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Commuting Differential and Difference Operators Associated to Complex Curves I

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Abstract. The purpose of this paper is to define functional realizations of the Khizhnik–Zamolochikov–Bernard (KZB) connection on the bundle of conformal blocks over the moduli space of curves.

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Introduction

In [7], B. Feigin and A. Stoyanovsky introduced functional realizations of the space of conformal blocks associated with a complex curve X and a semisimple Lie algebra $\bar{\mathfrak{g}}$. This space is defined as the set of $\mathfrak{g}^{out}$-invariant forms on an integrable $\mathfrak{g}$-module $L_{\Lambda,k}$ located at a point P_0 of X, $\mathfrak{g}^{out}$ being the Lie algebra of regular maps from $X - \{P_0\}$ to $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ the Kac–Moody Lie algebra at P_0. Feigin and Stoyanovsky associate to such a form ψ, the family of forms on a product of symmetric products of X

$$f_{FS}(z_j^{(i)}) = \left\langle \psi, \prod_{i=1}^{r} e_i(z_j^{(i)}) dz_j^{(i)} (wv_{top}^{(P_0)}) \right\rangle, \tag{1}$$

where r is the rank of $\bar{\mathfrak{g}}$, $e_i(z)dz$ are the currents associated with the simple nilpotent generators of $\bar{\mathfrak{g}}$, w is an affine Weyl group element and $v_{top}^{(P_0)}$ is the highest weight vector of $L_{\Lambda,k}$.

In this paper, we introduce the twisted conformal blocks $\psi_\lambda = \psi \circ e^{\sum_{i,a} \lambda_a^{(i)} h_i[r_a]}$. Here $(r_a)_{a=1,\dots,g}$ are functions on X, regular outside P_0, single-valued around a-cycles and all b-cycles except the ath, along which it increases by 1 (Section 1), and the h_i are the simple coroots of $\bar{\mathfrak{g}}$. The functions r_a are thus the analogues of the function θ'/θ in the elliptic case. ψ_λ is independent of the choice of the functions r_a.

For any v in $L_{\Lambda,k}$, the function $v \mapsto \langle \psi_\lambda, v \rangle$ is defined as a formal function in $\lambda = (\lambda_a^{(i)})$. We show (Theorem 2.1) that it is actually a holomorphic function in λ

with theta-like properties. This result relies on adelization of the representations $L_{\Lambda,k}$ (see [19]), reduction to the $\mathfrak{sl}_2$ case, formulas for the tame symbol and the identity $(f^k)' = - : hf^k :$ (see [14]). This generalizes a result obtained in [9] in the genus 1 case.

We then consider the forms

$$f(\lambda \mid z_j^{(i)}) = \left\langle \psi_\lambda, \prod_{i=1}^{r} e_i(z_j^{(i)}) dz_j^{(i)} (w v_{top}^{(P_0)}) \right\rangle. \tag{2}$$

These forms have the following geometric interpretation. It is known ([1,13]) that conformal blocks can be viewed as sections of a bundle on the moduli space $Bun_{\bar{G}}$ of $\bar{G}$; such sections are called generalized theta functions. In Section 3, we explain that the forms (1) of Feigin–Stoyanovsky can be viewed as generating functions for lifts of the generalized theta functions to a space, which in the case $\bar{\mathfrak{g}} = \mathfrak{sl}_n$ can be described as $Bun_{(n_i,P_0)}$, the moduli space of bundles with filtration $E_1 \subset E_2 \subset \cdots$ and associated graded isomorphic to $\oplus_i \mathcal{O}(n_i P_0)$, n_i some integer numbers. ›From this viewpoint, the twisted correlation functions (2) are generating functions for lifts of generalized theta-functions to the moduli space Bun_B of B-bundles over X, where B is the Borel subgroup of $\bar{G}$.

We then express the Knizhnik–Zamolodchikov–Bernard (KZB) connection in terms of the forms (2) (Section 4.3). Our treatment of the KZB connection follows [8]; the KZB connection is defined on the space of projective structures on curves of genus g. However, such a projective structure is canonically attached to the choice of a-cycles on the curve, via a bidifferential form $\widetilde{\omega}$ (see (7); this form appeared in [5], cor. 2.6). This allows to define the KZB connection as a projectively flat connection on the moduli space of curves with marked a-cycles, which is intermediate between the moduli space of curves and its universal cover. The KZB connection is expressed as the action of differential-evaluation operators $(T_z)_{z\in X}$ on the $f(\lambda \mid z_j^{(i)})$, which are forms on $J^0(X)^r \times \prod_i S^{n_i} X$ (differential in λ and residues and evaluation in the $\lambda_a^{(i)}$).

We also express the KZB connection in the directions given by variation of points in case of a curve with marked points (Section 4.4). Denote by $\widetilde{m}$ a quadruple $(X, [\{\zeta_i\}], P_i, \zeta_i)$ formed by a curve with projective structure, marked points and coordinates at these points, by $\psi(\widetilde{m})$ a conformal block associated to this complex structure, and by $f(\tilde{m})(\lambda \mid z_j^{(i)})$ the twisted correlation function associated with this conformal block according to (2). In the case $\bar{\mathfrak{g}} = \mathfrak{sl}_2$, the connection takes the form

$$2(k+2)\nabla_{\frac{\partial}{\partial P_i}} f(\tilde{m})(\lambda \mid z_\alpha) = 2(k+2)\frac{\partial}{\partial P_i} f(\tilde{m})(\lambda \mid z_\alpha) - (K_i f)(\tilde{m})(\lambda \mid z_\alpha)$$

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with

$$
\begin{aligned}
&(K_i f)(\tilde{m})(\lambda \mid z_\alpha)\\
&\quad = \Bigg[- \Lambda_i \sum_a \omega_a(P_i)\partial_{\lambda_a} + \Lambda_i \Bigg(\sum_{j\neq i} \Lambda_j G(P_j, P_i) - 2 \sum_\alpha G(z_\alpha, P_i) \Bigg) + \\
&\qquad + \Lambda_i^2 \phi(P_i) + 2\Lambda_i g_{2\lambda}(P_i) \Bigg] f(\tilde{m})(\lambda \mid z_\alpha)\\
&\qquad + \sum_\alpha \Bigg[- 2G_{2\lambda}(P_i, z_\alpha) + \Bigg(\sum_a \omega_a(z_\alpha)\partial_{\lambda_a} + 2 \sum_{\beta\neq\alpha} G(z_\alpha, z_\beta) \Bigg) - \\
&\qquad - 4G_{2\lambda}(P_i, z_\alpha)G(z_\alpha, P_i) + 2k d_{z_\alpha} G_{2\lambda}(z_\alpha, P_i) \Bigg] res_{z=P_i} f(\tilde{m})(\lambda \mid z, (z_\beta)_{\beta\neq\alpha}) \qquad (3)
\end{aligned}
$$

where the functions G and $G_{2\lambda}$ are (twisted) Green functions.

The relation to the usual formulation of the KZ connection in the rational case is the following. In that case, the KZ connection has the form

$$2(k+2)\nabla_{P_i}\psi(P_i) = 2(k+2)\partial_{P_i}\psi(P_i) - K_i^{rat}\psi(P_i), \qquad (4)$$

with $\psi(P_i)$ in a tensor product $\otimes_i V_{\Lambda_i}$ of lowest weight $\bar{\mathfrak{g}}$-modules, and

$$K_i^{rat} = \sum_{j\neq i} \frac{t^{(ij)}}{P_i - P_j}.$$

Equation (3) above may be viewed as the expression of the action of K_i on 'Bethe ansatz vectors' $\widetilde{e}(\zeta_1)\cdots\widetilde{e}(\zeta_k)(\otimes_i v_{\Lambda_i}^{bot})$, where $\widetilde{e}(z) = \sum_i e^{(i)}/(z - P_i)$. Extracting coefficients of $\prod(\zeta_i - z_j)^{a_{ij}}$ from (3), one recovers (4). The equation for the bottom component of $\psi(P_i)$ is simpler than (3) (see Equation (29)).

The operators $(T_z)_{z\in X}$ depend in a simple way on the level k. In Section 5, we show that these operators commute when k is critical, thus defining a commuting family of differential operators, acting on a finite-dimensional bundle over the degree zero part $J^0(X)$ of the Jacobian of X (Theorem 5.1). This is proved using a class of modules $W_{n|m,m'}$ generalizing the twisted Weyl modules.

In the case where there are no $z_j^{(i)}$, these operators take the form

$$
\begin{aligned}
(T_z f)(\lambda) = \Bigg(& \sum_{v=1}^{r} \Bigg(\sum_{a=1}^{g} \omega_a(z)\partial_{(h_v)_a} \Bigg)^2 + \sum_{\alpha\in\Delta_+} \sum_{a=1}^{g} D_z^{(\lambda,\alpha)} \omega_a(z)\partial_{(\alpha^\vee)_a} \\
& + k \sum_{\alpha\in\Delta_+} \omega_{(\lambda,\alpha)}(z) \Bigg) f(\lambda),
\end{aligned}
$$

where $(hv)_{v=1,\dots,\Gamma}$ is an orthonormal basis of the Cartan subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{g}}$, $(\omega_a)_{a=1,\dots,g}$ are the canonical differentials of X, Δ_+ is the set of positive roots of $\bar{\mathfrak{g}}$, λ is a collection $(\lambda_1, \dots, \lambda_g)$ of variables in $\bar{\mathfrak{h}}$, $\alpha^\vee$ is the coroot associated to the root α, $D_z^{(\lambda^{(i)})}$ is a

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connection depending on $(\lambda^{(i)})$ in $\mathbb{C}^g$, on the canonical bundle Ω_X (see (22)) and $\omega_{(\lambda^{(i)})}$ is a quadratic differential form depending on the same variables (see (24)).

We close the paper by explaining the link of the operators $(T_z)_{z\in X}$ with the Beilinson–Drinfeld (BD) operators (Rem. 14).

