ON GROUPS WITH SMALL ENGEL DEPTH

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Every finite group $G$ satisfies a law $[x, y^r] = [x, y^s]$ for some positive integers $r < s$. The minimal value of $r$ is called the depth of $G$. It is well known that groups of depth 1 are abelian. In this paper we prove the following. Let $G$ be a finite group of depth 2. Then $G/F(G)$ is supersoluble, metabelian and has abelian Sylow $p$-subgroups for all odd primes $p$. Moreover, $l_p(G) \leq 1$ for $p$ odd and $l_2(G^2) \leq 1$.

1. Introduction

If $G$ is a finite group, then there exist positive integers $r < s$ such that for all $x, y \in G$ the following holds: $[x, y^r] = [x, y^s]$. If $r$ is chosen minimal with respect to this property, we call $r$ the (Engel-) depth of $G$. Let $\mathcal{V}_r$ be the class of all finite groups of Engel depth less than or equal to $r$. Obviously, a finite nilpotent group belongs to $\mathcal{V}_r$ if and only if it satisfies the $r$th Engel condition.

In [7, Theorem 3.2] it has been proved that groups in $\mathcal{V}_1$ are abelian. By contrast, the groups $\text{PSL}(2, 5)$ and $\text{PSL}(2, 8)$ are of depth 3 (D. Nikolova, Personal Communication).

Here we consider groups of depth 2. It turns out that these groups are soluble. More precisely, we shall prove

THEOREM. Let $G$ be a finite group of depth 2. Then

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(a) $G/F(G)$ is supersoluble, metabelian and for all odd primes $p$ the Sylow $p$-subgroups of $G/F(G)$ are abelian,

(b) if $p$ is an odd prime, $l_p(G) \leq 1$; also $l_2(a^2) \leq 1$.

Unless otherwise stated, all groups considered in this paper are finite.

2. The structure of groups in $V_2$

This section is devoted to a proof of the main theorem mentioned in the introduction. We first note a simple observation that turns out to be very useful in the proofs.

**Lemma 1.** Let $H \in V_2$ and let $A$ be a nilpotent normal subgroup of $H$. Then for each $a \in A$ the normal closure $\langle a^H \rangle$ is abelian.

**Proof.** Let $b \in H$. By assumption, we have $[b, 2^a] = [b, 2^{k+a}]$ for some $k$. So $[b, 2^a] = [b, 2^{k+a}] = \gamma_{k+2}(A)$ for all positive integers $t$. As $A$ is nilpotent, we get $[b, 2^a] = 1$ and so $[a, a^b] = 1$. This implies that $\langle a^H \rangle$ is abelian.

We now prove that all groups in $V_2$ are soluble (this fact has been found independently by D. Nikolova). In order to do this, we examine the minimal simple groups (see [11]).

**Lemma 2.** The Suzuki groups $Sz(q)$ and $SL(3, 3)$ do not belong to $V_2$.

**Proof.** Let $G = Sz(q)$, let $A$ be a Sylow 2-subgroup of $G$ and let $H = N_G(A)$. Any element in $H$ of order $q - 1$ acts transitively on $(A/\Phi(A))^H$ and so for any $a \in A\Phi(A)$ we have $A = \langle a^H \rangle$. But $A$ is non-abelian and so $H \not\leq V_2$ by Lemma 1. This proves $G \not\leq V_2$.

The group $SL(3, 3)$ contains a subgroup $H$ isomorphic with $SL(2, 3)$. The same argument yields $SL(3, 3) \not\leq V_2$.

We now deal with the remaining minimal simple groups $G = PSL(2, q)$.
The search for suitable elements proving $G \not\leq V_2$ has been eased considerably by computer calculations performed on a TR440 at the Rechenzentrum der Universität Würzburg.

