 0 & \cos\frac{2\pi}{5} & \sin\frac{2\pi}{5}\\ 0 & 0 & \sin\frac{2\pi}{5} & -\cos\frac{2\pi}{5}\end{bmatrix}$$

and its properties are then easily checked.

2. For the curve of genus 2 with a group of automorphisms of order 24 the reflection τ is the reflection on the three geodesics from sides 12 to 3, 4 to 7 and 8 to 11. In the vector space spanned by

$$\{\theta_\alpha, \theta_{\nu\alpha}, \theta_{\mu\alpha}, \theta_{\nu\mu\alpha}\}$$

the matrix of τ is

$$B=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}.$$

Since $\Delta^{-1}B\Delta=B$ the properties

$$B^t(\Delta^t F\Delta)B=-\Delta^t F\Delta, \qquad B^2=I,$$

$B(\Delta^{-1}r\Delta)=-(\Delta^{-1}R\Delta)B$ are immediate.

3. For the surface of genus 3 with 168 automosphisms, let B be the reflection on the geodesic from the vertex at sides 10 and 11, through the center to the vertex at sides 3 and 4 and continuing as side 1. Since $BSB=S^{-1}$ and B is anti-linear over C, B fixes the eigenfunctions and therefore the matrix over the reals has in each eigenspace the form

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

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P. Universidad Católica
Casila 306 Correo 22
Santiago, Chile
E-mail: griera@mat.puc.cl