In a sequel to this paper, we will construct q-deformations of the operators T_z, by replacing the inclusion $U\mathfrak{g}^{out} \subset U\mathfrak{g}$ by some inclusion of quasi-Hopf algebras, which were introduced in work of one of us and V. Rubtsov ([4]). The outcome will be a commuting family of difference-evaluation operators, which may be viewed in the case of a rational curve as the Bethe ansatz formulation of the qKZ operators.

One may hope to obtain hypergeometric representation for solutions of the KZB equations formulated in Section 4.3. This may be related with the formulas of [11] expressing the scalar product on the space of conformal blocks.

1 Bases of Functions on X

Let X be a smooth, compact complex curve; denote by g its genus. Let P_0 be a point of X. Denote by $\mathcal{K}$ and $\mathcal{O}$ the completed local field and ring of X at P_0. Denote by $\Omega_{\mathcal{K}}$ and $\Omega_{\mathcal{O}}$ the spaces of differentials and regular differentials at the formal neighborhood of P_0. The residue defines a natural pairing between $\mathcal{K}$ and $\Omega_{\mathcal{K}}$.

In what follows, we will fix a system $(A_a, B_a)_{a=1,\dots,g}$ of a- and b-cycles on X. We will denote by γ_{A_a} and γ_{B_a} the corresponding deck transformations of the universal cover $\widetilde{X}$ of X, and by σ the projection from $\widetilde{X}$ to X.

Define $R_{(b)}$ as the set of functions f defined on $\widetilde{X}$, regular outside $\sigma^{-1}(P_0)$, such that there exist constant functions $\alpha_a(f)$ such that for any z in $\tilde{X} - \sigma^1(P_0)$ and any $a = 1, \dots, g$, we have $f(\gamma_{A_a} z) = f(z)$ and $f(\gamma_{B_a} z) = f(z) + \alpha_a(f)$. Let us also denote by R the space of functions on X, regular outside P_0.

PROPOSITION 1.1. *$R_{(b)} \cap \mathcal{O} = \mathbb{C}1$. R has codimension g in $R_{(b)}$. Moreover, $R_{(b)} + \mathcal{O} = \mathcal{K}$.*

Proof. The first point is clear: for any f in $R_{(b)} \cap \mathcal{O}$, df is a regular form with vanishing a-periods, and therefore vanishes.

To prove the second point, define $R_{(ab)}$ as the set of regular functions defined on the universal cover of $X - P_0$, such that $f(\gamma_{A_a} z) = f(z) + \beta_a(f)$ and $f(\gamma_{B_a} z) = f(z) + \alpha_a(f)$, with $\alpha_a(f)$ and $\beta_a(f)$ some constants. We will show that R has codimension $2g$ in $R_{(ab)}$. $R_{(ab)} \cap \mathcal{O}$ has dimension $g+1$ (it is spanned by the constants and the $\int_{P_0}^x \omega_a$). On the other hand, we have $R_{(ab)} + \mathcal{O} = \mathcal{K}$, because $\mathcal{K}/(R_{(ab)} + \mathcal{O})$ is zero (the differential maps it injectively to $Ker\ res/(\Omega_R + \Omega_{\mathcal{O}})$, where res is the residue map from $\Omega_{\mathcal{K}}$ to $\mathbb{C}$, which is the kernel of the residue map from $H^1(X, \Omega_X)$ to $\mathbb{C}$ and is therefore zero). We have an exact sequence $0 \to (R_{(ab)} \cap \mathcal{O})/(R \cap \mathcal{O}) \to R_{(ab)}/R \to (R_{(ab)} + \mathcal{O})/(R + \mathcal{O}) \to 0$, therefore $dim(R_{(ab)}/R) = dim(R_{(ab)} \cap \mathcal{O}/R \cap \mathcal{O}) + dim(\mathcal{K}/R + \mathcal{O}) = 2g$. Since $dim(R_{(ab)}/R) = dim(R_{(ab)}/R_b) + dim(R_{(b)}/R)$, we have $dim(R_{(ab)}/R_{(b)}) + dim(R_{(b)}/ R) = 2g$.

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On the other hand, $dim(R_{(ab)}/R_{(b)})$ and $dim(R_{(b)}/R)$ are both $\leqslant g$, because the maps $R_{(ab)}/R_{(b)} \to \mathbb{C}^g$ sending the class of f to $(\beta_a(f))_{a=1,\dots,g}$ and $R_{(b)}/R \to \mathbb{C}^g$ sending f to $(\alpha_a(f))_{a=1,\dots,g}$, are both injections.

It follows that $dim(R_{(ab)}/R_{(b)})$ and $dim(R_{(b)}/R)$ are both equal to g.

Finally, the fact that $\mathcal{K}/(\mathcal{O}+R)$ is equal to $H^1(X, \mathcal{O}_X)$ and has therefore dimension g implies the last point. $\square$

COROLLARY 1.1. *For $a = 1, \dots, g$, there exists a function r_a defined on $\widetilde{X}$, regular outside $\sigma^{-1}(P_0)$, with the properties*

$$r_a(\gamma_{A_b} z) = r_a(z), \quad r_a(\gamma_{B_b} z) = r_a(z) - \delta_{ab},$$

for $b = 1, \dots, g$ and z in $\tilde{X} - \sigma^{-1}(P_0)$. The functions r_a are well-defined up to addition of functions of R.

Fix a coordinate z at P_0. Let us denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$, by $(r_{i,0}^{in})$ a basis of $\mathfrak{m}$ and by $(r_i^{out}, 1)$ a basis of $R = H^0(X - P_0, \mathcal{O}_X)$, such that $res_{P_0} r_i^{out} dz/z = 0$. From Proposition 1.1. follows that we can fix functions $(r_a)_{a=1,\dots,g}$ of $R_{(b)}$ such that $res_{P_0} r_a\, dz/z = 0$, so that $(r_a, r_i^{out}, 1)$ is a basis of $R_{(b)}$ and $(r_{i,0}^{in}, r_a, r_i^{out}, 1)$ is a basis of $\mathcal{K}$.

Let $(\omega_a)_{a=1,\dots,g}$ be the basis of the space of holomorphic differentials $\Omega_{\mathcal{O}} \cap H^0(X - P_0, \Omega_X)$, dual to (r_a). We have

$$\frac{1}{2i\pi} \int_{A_a} \omega_b = \delta_{ab}.$$

We can fix families (ω_i^{in}) and (ω_i^{out}) in $\Omega_{\mathcal{O}}$ and $H^0(X - P_0, \Omega_X)$, so that $(\omega_i^{out}, \omega_a, \omega_i^{in}, dz/z)$ is the basis of $\Omega_{\mathcal{K}}$ dual to $(r_{i,0}^{in}, r_a, r_i^{out}, 1)$.

We associate with these dual bases the Green function defined as

$$G(z, w) = \sum_i \omega_i^{out}(z) r_{i,0}^{in}(w). \tag{5}$$

It is clear that G depends only on the choice of a-cycles in X.

Denote by $J(X)$ the Jacobian of X. It is the direct sum of its degree n components $J^n(X)$, with n integer, which are identified with the sets of classes of line bundles of degree n on X. Denote by Γ the lattice of periods of X, which we identify with a lattice in $\mathbb{C}^g$ via the basis dual to $(\omega_a)_{a=1,\dots,g}$. $J^0(X)$ is identified with the quotient $\mathbb{C}^g/\Gamma$, as follows: for some $\lambda = (\lambda_a)$ in $\mathbb{C}^g$, the corresponding line bundle is denoted by $\mathcal{L}_\lambda$. Sections of $\mathcal{L}_\lambda$, regular outside a finite subset S of X, are identified with the functions on the universal cover of X, regular outside the preimage of S, such that $f(\gamma_{A_a} z) = f(z)$ and $f(\gamma_{B_a} z) = e^{\lambda_a} f(z)$. Multiplication by the functions $\exp(\int^z \omega_a)$ identifies the spaces of sections of $\mathcal{L}_\lambda$ and $\mathcal{L}_{\lambda'}$, for λ and λ' in the same class of $\mathbb{C}^g/\Gamma$.

In what follows, we will set

$$R_\lambda = H^0(X - \{P_0\}, \mathcal{L}_\lambda). \tag{6}$$

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Let λ be a nonzero element in $J^0(X)$. We may identify $H^0(X - \{P_0\}, \Omega_X \otimes \mathcal{L}_\lambda)$ with the space of differentials ω on the universal cover of X, regular outside the preimage of P_0, such that $\gamma^*_{A_a}(\omega) = \omega$ and $\gamma^*_{B_a}(\omega) = e^{\lambda_i}\omega$ for $a = 1, \ldots, g$. The space $H^0(X, \Omega_X \otimes \mathcal{L}_\lambda)$ may be identified with the intersection $\Omega_{\mathcal{O}} \cap H^0(X - \{P_0\}, \Omega_X \otimes \mathcal{L}_\lambda)$. By the Riemann–Roch theorem, it has dimension $g - 1$. Let $(\omega_{a;\lambda})_{a=1,\ldots,g-1}$ be a basis of this space. We may complete it to a basis $(\omega^{out}_{i;\lambda}, \omega_{a;\lambda}, \omega^{in}_i)$ of $\Omega_{\mathcal{K}}$, such that $(\omega^{out}_{i;\lambda}, \omega_{a;\lambda})$ is a basis of $H^0(X - \{P_0\}, \Omega_X \otimes \mathcal{L}_\lambda)$ and $(\omega_{a;\lambda}, \omega^{in}_i)$ is a basis of $\Omega_{\mathcal{O}}$. Moreover, we may assume that the ω^{in}_i have a zero of order $\geqslant g - 1$ ar P_0 (for example, we may choose $\omega^{in}_i = z^{g-1+i}dz$, $i \geqslant 0$).