**Lemma 3.** Let $q \geq 4$ be a prime power. Then $\text{PSL}(2, q) \not\leq V_2$.

**Proof.** Because of the isomorphism $\text{PSL}(2, 5) \cong \text{PSL}(2, 4)$ we may assume $q \not= 5$. Let $e \in \text{GF}(q)$ with $e^2 \not= \pm 1$. Let

$$
\begin{bmatrix}
-e^2(e^2+1)(e^2-1)^{-1} & (e^2-1)^{-3} \\
-e^2(e^2-1)^2 & e^{-2}(e^2+1)^{-1}
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
e^{-1} & 0 \\
0 & e
\end{bmatrix}
$$

A straightforward computation yields

$$
[x, e^ky] = \begin{bmatrix}
e^{-2} & e^{-2}(e^2-1)^{-1} \\
0 & e^2
\end{bmatrix}
$$

So for any $k \geq 3$, we have $[x, e^ky] = \begin{bmatrix}1 & \ast \\0 & 1\end{bmatrix}$. As $e^2 \not= \pm 1$ we have shown $[x, e^ky] \not= \pm [x, e^ky]$ for all $k \geq 3$. Hence $\text{PSL}(2, q) \not\leq V_2$.

We now prove the first part of our main theorem.

**Theorem A.** Let $G \in V_2$. Then $G/F(G)$ is supersoluble.

**Proof.** Let $G$ be a minimal counterexample. Lemma 2, Lemma 3 and [11] imply that $G$ is soluble. By [2, 2.9] we know that $G$ is a split extension of its unique minimal normal subgroup $N$ by a complement $Q$ and all proper subgroups of $Q$ are supersoluble. From [5] we infer that $Q$ has a unique normal Sylow subgroup $A$ possessing a complement $B$ in $Q$. Moreover, $A/\Phi(A)$ is an irreducible $B$-module and $A$ is noncyclic. Also $\Phi(A) \leq Z(A)$.

We first show that $A$ is elementary abelian. Let $a \in A \setminus \Phi(A)$. By
Lemma 1 we know that \( \langle a^B \rangle \) is abelian. As \( B \) acts irreducibly on \( A/\Phi(A) \), we have \( A = \langle a^B \rangle \cdot \Phi(A) = \langle a^B \rangle \) and so \( A \) is abelian. The proof of part (f) of [5, Satz 1] now yields that \( A \) is elementary.

Let \( 1 \neq a \in A \) and let \( n \in N \) and \( b \in B \) be arbitrary. Then \([b, na] = [b, a][b, n]^a \) and so

\[
[b, 2^na] = ([b, a][b, n]^a, na]
= [b, a, na]^a[b, n]^a
= ([b, a, a][b, a, n]^a)[b, n]^a[b, n]^a
= [b, a, n]^a[b, n]^a
\]
as \( [b, a, a] = 1 \).

From \([b, 2^na] \in N\) we obtain by a straightforward computation

\[
[b, 2^{k+\kappa}na] = [b, a, n, \kappa a][b, n, a, \kappa a]^a.
\]

As \( G \in V_2 \), there exists some \( k \) with

\[
[b, a, n] \cdot [b, n, a] = [b, a, n, \kappa a][b, n, a, \kappa a].
\]

In particular, we get

\[
[b, a, n][b, n, a] \in \langle N, a \rangle
\]
and so

\[
[b, a, n] \in \langle N, a \rangle.
\]

Hence \([n, [b, a]] = [n, a^{-b}a] \in \langle N, a \rangle\) and finally \([n, a^{-b}] \in \langle N, a \rangle\).

As \( n \in N \) has been chosen arbitrarily, we get \([N, a^{-b}] \leq \langle N, a \rangle\).

The latter holds for any \( b \in B \) and so \([N, a^{-b}] \leq [N, a] \) for all choices \( b \in B \). As \( B \) acts irreducibly on \( A \), we have \( A = \langle a^B \rangle \) and so we arrive at \( N = \langle N, a \rangle \leq [N, a] \). This implies \( C^*_N(a) = 1 \).

Hence every nonidentity element of \( A \) acts fixed point freely on \( N \) and so \( A \) is cyclic. This, however, contradicts the structure of \( A \).
Using Theorem A, we can now prove

**THEOREM B.** Let \( G \in V_2 \). Then for all odd primes \( p \), the quotient \( G/F(G) \) has abelian Sylow \( p \)-subgroups.

**Proof.** Let \( p \) be an odd prime and let \( G \) be a counterexample of least possible order. From [2, 2.9] we infer that \( G \) is a split extension of a uniquely determined minimal normal subgroup \( N = F(G) \) by a complement \( Q \). Moreover, all proper subgroups of \( Q \) have abelian Sylow \( p \)-subgroups. This implies that \( Q \) is a nonabelian \( p \)-group all of whose proper subgroups are abelian. So \( Q \) is nilpotent of class two by a result of Rédei [8, p. 309]. Also, \( N \) is a \( p' \)-group.