Let $(r^{in}_i, r_{a;-\lambda}, r^{out}_{i;-\lambda})$ be the basis of $\mathcal{K}$ dual to $(\omega^{out}_{i;\lambda}, \omega_{a;\lambda}, \omega^{in}_i)$. Then (r^{in}_i) is a basis of $\mathcal{O}$ and $(r^{out}_{i;-\lambda})$ is a basis of $H^0(X - \{P_0\}, \mathcal{L}^{-1}_\lambda)$. The assumption on zeroes of the ω^{in}_i implies that the $r_{a;-\lambda}$ have poles at P_0 of order $\leqslant g - 1$.

The twisted Green function defined by these bases is

$$G_\lambda(z, w) = \sum_{a=1}^{g-1} \omega_{a;\lambda}(z) r_{a;-\lambda}(w) + \sum_i \omega^{out}_{i;\lambda}(z) r^{in}_i(w). \tag{7}$$

Remark 1. Expression of the Green functions. We may set

$$r_a(z) = \partial_{\varepsilon_a} \ln \Theta(-A(z) + gA(P_0) - \Delta),$$

where Θ is the Riemann theta-function on $J^0(X)$, $\Delta \in J^{g-1}(X)$ is the vector of Riemann constants of X, ε_a is the ath basis vector of $\mathbb{C}^g$ and A is the Abel map from X to $J^1(X)$.

A formula for G_λ is

$$G_\lambda(z, w) = \frac{\Theta(A(z) - A(w) + (g-1)A(P_0) - \lambda - \Delta)}{\Theta(A(z) - A(w) + (g-1)A(P_0) - \Delta)\Theta((g-1)A(P_0) - \lambda - \Delta)} \times$$
$$\times \sum_{i=1}^{g} \frac{\partial\Theta}{\partial\lambda_a}((g-1)A(P_0) - \Delta)\omega_a(z);$$

$G_\lambda(z, w)$ is a λ-twisted differential in z, with simple pole at $z = w$ and residue 1, and a zero of order $g - 1$ at P_0; it is also a $(-\lambda)$-twisted function in w, with simple poles at $w = z$ and a pole of order $g - 1$ at $w = P_0$. This is because

$$\sum_{i=1}^{g} \frac{\partial\Theta}{\partial\lambda_a}((g-1)P_0 - \Delta)\omega_a(z),$$

which is equal to $-d_z\Theta(w - z + (g-1)P_0 - \Delta)_{|w=z}$, is a holomorphic differential with a zero of order $g - 1$ at P_0. For z, w fixed, $G_\lambda(z, w)$ is a meromorphic function in P_0. One may replace $(g-1)P_0$ by any effective divisor $Q = \sum_i n_i Q_i$ of degree $g - 1$ in the definition of G_λ, and obtain this way $G^Q_\lambda(z, w)$, a λ-twisted differential in z, with simple pole at w and a zero of order n_i at each Q_i, which is also a $(-\lambda)$-twisted function in w, with a simple pole at z and poles of order n_i at $w = Q_i$, and is a meromorphic function in the Q_i.

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A formula for $G(z, w)$ is

$$G(z, w) = d_z \ln \Theta(A(w) - A(z) + (g-1)A(P_0) - \Delta) - d_z \ln \Theta(gA(P_0) - A(z) - \Delta).$$

$G(z, w)$ is a differential in z with simple pole at w and residue 1; simple pole at $z = P_0$, regular at other points, and such that $\int_{A_a} G(\cdot, w) = 0$ for w near P_0; and a function in w, multivalued in w around b-cycles, such that $G(z, \gamma_{B_a} w) = G(z, w) + \omega_a(z)$, vanishing for $w = P_0$, with simple pole at $w = z$, and regular at other points.

These properties of P_0 imply that two $G(z, w)$ attached to different points P_0 differ by a form in z, constant in w. In what follows, we will set

$$\widetilde{\omega}(z, w) = d_w G(z, w). \tag{8}$$

$\widetilde{\omega}(z, w)$ is a bidifferential form in z, w with the local expansion at any point of

$$X, \widetilde{\omega}(z, w) = \frac{dzdw}{(z-w)^2} + r(z)dzdw + \mathrm{O}(z-w)dzdw.$$

$\widetilde{\omega}$ is symmetric in z and w, because $\widetilde{\omega}(z, w) - \widetilde{\omega}(w, z)$ has no poles and for w near

$$P_0, \int_{A_a} \widetilde{\omega}(\cdot, w) - \widetilde{\omega}(w, \cdot) = d_w \int_{A_a} G(\cdot, w) - (G(w, \gamma_{A_a} z) - G(w, z)) = 0$$

because $\int_{A_a} G(\cdot, w) = 0$ and because $G(w, \cdot)$ is single-valued along a-cycles. The fact that $\widetilde{\omega}$ is symmetric can also be viewed as a consequence of the expression $\widetilde{\omega} = d_z d_w \ln \Theta(A(w) - A(z) + \delta - \Delta)$ where δ in $J^{g-1}(X)$ is some odd theta-divisor.

2. Twisted Conformal Blocks

2.1. TWISTED CONFORMAL BLOCKS

Let $\bar{\mathfrak{g}}$ be a simple complex Lie algebra. Let us set $\mathfrak{g} = (\bar{\mathfrak{g}} \otimes \mathcal{K}) \oplus \mathbb{C}K$, $\mathfrak{g}^{in} = (\bar{\mathfrak{g}} \otimes \mathcal{O}) \oplus \mathbb{C}K$, $\mathfrak{g}^{out} = \bar{\mathfrak{g}} \otimes R$. For x in $\bar{\mathfrak{g}}$, ε in $\mathcal{K}$, we set $x[\varepsilon] = (x \otimes \varepsilon, 0)$; the commutation rules on $\mathfrak{g}$ are then

$$[x[\varepsilon], y[\varepsilon']] = [x, y][\varepsilon\varepsilon'] + K\langle d\varepsilon, \varepsilon'\rangle(x|y),$$

with $(\cdot|\cdot)$ the invariant scalar product on $\bar{\mathfrak{g}}$ such that $(\theta^\vee|\theta^\vee) = 2$, where $\theta^\vee$ is the coroot associated to a maximal root θ, and $\langle\omega, \varepsilon\rangle = res_{P_0}(\omega\varepsilon)$. We view $\mathfrak{g}^{out}$ as a subalgebra of $\mathfrak{g}$, using the embedding $x \otimes p \mapsto x[p]$.

Let V be a $\mathfrak{g}$-module of level k, and let ψ be a $\mathfrak{g}^{out}$-invariant linear form on V. Fix a Cartan decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{n}}_-$. Let r be the rank of $\bar{\mathfrak{g}}$. Let Δ be the set of roots of $\bar{\mathfrak{g}}$, and define the positive roots as those associated with $\bar{\mathfrak{n}}_+$. For each α in Δ, define $\bar{\mathfrak{g}}_\alpha$ as the root subspace of $\bar{\mathfrak{g}}$ associated with α. For each simple root α_i, let us fix e_i, h_i and f_i in $\bar{\mathfrak{g}}_{\alpha_i}$, $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{g}}_{-\alpha_i}$, such that (e_i, h_i, f_i) is an $\mathfrak{sl}_2$-triple.

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Let $(r_a)_{a=1,\ldots,g}$ be as in Corollary 1.1 and let $(\lambda_a^{(i)})_{a=1,\ldots,g,i=1,\ldots,r}$ be formal variables and define the linear form ψ_λ in V

$$\langle \psi_\lambda, v\rangle = \left\langle \psi, e^{\sum_{i,a} \lambda_a^{(i)} h_i[r_a]} v\right\rangle. \tag{9}$$

This form is independent of the choice of the r_a, because $[h_i[r], h_j[r_a]] = 0$ for r in R.

In the case where V is an integrable module, one expects that one can make sense of (8) for complex λ. If one wished to argue that the action of $\mathfrak{g}$ on V lifts to a projective action of the associated Kac–Moody group, one would meet the difficulty that the functions $\exp(\sum_a \lambda_a^{(i)} r_a)$ have essential singularities at P_0, so that we cannot view $\exp(\sum_{a,i} \lambda_a^{(i)} h_i[r_a])$ as an element of the Kac–Moody group.

However, we have:

THEOREM 2.1. *For ψ a $\mathfrak{g}^{out}$-invariant form on $L_{\Lambda,k}$, the form $\psi_\lambda = \psi \circ \exp(\sum_{i,a} \lambda_a^{(i)} h_i[r_a])$ on $L_{\Lambda,k}$ has the following properties:*

(1) *For any v in $L_{\Lambda,k}$, the function $\langle \psi_\lambda, v\rangle$ is the formal expansion at* 0 *of an analytic function in λ, which satisfies the equations*

$$\partial_{\lambda_a^{(i)}} \langle \psi_\lambda, v\rangle = \langle \psi_\lambda, h_i[r_a] v\rangle,$$

$a = 1, \ldots, g, i = 1, \ldots, r.$

(2) *Set $\lambda_a = \sum_i \lambda_a^{(i)} h_i$. Set $\lambda = (\lambda_1, \ldots, \lambda_g)$ and*

$$\mathfrak{g}_\lambda^{out} = (\bar{\mathfrak{h}} \otimes R) \oplus \oplus_{\alpha\in\Delta} (\bar{\mathfrak{g}}_\alpha \otimes R_{\langle\alpha,\lambda_1\rangle,\ldots,\langle\alpha,\lambda_g\rangle}).$$

Then ψ_λ is a $\mathfrak{g}_\lambda^{out}$-invariant form on $L_{\Lambda,k}$.

(3) *For any v in $L_{\Lambda,k}$, the function $\lambda \mapsto \langle \psi_\lambda, v\rangle$ has the following theta-like behavior. Set $\omega_{ab} = \int_{B_b} \omega_a$, $\zeta_a(z) = \int_{P_0}^z \omega_a$, and $\Omega_a = \sum_b \omega_{ab}\delta_b$, with δ_a the a-th basis vector of $\mathbb{C}^g$. Then*

$$\left\langle \psi_{\lambda^{(1)},\ldots,\lambda^{(l)}+2i\pi\delta_a,\ldots,\lambda^{(r)}}, v\right\rangle = \left\langle \psi_{\lambda^{(1)},\ldots,\lambda^{(r)}}, v\right\rangle$$

and

$$\left\langle \psi_{\lambda^{(1)},\ldots,\lambda^{(l)}+2i\pi\Omega_a,\ldots,\lambda^{(r)}}, v\right\rangle = e^{-k(h_l|\lambda_a)-\frac{1}{2}i\pi k\omega_{aa}(h_l|h_l)} \left\langle \psi_{\lambda^{(1)},\ldots,\lambda^{(r)}}, e^{h_l[\zeta_a]} v\right\rangle,$$

where $\lambda_a = \sum_{i=1}^r \lambda_a^{(i)} h_i$.