We claim that every nonidentity element of \( Q \) acts fixed point freely on \( N \). Indeed, let \( 1 \neq b \in Q \) with \( C_N(b) \neq 1 \) be given. As \( Q \) acts faithfully and irreducibly on \( N \), we have \( b \notin Z(Q) \). So there exists \( a \in Q \) with \( z = [a, b] \neq 1 \). Moreover, \( z \in Z(Q) \).

Let \( n \in C_N(b) \). We now compute \([a, n^k b]\). First

\[
[a, n^k b] = an^k \quad \text{for some } n^k \in N.
\]

As \( Q \) is nilpotent of class two, we have \([a, n^k b] \in N \) for all \( k \geq 2 \).

As \( G \in V_2 \), there exists some positive integer \( d \) such that

\[
[a, 2n^k b] = [a, 2^d n^k b].
\]

Let \( n_2 = [a, 1+dn^k b] \). Then

\[
[sn_1, nb] = [n_2, nb].
\]

Hence \( sn_1 n_2^{-1} \in C_k(nb) \).

As \( nb = bn \) and the orders of \( n \) and \( b \) are coprime, we have \( n \in (nb) \). So \( sn_1 n_2^{-1} \) centralizes \( n \). From this we finally get \( n \in C_N(z) \). This implies \( C_N(b) \leq C_N(z) = 1 \) which contradicts the choice of \( b \).

From [6, Theorem 10.3.1, p. 339] we conclude that \( Q \) is cyclic. This contradicts the structure of \( Q \).

**COROLLARY.** Let \( G \in V_2 \). Then \( G/F(G) \) is metabelian.

**Proof.** Theorem A implies that \( Q = G/F(G) \) is supersoluble, and so \( Q' \) is nilpotent. By Theorem B, all Sylow subgroups of odd order of \( Q' \)
are abelian. Let $S$ be a Sylow 2-subgroup of $Q$. As $G \in V_2$, $S$ satisfies the second Engel condition and so is nilpotent of class two. Hence $S'$ is abelian. As $Q$ is 2-nilpotent, $S'$ is a Sylow 2-subgroup of $Q'$. So $Q'$ is abelian and the result follows.

From this we can deduce a property of infinite soluble groups of depth two.

**COROLLARY.** Let $G$ be poly- (abelian or finite). Assume that for any $x, y \in G$ there exists some positive integer $s = s(x, y) > 2$ such that $[x, 2^y] = [x, y]$. Then $G$ is (2-Engel)-by-metabelian.

**Proof.** Let $U$ be a finitely generated subgroup of $G$. From [4, Theorem B] we infer that $U$ is finite-by-nilpotent, and so $U$ is residually finite. Every finite quotient of $U$ belongs to the variety $\mathcal{V}$ of all (2-Engel)-by-metabelian groups. This implies $U \in \mathcal{V}$ and so $G \in \mathcal{V}$.

The remainder of our main theorem now follows from

**THEOREM C.** Let $G \in V_2$. Then

(a) $l_p(G) \leq 1$ for all odd primes $p$,

(b) $l_2(G^2) \leq 1$.

**Proof.** (a) Let $G$ be a counterexample of least possible order. By [8, p. 693], $G$ is a split extension of its unique minimal normal subgroup $N = F(G)$, which is a $p$-group, by a complement $Q$. By the Hall-Higman reduction (see [1, p. 258]), $Q$ is a split extension of a normal Sylow $q$-subgroup $A$ of $Q$ by a $p$-group $B$ acting irreducibly on $A/\Phi(A)$. From Theorem A we infer that $Q$ is supersoluble and hence $A$ is cyclic. As all nilpotent subgroups of $G$ satisfy the second Engel condition, every $p$-element of $Q$ acts as a linear map on $N$ with minimal polynomial dividing $(-1+\lambda)^2$. The result now follows from [6, Theorem 11.1.1, p. 359] as $G$ has abelian Sylow $r$-subgroups for all primes $r \neq p$.

(b) Let $F$ be the class of all extensions of groups having 2-length one by elementary abelian 2-groups. As the product of a subgroup closed saturated formation containing all nilpotent groups with any formation is
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saturated, we see that $F$ is saturated.