Proof. See the appendix.

2.2. TWISTED CORRELATION FUNCTIONS IN THE $\mathfrak{sl}_2$ CASE

In this section, we assume that $\bar{\mathfrak{g}} = \mathfrak{sl}_2$ and $\Lambda = 0$. Let ψ be a $\mathfrak{g}^{out}$-invariant form on $L_{0,k}$. Let z be a local coordinate at P_0 and set $e(w) = \sum_{i\in\mathbb{Z}} e[z^i] w^{-i-1} dw$. For n a positive integer, set $n = ak + b$, $0 \leqslant b < k$, and $v_n = f[z^{-2a-1}]^b v_{[a]}$, with $v_{[a]}$ as in Lemma 6.3. We have $h[1]v_n = -2nv_n$. Set $f(\lambda \mid z_1, \ldots, z_n) = \langle \psi_\lambda, e(z_1)dz_1 \cdots e(z_n)dz_n v_n\rangle$.

PROPOSITION 2.1 (see [7]). *The form* $f(\lambda \mid z_1, \dots, z_n)$ *depends analytically on* λ *in* $J^0(X)$ *and the* z_i *in* $X - \{P_0\}$. *It satisfies the relations*

$$f(\lambda + 2i\pi\delta_a \mid z_1, \dots, z_n) = f(\lambda \mid z_1, \dots, z_n) \tag{10}$$

and

$$f(\lambda + 2i\pi\Omega_a \mid z_1, \dots, z_n) = e^{-k(h|h)\lambda_a - \frac{1}{2}i\pi k\omega_{aa}(h|h)} e^{2\sum_{l=1}^{n}\int_{P_0}^{z_l}\omega_a} f(\lambda \mid z_1, \dots z_n). \tag{11}$$

Moreover, it depends on z_i *as a section of* $\Omega_X\mathcal{L}_{-2\lambda}$, *regular on* X *except for a pole of order* $\leqslant 2a + 2 - 2\delta_{b,0}$ *at* P_0; *it is symmetric in the* z_i, *and vanishes if* $k+1$ *variables* z_i *coincide.*

Proof. The proof is analogous to that of [7]. Identities (9) and (10) follow from Theorem 2.1, (3) and from the commutation relation $[h[\zeta_a], e(z_i)] = 2\zeta_a(z_i)e(z_i)$. □

Since $(h|h) = 2$, we have

$$f(\lambda \mid z_1, \dots, z_n) = \sum_{l=1}^{(2k)^g} \Theta_{2k}^{[l]}\left(\lambda + \frac{1}{k}\sum_{i=1}^{n} A(z_i)\right) f^{[l]}(z_1, \cdots, z_n),$$

where the $\Theta_{2k}^{[l]}$ are a basis of the space of $2k$th order theta functions on $J^0(X)$.

Remark 2. If $f_{FS}(z_1, \dots, z_n)$ are the forms introduced in [7], then $f(0 \mid z, \dots, z_n)$ coincides with $f_{FS}(z_1, \dots, z_n)$. It is not clear what are the functional properties of the $f^{[l]}(z_1, \cdots, z_n)$, and how to obtain the $f^{[l]}(z_1, \cdots, z_n)$ directly from $f_{FS}(z_1, \dots, z_n)$.

Remark 3. The forms $f(\lambda \mid z_1, , z_n)$ provided by conformal blocks also satisfy some vanishing conditions at $\lambda = 0$ (see [9]). These conditions, together with the functional properties of Proposition 2.1, should probably characterize these forms.

3 Lifts of Generalized Theta-Functions to Bun_B

It follows from the works [1, 13] follows that conformal blocks may be viewed as the space of sections of a line bundle on the moduli space $Bun_{\bar{G}}$ of principal $\bar{G}$-bundles over an complex curve X, for $\bar{G}$ the simply connected group associated with $\bar{\mathfrak{g}}$. This identification is as follows: $Bun_{\bar{G}}$ is identified with the double coset $\bar{G}(R)\backslash\bar{G}(\mathcal{K})/\bar{G}(\mathcal{O})$, with $\mathcal{K}$ the local field at some point P_0 of X, $\mathcal{O}$ the local ring at P_0 and R the ring of functions regular outside P_0. For k integer $\geqslant 0$, the level k vacuum representation $L_{0,k}$ of the Kac–Moody algebra $\mathfrak{g}$ associated with $\bar{\mathfrak{g}}$ carries a projective representation of $\bar{G}(\mathcal{K})$. Fix a lift $x \longmapsto \tilde{x}$ of $\bar{G}(\mathcal{K})$ to its universal central extension. Let $\mathfrak{g}^{out}$ be the Lie algebra $\bar{\mathfrak{g}} \otimes R$. To each $\mathfrak{g}^{out}$-invariant form ψ^{out} on $L_{0,k}$ is associated the function

$$g \longmapsto \langle\psi^{out}, \tilde{g}v_{top}\rangle \tag{12}$$

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on $\bar{G}(\mathcal{K})$, where v_{top} is the vacuum vector of $L_{0,k}$, which is a section of a power of the determinant bundle over $Bun_{\bar{G}}$. This construction can be extended to the case of marked points and integrable representations other than $L_{0,k}$. In what follows, we will consider the situation of some integrable module $L_{\Lambda,k}$ at P_0, with highest weight vector $v_{top}^{(P_0)}$.

It was proposed to study these functions through their lifts to moduli spaces of flags of bundles ([3, 18]). In [7], Feigin and Stoyanovsky studied the lift of conformal blocks to a space, which in the case $\bar{\mathfrak{g}} = \mathfrak{sl}_n$ can be described as $Bun_{(n_i,P_0)}$, the moduli space of bundles with filtration $E_1 \subset E_2 \subset \cdots$ and associated graded isomorphic to $\oplus_i \mathcal{O}(n_i P_0)$, n_i some integer numbers. Since this space is isomorphic to $N(R)\backslash N(\mathcal{K}) diag(z^{n_i})/N(\mathcal{O})$, with N the maximal unipotent subgroup of $\bar{G}$, lifts of functions provided by the conformal blocks are the

$$\left\langle \psi^{out}, n_{\mathcal{K}}(w v_{top}^{(P_0)}) \right\rangle, \tag{13}$$

$n_{\mathcal{K}}$ in $N(\mathcal{K})$, $w = diag(z^{n_i})$ an affine Weyl group translation. Generating functions for these quantities are the forms

$$\left\langle \psi^{out}, \prod_{i \text{ simple}} \prod_{j=1}^{n_j} e_i(z_j^{(i)}) dz_j^{(i)} (w v_{top}^{(P_0)}) \right\rangle,$$

where $e_i(z)dz$ are the currents associated to the nilpotent generators e_i attached to the simple roots of $\bar{\mathfrak{g}}$. In [7], Feigin and Stoyanovsky characterized the functional properties of these forms.

Let us study the lift of functions (12) to Bun_B, the moduli space of B-bundles over X, where B is the Borel subgroup of $\bar{G}$ containing N. Bun_B can be described as the double quotient $B(K)\backslash B(\mathbb{A})/B(\mathcal{O}_{\mathbb{A}})$, where K is the function field $\mathbb{C}(X)$, $\mathbb{A}$ is the adeles ring of X and $\mathcal{O}_{\mathbb{A}}$ its subring of integral adeles. To make sense of the analogue of (13) for the space of B-bundles, one should replace the representation at P_0 by its 'adelic' version $L^{\mathbb{A}}$, which is its restricted tensor product with vacuum representations at the points of $X - \{P_0\}$. To ψ^{out} is then associated a $\bar{\mathfrak{g}} \otimes K$-invariant form $\psi^{\mathbb{A}}$ (see Lemma 6.1). In the case of B-bundles, lifts of the functions on $Bun_{\bar{G}}$ provided by conformal blocks are the

$$b \mapsto \langle \psi^{\mathbb{A}}, b v_{top}^{\mathbb{A}} \rangle, \tag{14}$$

for $b \in B(\mathbb{A})$, $v_{top}^{\mathbb{A}}$ the product of the highest weight vector of the module at P_0 with the vacuum vectors at other points. b can be decomposed as a product ntw, with n in $N(\mathbb{A})$, t in $T(\mathbb{A})$ with all components of degree zero (T is the Cartan subgroup associated to B; the degree in $\mathbb{A}^\times$ is defined as the sum of the valuations of all components) and w a product of affine Weyl group translations. In the case $\bar{\mathfrak{g}} = \mathfrak{sl}_n$, b represents a filtered bundle whose associated graded is a sum of line bundles, associated to the projections in the Jacobian $J(X) = K^\times\backslash\mathbb{A}^\times/\mathcal{O}_{\mathbb{A}}^\times$ of the components of tw.

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The computation of (14) may be done as follows. $wv_{top}^{\mathbb{A}}$ is an extremal vector of $L^{\mathbb{A}}$. n may be replaced by an element $n_{\mathcal{K}}$ of $N(\mathbb{A})$ with only nontrivial component at P_0. The map $\lambda\mapsto f(\lambda)$ of Section A.2 is a section of the projection map $K^{\times}\backslash(\mathbb{A}^{\times})^0 \to J^0(X)$ (the ^{0} denotes the degree zero parts). t can be decomposed as $t^{out}t_{\lambda}t^{in}$, t^{out} in $T(K)$, t^{in} in $T(\mathcal{O}_{\mathbb{A}})$ and $t_{\lambda} = \prod_i t_i[f_{\lambda^{(i)}}]$, t_i the subgroups of $\bar{G}$ associated to the simple coroots of $\bar{\mathfrak{g}}$. Then (14) is equal to $(t^{in}, n_{\mathcal{K}})\langle\psi^{\mathbb{A}}, t_{\lambda}n_{\mathcal{K}}(wv_{top}^{\mathbb{A}})\rangle$ (where $(,)$ denotes the group commutator).

Therefore to compute (14), it suffices to compute the

$$\left\langle \psi^{\mathbb{A}}, \prod_{i=1}^{r} t_i[f_{\lambda^{(i)}}] \prod_{i=1}^{r} e_i[\varepsilon_1^{(i)}]\cdots e_i[\varepsilon_{n_i}^{(i)}](wv_{top}^{\mathbb{A}})\right\rangle, \tag{15}$$

where r is the rank of $\bar{\mathfrak{g}}$. In Thm. 2.1, we study the linear form

$$v\mapsto\left\langle \psi^{\mathbb{A}}, \prod_{i=1}^{r} t_i[f_{\lambda^{(i)}}](v\otimes \otimes_{x\in X-\{P_0\}} v_{top})\right\rangle, \tag{16}$$

for v in $L_{\Lambda,k}$.

From Theorem 2.1 follows that the expansion at $(\lambda_a^{(i)}) = 0$ of (17) is equal (up to multiplication by a phase factor) to

$$\left\langle \psi^{out}, e^{\sum_{i,a}\lambda_a^{(i)}h_i[r_a]} \prod_{i} e_i[\varepsilon_1^{(i)}]\cdots e_i[\varepsilon_{n_i}^{(i)}](wv_{top}^{(P_0)})\right\rangle. \tag{17}$$

Generating functions for (17) are the forms (2).

The interest of expressing (14) in the form (17) is that the latter expression is computed in a single module located at P_0. When the $\lambda_a^{(i)}$ are formal, (17) also makes sense in arbitrary modules. What we will do now is compute the action of the Sugawara tensor on these correlation functions.

4. Expression of the KZB Connection

4.1. ACTION OF THE SUGAWARA TENSOR ON THE TWISTED CORRELATION FUNCTIONS ($\bar{\mathfrak{g}} = \mathfrak{sl}_2$)

In this section, we treat the case $\bar{\mathfrak{g}} = \mathfrak{sl}_2$. Let n be an integer and let v_n be a vector of $L_{\Lambda,k}$ such that $h[1]v_n = -2nv_n$, $h[t^k]v_n = 0$ for $k > 0$ and $f[t^k]v_n = 0$ for $k \geqslant -(g-1)$. An example of v_n is in the vacuum module $L_{0,k}$, the extremal vector $f[t^{-(2a-1)}]^k \cdots f[z^{-1}]^k v_{top}$, with $2a+1 \geqslant g-1$.

In what follows, we will denote by $x(z)$ the series $\sum_{i\in\mathbb{Z}} x[t^i]z^{-i-1}dz$, for x in $\bar{\mathfrak{g}}$.

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The expression for the Sugawara tensor is

$$
\begin{aligned}
&2(k+2)T_{\widetilde{\omega}}(z)\\
&\quad=\lim_{z\to z'}\left[e(z)f(z')+f(z)e(z')+\frac{1}{2}h(z)h(z')-3k\widetilde{\omega}(z,z')\right],
\end{aligned}
\tag{18}
$$

with $\widetilde{\omega}$ as in (7). It is used to define the KZB connection in Section 4.3.

4.1.1. *Action of the Currents on the Correlation Functions*

Assume that m is $\leqslant -(g-1)$. Let us compute some correlation functions in $L_{\Lambda,k}$.

LEMMA 4.1 *We have*

$$
\begin{aligned}
&\langle\psi_\lambda, h(z)e(z_1)\cdots e(z_n)v_n\rangle\\
&\quad=\left(\sum_a \omega_a(z)\partial_{\lambda_a}+2\sum_{\alpha=1}^{n}G(z,z_\alpha)\right)f(\lambda\mid z_1,\ldots,z_n)
\end{aligned}
$$

where $G(z,z_\alpha)$ is as in (5).

Proof. Let us write $h(z)=\sum_i h[r_i^{out}]\omega_i^{in}+\sum_a h[r_a]\omega_a+\sum_i h[r_{i,0}^{in}]\omega_i^{out}$. The contribution of the first term of this sum is zero by invariance of ψ_λ, the contribution of the second part is the differential part. The contribution of the third part is

$$
\begin{aligned}
&\sum_i\langle\psi_\lambda, h[r_{i,0}^{in}]e(z_1)\cdots e(z_n)v_n\rangle\omega_i^{out}(z)\\
&\quad=\sum_i\sum_{j=1}^{n}2r_{i,0}^{in}(z_j)\langle\psi_\lambda, e(z_1)\cdots e(z_n)v_n\rangle\omega_i^{out}(z)+\\
&\qquad+\sum_i\langle\psi_\lambda, e(z_1)\cdots e(z_n)h[r_{i,0}^{in}]v_n\rangle\omega_i^{out}(z)\\
&\quad=\sum_i\sum_{j=1}^{n}2G(z,z_j)\langle\psi_\lambda, e(z_1)\cdots e(z_n)v_n\rangle,
\end{aligned}
$$

because v_n is annihilated by the positive Cartan modes. □

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LEMMA 4.2. *We have*

$$
\begin{aligned}
&\langle\psi_\lambda, f(z)e(z_1)\cdots e(z_{n+1})v_n\rangle \\
&\quad = -\sum_\alpha G_{2\lambda}(z, z_\alpha)\left(\sum_a \omega_a(z_\alpha)\partial_{\lambda_a} + 2\sum_{\beta\neq\alpha} G(z_\alpha, z_\beta)\right)\times \\
&\qquad \times \langle\psi_\lambda, e(z_1)\cdots e(z_\alpha - 1)e(z_\alpha + 1)\cdots e(z_{n+1})v_n\rangle + \\
&\qquad + k\sum_{\alpha=1}^{n+1} d_{z_\alpha} G_{2\lambda}(z, z_\alpha)\langle\psi_\lambda, e(z_1)\cdots e(z_\alpha - 1)e(z_\alpha + 1)\cdots e(z_{n+1})v_n\rangle,
\end{aligned}
\tag{19}
$$

with $G_{2\lambda}(z, z_\alpha)$ *as in (7).*

Proof. Write

$$f(z) = \sum_i f[r^{out}_{i;-2\lambda}]\omega^{in}_i(z) + \sum_a f[r_{a;-2\lambda}]\omega_{a;2\lambda}(z) + \sum_i f[r^{in}_i]\omega^{out}_{i;2\lambda}(z).$$

The contribution of the first term is zero by invariance of ψ_λ. The contribution of the next two terms is

$$
\begin{aligned}
&\sum_a\sum_{\alpha=1}^{n+1}\langle\psi_\lambda, e(z_1)\cdots(-r_{a;-2\lambda}(z_\alpha)h(z_\alpha) + kdr_{a;-2\lambda}(z_\alpha))\cdots e(z_{n+1})v_n\rangle\omega_{a;2\lambda}(z) + \\
&\quad + \sum_i\sum_{\alpha=1}^{n+1}\langle\psi_\lambda, e(z_1)\cdots(-r^{in}_i(z_\alpha)h(z_\alpha) + kdr^{in}_i(z_\alpha))\cdots e(z_{n+1})v_n\rangle\omega^{out}_{i;2\lambda}(z).
\end{aligned}
\tag{20}
$$

because of the relation

$$[f[\varepsilon], e(z)] = -\varepsilon(z)h(z) + kd\varepsilon(z),$$

and because we have $f[r^{in}_i]v_n = f[r_{a;2\lambda}]v_n = 0$; the latter equality is because the $r_{a;2\lambda}$ have poles of order $\leqslant g-1$ at P_0.

Equation (20) is then equal to

$$
\begin{aligned}
&\sum_{\alpha=1}^{n+1}[-G_{2\lambda}(z, z_\alpha)]\langle\psi_\lambda, h(z_\alpha)e(z_1)\cdots\check{\alpha}\cdots e(z_{n+1})v_n\rangle + \\
&\quad + \sum_{\alpha=1}^{n+1} kd_{z_\alpha}G_{2\lambda}(z, z_\alpha)]\langle\psi_\lambda, e(z_1)\cdots\check{\alpha}\cdots e(z_{n+1})v_n\rangle.
\end{aligned}
\tag{21}
$$

Applying Lemma 4.1 to the first sum, one gets (19). □

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4.1.2. *Action of the Sugawara Tensor on the Correlation Functions*

Let us compute now

$$\langle\psi_\lambda, h(z)h(z')e(z_1)\cdots e(z_n)v_n\rangle.$$

This is equal to

$$\sum_a\langle\psi_\lambda, h[r_a]h(z')e(z_1)\cdots e(z_n)v_n\rangle\omega^a(z)+ \\ +\sum_i\langle\psi_\lambda, h[r^{in}_{i,0}]h(z')e(z_1)\cdots e(z_n)v_n\rangle\omega^{out}_i(z)$$

that is

$$\sum_a\omega^a(z)\partial_{\lambda_a}\langle\psi_\lambda, h(z')e(z_1)\cdots e(z_n)v_n\rangle+ \\ +\sum_i\langle\psi_\lambda, h(z')h[r^{in}_{i,0}]e(z_1)\cdots e(z_n)v_n\rangle\omega^{out}_i(z)+ \\ +2kd_{z'}G(z,z')\langle\psi_\lambda, e(z_1)\cdots e(z_n)v_n\rangle.$$

The second line is equal to

$$\sum_{\alpha=1}^n 2G(z,z_\alpha)\langle\psi_\lambda, h(z')e(z_1)\cdots e(z_n)v_n\rangle.$$