Let $G$ be a minimal counterexample. Again $G$ is a split extension of a minimal normal subgroup $N = F(G)$ by a complement $Q$ acting faithfully on $N$. Clearly $N$ is an elementary abelian 2-group. From Theorem A we infer that $Q$ is supersoluble so that, in particular, $Q$ is 2-nilpotent. Let $x \in Q$ be a 2-element. Then $(N, x)$ is a second Engel group and so a straightforward computation shows that $x$ is an involution. This proves $l_2(\sigma^2) = 1$ contradicting our choice of $G$.

3. Some groups of small depth

In the sequel a collection of examples may be found which illustrate that some stronger versions of the above theorems cease to be true. For example, the class $V_2$ does not contain all metabelian groups as there are metabelian $p$-groups of arbitrary Engel length. However

**PROPOSITION 1** ([9]). Let $G$ be an extension of an abelian normal subgroup $N$ by an abelian group $Q$. If the orders of $N$ and $Q$ are coprime, then $G \in V_2$.

Proof. Let $x, y \in G$. Then $N = C_N(y) \times [N, y] = N_1 \times N_2$. We have $[x, y] = n_1 n_2$ for some $n_1 \in N_1$. So $[x, 2y] = [n_2, y] \in N_2$. As $y$ acts fixed point freely on $N_2$, we infer from [3, Lemma 4] that there exists some positive integer $d = d(x, y)$ with $n_2 = [n_2, d]$. Hence $[x, 2y] = [x, 2+dy]$. Let $D$ be the least common multiple of all $d(x, y)$. Then $[x, 2y] = [x, 2+Dy]$ for all $x, y \in G$.

An obvious generalization of Proposition 1 to groups of higher derived length does not seem to be at hand as is shown by the following example which has been computed on a TR 440 at the Rechenzentrum der Universität Würzburg.

**EXAMPLE.** Let $G$ be generated by elements $n_1, \ldots, n_5$, $a_1, \ldots, a_5$, $b$ subject to the following defining relations:
Let $x = b$ and $y = n_1 a_1 b$. Then $[x, 50y] = [x, 50y]$ but $[x, \kappa y] \neq [x, \kappa y]$ for all $k > 4$. So the depth of $G$ is at least 5, but $G$ has derived length 3.

Another series of groups of depth 2 may be found among Frobenius groups.

**PROPOSITION 2.** Let $G$ be a Frobenius group with kernel $N$ and complement $Q$. If $N$ is abelian and $Q$ is metacyclic then $G \in V_2$.

**Proof.** This follows from [3, Lemma 4].

A similar sort of argument proves that any extension of an elementary abelian 2-group by the dihedral group of order $2p$, where $p$ is any odd prime, has depth 2. So groups in $V_2$ need not be metanilpotent.

We end with some speculations concerning the general situation. In view of the first corollary to Theorem B one might ask whether there is a bound $f(r)$ depending on $r$ such that for any soluble group in $V_r$ the quotient $G/F(G)$ has derived length less than or equal to $f(r)$. Or, in view of Theorem A, are the ranks of the chief factors of $G/F(G)$ bounded by some function of $r$? The answer to both questions, however, is negative in general.

**EXAMPLE.** Let $n$ be any positive integer. By [10] there exist finite groups of exponent 4 and derived length $n$. Let $Q$ be such a group of least possible order. Then $Z(Q)$ is cyclic and so there exists a faithful and irreducible $GF(p)$ $Q$-module $N$ ($p$ denotes any odd prime). Let $G$
be the split extension of $N$ by $Q$. Now [12] implies that $Q$ satisfies the 4th Engel condition and so an argument similar to that one used in the proof of Proposition 1 shows $G \in V_5$.

By an analogous construction using a split extension of some faithful and irreducible $\mathbb{G}P(q)$ $G$-module $M$ by $G$ it is possible to disprove the second statement.

Presumably it is essential in this example that the groups under consideration are not generated by two elements. A positive answer to any of these questions for two-generator groups would establish the following CONJECTURE. There exists a function $F$ such that every soluble group in $V_p$ has Fitting length at most $F(r)$.

References


[9] D. Nikolova [D. Nikolova], "Тождества в метабелевых многообразиях $A_{k_1}$" [Identities in the metabelian variety $A_{k_1}$], Serdica (to appear).


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