Applying Lemma 4.1 to the first two sums, we find

$$2(k+2)\langle\psi_\lambda, h(z)h(z')e(z_1)\cdots e(z_n)v_n\rangle \\ = 2kd_{z'}G(z,z')f(\lambda\mid z_1,\ldots,z_n) \\ +\left(\sum_a\omega_a(z)\partial_{\lambda_a}+2\sum_{\alpha=1}^n G(z,z_\alpha)\right)^2 \\ \times f(\lambda\mid z_1,\ldots,z_n)+\mathrm{O}(z,z').$$

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On the other hand, we have, by Lemma 4.2,

$$
\begin{aligned}
&\langle\psi_\lambda, f(z')e(z)e(z_1)\cdots e(z_n)v_n\rangle\rangle\\
&\quad= -G_{2\lambda}(z',z)\left(\sum_a \omega_a(z)\partial_{\lambda_a} + \sum_{\alpha=1}^n 2G(z,z_\alpha)\right)f(\lambda\mid z_1,\dots,z_n)-\\
&\qquad -\sum_{\alpha=1}^n G_{2\lambda}(z',z_\alpha)\left[\sum_a \omega_a(z_\alpha)\partial_{\lambda_a} + \sum_{\beta=1,\beta\neq\alpha}^n 2G(z_\alpha,z_\beta)\right]\times\\
&\qquad \times f(\lambda\mid z_1,\dots,z,\dots,z_n)-\\
&\qquad -\sum_{\alpha=1}^n G_{2\lambda}(z',z_\alpha)2G(z_\alpha,z)f(\lambda\mid z_1,\dots,z,\dots,z_n)+\\
&\qquad + kd_z(G_{2\lambda}(z',z))f(\lambda\mid z_1,\dots,z_n)+\\
&\qquad + k\sum_{\alpha=1}^n d_{z_\alpha}(G_{2\lambda}(z',z_\alpha))ff(\lambda\mid z_1,\dots,z,\dots,z_n)
\end{aligned}
$$

so that

$$
\begin{aligned}
&\langle\psi_\lambda, (e(z)f(z')+f(z)e(z'))e(z_1)\cdots e(z_n)v_n\rangle\\
&\quad= \left(\sum_a D_z^{(2\lambda)}\omega_a(z)\partial_{\lambda_a} + \sum_{\alpha=1}^n 2(D_z^{(2\lambda)}\otimes 1)G(z,z_\alpha)\right)f(\lambda\mid z_1,\dots,z_n)-\\
&\qquad -2\sum_{\alpha=1}^n G_{2\lambda}(z,z_\alpha)\left[\sum_a \omega_a(z_\alpha)\partial_{\lambda_a} + \sum_{\beta=1,\beta\neq\alpha}^n 2G(z_\alpha,z_\beta)\right]\times\\
&\qquad \times f(\lambda\mid z_1,\dots,z,\dots,z_n)-\\
&\qquad -4\sum_{\alpha=1}^n G_{2\lambda}(z,z_\alpha)G(z_\alpha,z)f(\lambda\mid z_1,\dots,z,\dots,z_n)+\\
&\qquad + k[d_z(G_{2\lambda}(z',z)) + d_{z'}(G_{2\lambda}(z,z'))]f(\lambda\mid z_1,\dots,z_n)+\\
&\qquad + 2k\sum_{\alpha=1}^n d_{z_\alpha}(G_{2\lambda}(z,z_\alpha))f(\lambda\mid z_1,\dots,z_n) + \mathrm{O}(z-z')
\end{aligned}
$$

(with z in the αth place in the right-hand side, where $D_z^{(\lambda)}(\omega)$ is defined by

$$D_z^{(\lambda)}(\omega)(z) = -\lim_{z\to z'}(\omega(z')G_\lambda(z,z') + \omega(z)G_\lambda(z',z)); \tag{22}$$

$D_z^{(\lambda)}$ defines a connection from the bundle Ω_X to Ω_X^2.

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Set

$$
\begin{aligned}
&(T_z f)(\lambda \mid z_1, \ldots, z_n)\\
&= \left[\frac{1}{2}\left(\sum_a \omega_a(z)\partial_{\lambda_a} + 2\sum_\alpha G(z, z_\alpha)\right)^2 + \right.\\
&\quad \left. + \sum_a D_z^{(2\lambda)}\omega_a(z)\partial_{\lambda_a} + 2\sum_\alpha (D_z^{(2\lambda)} \otimes 1)(G(z, z_\alpha)) + k\omega_{2\lambda}(z)\right] f_\lambda(z_1, \cdots, z_n) +\\
&\quad + \sum_{\alpha=1}^n \left[-2G_{2\lambda}(z, z_\alpha)\left(\sum_a \omega_a(z_\alpha)\partial_{\lambda_a} + 2\sum_{\beta\neq\alpha} G(z_\alpha, z_\beta)\right) + \right.\\
&\quad \left. + \big(-4G_{2\lambda}(z, z_\alpha)G(z_\alpha, z) + 2k d_{z_\alpha} G_{2\lambda}(z, z_\alpha)\big)\right] f(\lambda \mid z_1, \ldots, z_n).
\end{aligned} \tag{23}
$$

where z is in the αth position in the right-hand side and we set

$$\omega_\lambda(z) = \lim_{z\to z'}(d_{z'}G_\lambda(z, z') + d_z G_\lambda(z', z) - 2d_{z'}G(z, z')). \tag{24}$$

Then

PROPOSITION 4.1. *Let us set*

$$f(\lambda \mid z_1, \ldots, z_n) = \langle \psi_\lambda, e(z_1)\cdots e(z_n)v_n\rangle$$

We have

$$\langle \psi_\lambda, T_{\underset{\omega}{\sim}}(z)e(z_1)\cdots e(z_n)v_n\rangle = (T_z f_\lambda)(z_1, \cdots, z_n).$$

Remark 4. It would be interesting to have an expression of the action of $T(z)$ directly in terms of the $f_{FS}(z_1, \ldots, z_n)$. For this, one would need either to understand the correspondence of Rem. 2, or how to express the $T[z^p]v_n$ as combinations of the $e[z^{i_1}]\cdots e[z^{i_l}]v_{n+l}$.

4.2. ACTION OF THE SUGAWARA TENSOR IN THE GENERAL CASE

In this section, we show how the expression of the operators T_z is modified in the case of a general semisimple $\bar{\mathfrak{g}}$. For any α in Δ_+, let e_α, f_α and $\alpha^\vee$ be in $\bar{\mathfrak{g}}_\alpha$, $\bar{\mathfrak{g}}_{-\alpha}$ and $\bar{\mathfrak{h}}$ forming a standard $\mathfrak{sl}_2$-triple, and let $(a_{ij})_{1\leqslant i,j\leqslant r}$ be the Cartan matrix of $\bar{\mathfrak{g}}$.

For $i_1, \ldots, i_s$ in $\{1, \ldots, r\}$, such that $\sum_{j=1}^s \alpha_{i_j} < \alpha$, define the number $n_{\alpha;i_1,\ldots,i_s}$ by the equality

$$[[[f_\alpha, e_{i_1}], e_{i_2}]\cdots, e_{i_s}] = n_{\alpha;i_1\ldots i_s} f_{\alpha-\sum_{j=1}^s \alpha_{i_j}};$$

for α, β in Δ_+, such that $\alpha - \beta$ belongs to Δ_+, define the number $N_{\alpha\beta}$ by the equality

$$[f_{\alpha-\beta}, e_\alpha] = N_{\alpha\beta} e_\beta;$$

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define $v_{i_1\dots i_s}$ by the equality

$$[[e_{i_1}, e_{i_2}], \cdots, e_{i_s}] = v_{i_1\dots i_k} e_{\sum_{j=1}^{s} e_{\alpha_{i_j}}}.$$

As we have seen, one may attach to a $\mathfrak{g}^{out}$-invariant form ψ on any $\mathfrak{g}$-module V, the forms

$$f(\lambda \mid z_u^{(i)})_{1 \leqslant i \leqslant r, u\in I_i^-} = \left\langle \psi, e^{\sum_{a,i} \lambda_a^{(i)} h_i[r_a]} \prod_{i=1}^{r} \prod_{u\in I_i} e_i(z_u^{(i)}) dz_u^{(i)} v \right\rangle, \tag{25}$$

where the I_i are finite sets attached to $i = 1, \dots, r$ and v is a vector in V with the suitable weight. The $\lambda_a^{(i)}$ are formal variables. We attach to them the family $(\lambda_a)_{1 \leqslant a \leqslant g}$ of formal elements of $\bar{\mathfrak{h}}^g$, where $\lambda_a = \sum_i \lambda_a^{(i)} h_i$. For μ in $\bar{\mathfrak{h}}^*$, we set $(\mu, \lambda) = (\mu, \lambda_a)_{1 \leqslant a \leqslant g}$.

The form $f(\lambda \mid z_u^{(i)})_{1 \leqslant i \leqslant r, u\in I_i^-}$ depends on the $z_u^{(i)}$ as a section of $\Omega_X \otimes \mathcal{L}_{-(\alpha_i,\lambda)}$, regular on X outside P_0 and the $z_v^{(j)}$ for the j such that $a_{ij} < 0$. It is symmetric in the $z_u^{(i)}$ for each i, with simple poles at the diagonals $z_u^{(i)} = z_v^{(j)}$ when $a_{ij} < 0$, and satisfies

$$res_{z_{u_1}^{(i)}=z_v^{(j)}} res_{z_{u_2}^{(i)}=z_v^{(j)}} \cdots res_{z_{u_{1-a_{ij}}}^{(i)}=z_v^{(j)}} f(\lambda \mid z_u^{(i)}) = 0$$

for v in I_j and $u_1, \dots, u_{1-a_{ij}}$ distinct in I_j (see [7]); this is a translation of the Serre relations, using the identities $res_{z=z'} \langle \psi, x(z) y(z') v \rangle = \langle \psi, [x, y](z') v \rangle$.

Assume that v is annihilated by the positive Cartan modes $h_\nu[z^i]$, $i \geqslant 1$, $\nu = 1, \dots, r$ and the $f_i[z^i]$, $i \geqslant -(g-1)$; let $(h_\nu)_{1 \leqslant \nu \leqslant r}$ be an orthonormal basis of $\bar{\mathfrak{h}}$ and define the Sugawara tensor as

$$\begin{aligned} &2(k + h^\vee) T_{\widetilde{\omega}}(z) \\ &\quad = \lim_{z' \to z} \left(\sum_{\nu=1}^{r} h_\nu(z) h_\nu(z') + \sum_{\alpha\in\Delta_+} \Big(f_\alpha(z) e_\alpha(z') + e_\alpha(z) f_\alpha(z') \Big) - k(dim\bar{\mathfrak{g}}) \widetilde{\omega}(z, z') \right), \end{aligned}$$

with $h^\vee$ the dual Coxeter number of $\bar{\mathfrak{g}}$.

Let P (resp. P') be the set of sequences $p = (i_1, \dots, i_s)$ such that $\alpha = \sum_{j=1}^{s} \alpha_{i_j}$ (resp. $\alpha > \sum_{j=1}^{s} \alpha_{i_j}$). The sequence (u_j) is associated to P if it is a sequence of pairwise different elements of $\cup_i I_i$, such that u_k belongs to I_{i_k}. We denote by S_i the subset of I_i formed by all u_j such that i_j is equal to i.

PROPOSITION 4.2. *The action of* $T_{\tilde\omega}(z)$ *on the correlation function (25) is given by*

$$
\begin{aligned}
&2(k+h^\vee)\langle\psi_\lambda, T_{\tilde\omega}(z)\prod_{i=1}^{r}\prod_{u\in I_i} e_i(z_u^{(i)})v\rangle\\
&=\Bigg[\sum_v\Bigg(\sum_a \omega_a(z)\partial_{(h_v)_a}+\sum_i\sum_{u\in I_i}(\alpha_i,h_v)G(z,z_u^{(i)})\Bigg)^2\\
&\quad+\sum_{\alpha\in\Delta_+}\Bigg(\sum_a D_z^{(\alpha,\lambda)}\omega_a(z)\partial_{(\alpha^\vee)_a}+\sum_i\sum_{u\in I_i}(\alpha_i,\alpha^\vee)D_z^{(\alpha,\lambda)}G(z,z_u^{(i)})\Bigg)\\
&\quad+k\sum_{\alpha\in\Delta_+}\omega_{(\alpha,\lambda)}(z)\Bigg]f(\lambda\mid z_u^{(i)})+\sum_{p'\in P'}n_{\alpha;i_1\ldots i_sN_\alpha;\alpha_i1+\cdots+\alpha_is}/\nu_{i_1\ldots i_s}\\
&\quad\sum_{(u_i)\text{ associated to }p'} G_{(\alpha,\lambda)}(z,z_{u_1}^{(i_1)})G_{(\alpha-\alpha_i1,\lambda)}(z_{u_1}^{(i_1)},z_{u_2}^{(i_2)})\cdots\\
&\quad G_{(\alpha-(\alpha_i1+\cdots+\alpha_{i_s}),\lambda)}(z_{u_s}^{(i_s)},z)\quad \operatorname{res}_{z_{u_1}^{(i_1)}=z_{u_2}^{(i_2)}}\operatorname{res}_{z_{u_2}^{(i_2)}=z_{u_3}^{(i_3)}}\cdots\\
&\quad\operatorname{res}_{z_{u_{s-1}}^{(i_{s-1})}=z}f(\lambda\mid z_u^{(i)})_{|z_{u_s}^{(i_s)}=z}-\sum_{p\in P}n_{\alpha;i_1\ldots i_{s-1}}/\nu_{i_1\ldots i_s}\\
&\quad\sum_{(u_i)\text{ associated to }p} G_{(\alpha,\lambda)}(z,z_{u_1}^{(i_1)})G_{(\alpha-\alpha_{i_1},\lambda)}(z_{u_1}^{(i_1)},z_{u_2}^{(i_2)})\cdots\\
&\quad G_{(\alpha-(\alpha_{i_1}+\cdots+\alpha_{i_{s-1}}),\lambda)}(z_{i_{s-1}}^{(i_{s-1})},z_{u_s}^{(i_s)})\\
&\quad\Bigg[\sum_a\omega_a(z_{u_s}^{(i_s)})\partial_{(\alpha_{i_s}^\vee)_a}\sum_i\sum_{u\in I_i-S_i}(\alpha_i,\alpha_{i_s}^\vee)G(z_{u_s}^{(i_s)},z_u^{(i)})+(\alpha,\alpha_{i_s}^\vee)G(z_{u_s}^{(i_s)},z)\Bigg]\\
&\quad\operatorname{res}_{z_{u_1}^{(i_1)}=z_{u_2}^{(i_2)}}\operatorname{res}_{z_{u_2}^{(i_2)}=z_{u_3}^{(i_3)}}\ldots\operatorname{res}_{z_{u_{s-1}}^{(i_{s-1})}=z}f(\lambda\mid z_u^{(i)})_{|z_{u_s}^{(i_s)}=z}\\
&\quad+\sum_{p\in P}n_{\alpha;i_1\ldots i_s-1}/\nu_{i_1\ldots i_s}\\
&\quad\sum_{(u_i)\text{ associated to }p} G_{(\alpha,\lambda)}(z,z_{u_1}^{(i_1)})G_{(\alpha-\alpha_i1,\lambda)}(z_{u_1}^{(i_1)},z_{u_2}^{(i_2)})\cdots\\
&\quad G_{(\alpha-(\alpha_{i_1}+\cdots+\alpha_{i_{s-1}}),\lambda)}(z_{u_{s-1}}^{(i_{s-1})},z_{u_s}^{(i_s)})\ kd_zG_{(\alpha-(\alpha_{i_1}+\cdots+\alpha_{i_s}),\lambda)}(z_{u_{s-1}}^{(i_{s-1})},z)\\
&\quad\operatorname{res}_{z_{u_1}^{(i_1)}=z_{u_2}^{(i_2)}}\operatorname{res}_{z_{u_2}^{(i_2)}=z_{u_3}^{(i_3)}}\ldots\operatorname{res}_{z_{u_{s-1}}^{(i_{s-1})}=z}f(\lambda\mid z_u^{(i)})_{|z_{u_s}^{(i_s)}=z},
\end{aligned}
\tag{26}
$$

where, for x in $\bar{\mathfrak{h}}$, we denote by x_a the element $(0,\ldots,x,\ldots,0)$ *of* $\bar{\mathfrak{h}}^g$ *(x at the ath place); and by ∂_h the partial derivative in $\bar{\mathfrak{h}}^g$ in the direction of h, for h in $\bar{\mathfrak{h}}^g$.*

Remark 5. The set P and its associated sequences appeared in the work [16] on integral formulas for the KZ equations.

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4.3. EXPRESSION OF THE KZB CONNECTION

Denote by $Proj_g^{(1)}$ the moduli space of quadruples $\widetilde{m} = (X, [\{\zeta_\alpha\}], P_0, z)$, where X is a curve of genus g, $[\{\zeta_\alpha\}]$ is a projective atlas of X (that is an atlas whose transition functions are projective transformations), P_0 a point of X and z a coordinate of the atlas with origin at P_0. A local coordinate related to some z_α by a projective transformations will be called a projective coordinate.

For each representation V of $\mathfrak{g}^{out}$, we may form the bundle $CB(V)$ over $Proj_g^{(1)}$, whose fiber at $\widetilde{m}$ is defined as the space of $\mathfrak{g}^{out}$-invariant forms on V.

A projectively flat connection on the bundle $CB(V)$ is defined as follows. Let $\widetilde{m} \mapsto \psi(\widetilde{m})$ be a local section of $CB(V)$. Let $\delta\widetilde{m}$ be a variation of $\widetilde{m}$. Then

$$\nabla_{\delta\widetilde{m}}\psi = \partial_{\delta\widetilde{m}}\psi - \psi \circ T_0[\xi(\delta\widetilde{m})], \tag{27}$$

where the equality is in V^* and $\xi(\delta\widetilde{m})$ is the element of $\mathbb{C}((z))\partial_z$ induced by $\delta\widetilde{m}$ (for any moduli $\widetilde{m}$, we have a ring $R_{\widetilde{m}}$ contained in $\mathbb{C}((z))$, and we set $R_{\widetilde{m}+\delta\widetilde{m}} = (1 + \xi(\delta\widetilde{m}))R_{\widetilde{m}}$). We set $T_0[\xi] = res_{P_0}(T_0(z)dz^2\xi(\delta\widetilde{m})(z)\partial_z)$, with $T_0(z)$ defined as $T_{\widetilde{\omega}}(z)$ in (18) replacing $\widetilde{\omega}$ by $dzdw/(z-w)^2$.

This connection is well-defined, preserves $CB(V)$ and is projectively flat (see [19]).

The form $\widetilde{\omega}$ defined by (8) depends only on the choice of a-cycles. On the other hand, this form determines a projective structure on X. Indeed, it is known that there is a bijective correspondence between bidifferential forms near the diagonal with behavior $dzdw/(z-w)^2 + r(z)dzdw + o(z-w)dzdw$, with r/z regular, up to addition of regular bidifferential forms vanishing on the diagonal, and projective structures on X. The correspondence associates to the projective atlas $[\{\zeta_\alpha\}]$ the form $d_{\zeta_\alpha}d_{\zeta'_\alpha}\ln(\zeta_\alpha - \zeta'_\alpha)$. Conversely, the projective coordinate ζ associated to the bidifferential form $dzdz'/(z-z')^2 + r(z)dzdz' + o(z-z')dzdz'$ is determined by the equation $S(\zeta, z) = -6r(z)$, where $S(\zeta, z)$ is the Schwarzian derivative of ζ with respect to z. Then $T_0(\zeta)(d\zeta)^2$, computed in a projective coordinate determined by $\widetilde{\omega}$, gets identified with $T_{\widetilde{\omega}}(z)(dz)^2$.

Let us define $\mathcal{M}_g^{(a)}$ as the moduli space of genus g curves with marked homology classes of a-cycles. $\widetilde{\omega}$ defines a map from $\mathcal{M}_g^{(a)}$ to $Proj_g$, such that its composition with projection of $Proj_g$ to $\mathcal{M}_g$ coincides with the projection of $\mathcal{M}_g^{(a)}$ on $\mathcal{M}_g$.

Define $\mathcal{M}_g^{(a)(1)}$ as the fibered product of $\mathcal{M}_g^{(a)}$ with $Proj_g^{(1)}$ over $Proj_g$. The KZB connection is defined on $Proj_g^{(1)}$, and it induces a connection on $\mathcal{M}_g^{(a)(1)}$, using the map from $\mathcal{M}_g^{(a)(1)}$ to $Proj_g^{(1)}$. This connection can be expressed as follows.

Let us express the connection induced by (27) in terms of correlation functions. For any formal vector field $\xi = \xi(z)\partial_z$ in $\mathbb{C}((z))\partial_z$, let ε be an indeterminate with $\varepsilon^2 = 0$ and $R_\varepsilon = (1 + \varepsilon\xi)R$; let $\Omega_R \subset \Omega_{\mathcal{K}}$ be the space of differentials of R and Ω_{R_ε} the space of differentials of R_ε. Then Ω_{R_ε} is equal to $(1 + \varepsilon\mathcal{L}_\xi)(\Omega_R)$, where

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$\mathcal{L}_\xi$ is the Lie derivative associated to ξ. Similarly, we have $dR_\varepsilon = (1+\varepsilon\mathcal{L}_\xi)(dR)$. Therefore, $1+\varepsilon\mathcal{L}_\xi$ induces a map from Ω_R/dR to $\Omega_{R_\varepsilon}/dR_\varepsilon$. Bases of these spaces are the classes of the ω_a and dr_a. On the other hand, we have the formula $\int_{\gamma'}(1+\varepsilon\mathcal{L}_\xi)(\omega) = \int_\gamma \omega$ for any cycle γ of X, deformed to γ' and any ω in Ω_R. Therefore, we have $\mathcal{L}_\xi(dr_a) = 0 \bmod dR$ and $\mathcal{L}_\xi\omega_a = \sum_b \delta\tau_{ab} dr_b \bmod dR$, where $\delta\tau_{ab}$ is the variation of the period matrix corresponding to $\delta\tilde{m}$.

We have obtained:

PROPOSITION 4.3. *Let $\widetilde{m}\mapsto\psi(\widetilde{m})$ be a section of the bundle $\mathcal{F}^{(n)}(m')$ over $\mathcal{M}_g^{(a)(1)}$, then the KZB connection is expressed as*

$$\nabla_{\delta\widetilde{m}} f(\widetilde{m})_\lambda(z_1,\ldots,z_n) = \partial_{\delta\widetilde{m}} f(\widetilde{m})_\lambda(z_1,\ldots,z_n) - \langle\psi_\lambda, T[\xi(\delta\widetilde{m})]e(z_1)\cdots e(z_n)v_n\rangle,$$

where $\langle\psi_\lambda, \frac{1}{k+2}T[\xi(\delta\widetilde{m})]e(z_1)\cdots e(z_n)v_n\rangle$ can be computed using (23).

Remark 6. The fact that the action of $T(z)$ preserves the vanishing conditions of Feigin and Stoyanovsky (vanishing on codimension k diagonals) probably again follows from the identity $(e^k)' =: he^k :$.

4.4. MOTION OF MARKED POINTS ($\mathfrak{sl}_2$ CASE)

In this section, we indicate how the above results are changed in the case of curves with marked points. Let $(P_i)_{i=1,\ldots,N}$ be marked points on X, distinct from P_0. Attach to each P_i the weight Λ_i and the evaluation Verma module V_{Λ_i}. V_{Λ_i} is generated by the vector $v_{-\Lambda_i}$ such that $hv_{-\Lambda_i} = -\Lambda_i v_{-\Lambda_i}$, and $fv_{-\Lambda_i} = 0$. Set again $\psi_\lambda = \psi \circ e^{\sum_a \lambda_a h[r_a]}$ and

$$f(\lambda \mid z_1,\ldots,z_m) = \langle\psi_\lambda, (e(z_1)dz_1\cdots e(z_m)dz_m v_n) \otimes v_{-\Lambda_1} \otimes \cdots \otimes v_{-\Lambda_N}\rangle,$$

$m = n - \frac{1}{2}\sum_i \Lambda_i$.

$f_\lambda(z_1,\cdots,z_m)$ depends on the z_α as a section of $\Omega_X\mathcal{L}_{2\lambda}$, regular outside P_0 and with simple poles at the P_i.

For w_i in V_{Λ_i}, the values of the $\langle\psi_\lambda, (e(z_1)\cdots e(z_m)v_n)\otimes(\otimes_{i=1}^N w_i)\rangle$ can be recovered from $f(\lambda \mid z_1,\ldots,z_m)$ using the rule

$$\begin{aligned}&\mathrm{res}_{z=P_i}\langle\psi_\lambda, (e(z)dz e(z_1)dz_1\cdots e(z_m)dz_m v_n)\otimes(\otimes_{i=1}^N w_i)\rangle\\ &\quad = -\langle\psi_\lambda, (e(z_1)dz_1\cdots e(z_m)dz_m v_n)\otimes e^{(i)}(\otimes_{i=1}^N w_i)\rangle.\end{aligned}$$

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The action of the Sugawara tensor is expressed as

$$\begin{aligned}&\Big\langle \psi_\lambda, \Big(T_{\widetilde{\omega}}(z)e(z_1)dz_1\cdots e(z_m)dz_m v_n\Big)\otimes(\otimes_{i=1}^N v_{-\Lambda_i})\Big\rangle\\ &\quad=\left[\frac{1}{2}\left(\sum_a \omega_a(z)\partial_{\lambda_a}+2\sum_\alpha G(z,z_\alpha)-\sum_i \Lambda_i G(z,P_i)\right)^2+\right.\\ &\qquad+\sum_a D_z^{(2\lambda)}\omega_a(z)\partial_{\lambda_a}+2\sum_\alpha D_z^{(2\lambda)}G(z.z_\alpha)-\\ &\qquad\left.-\sum_i \Lambda_i D_z^{(2\lambda)}G(z,P_i)+k\omega_{2\lambda}(z)\right]f(\lambda\mid z_1,\dots,z_n)+\\ &\qquad+\sum_{\alpha=1}^n\left[-2G_{2\lambda}(z,z_\alpha)\left(\sum_a \omega_a(z_\alpha)\partial_{\lambda_a}+2\sum_{\beta\neq\alpha}G(z_\alpha,z_\beta)-\sum_i\Lambda_i G(z_\alpha,P_i)\right)+\right.\\ &\qquad\left.+\big(-4G_{2\lambda}(z,z_\alpha)G(z_\alpha,z)+2kd_{z_\alpha}G_{2\lambda}(z,z_\alpha)\big)\right]f(\lambda\mid z_1,\dots,z,\dots,z_n).\end{aligned}$$

When $k=-2$, the right-handside of this formula is the expression for a commuting family of differential-difference operators, or alternatively, for a commuting family of differential operators acting on some finite-dimensional bundle over $J^0(X)$.

The KZB connection is now a connection over the bundle of conformal blocks over $Proj_g^{(n)}$, which is the set of quadruples $\widetilde{m}=(X,[\{\zeta_\alpha\}],P_i,\zeta_i)$ of curves with projective structure, n marked points and flat coordinates vanishing at these points.

The vector fields $\zeta_i\partial/\partial\zeta_i$ describing the changes of coordinates fixing the points, and $\partial/\partial P_i$ describing the changes of points in the fixed coordinate, are respectively given by the action of Sugawara elements corresponding to vector fields $\xi_{\frac{\partial}{\partial P_i}}$ equal to $\partial/\partial\zeta_i$ at P_i and $o(\zeta_j)$ at P_j, and $\xi_{\zeta_i\frac{\partial}{\partial\zeta_i}}$ equal to $\zeta_i\partial/\partial\zeta_i$ at P_i and $o(\zeta_j)$ at P_j.

Set

$$G(z,w)=dz/(z-w)+\phi(z)dz+o(z-w)dz,$$

so that $G_\lambda(z,w)dz=dz/(z-w)+g_\lambda(z)dz$, with

$$\begin{aligned}g_\lambda(z)dz&=\phi(z)dz+\sum_{a=1}^g\omega_a(z)(\partial_{\varepsilon_a}\ln\Theta(-\lambda+(g-1)A(P_0)-\Delta)-\\ &\quad-\ \partial_{\varepsilon_a}\ln\Theta(gA(P_0)-A(z)-\Delta)).\end{aligned}$$

PROPOSITION 4.4. *The KZB connection is expressed, in the direction of variation of coordinates at* P_i *by*

$$2(k+2)\nabla_{\zeta_i\frac{\partial}{\partial\zeta_i}}f(\tilde{m})(\lambda\mid z_\alpha)=2(k+2)\zeta_i\frac{\partial}{\partial\zeta_i}f(\tilde{m})(\lambda\mid z_\alpha)-\frac{1}{2}\Lambda_i(\Lambda_i+2)f(\tilde{m})(\lambda\mid z_\alpha).$$

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and in the direction of variation of P_i, by

$$\begin{aligned}
&2(k+2)\nabla_{\frac{\partial}{\partial P_i}} f(\tilde{m})(\lambda \mid z_\alpha)\\
&\quad = 2(k+2)\frac{\partial}{\partial P_i} f(\tilde{m})(\lambda \mid z_\alpha) - \Bigg[-\Lambda_i \sum_a \omega_a(P_i)\partial_{\lambda_a}\\
&\qquad + \Lambda_i\Bigg(\sum_{j\neq i}\Lambda_j G(P_i,P_j) - 2\sum_\alpha G(P_i,z_\alpha)\Bigg)\\
&\qquad + \Lambda_i^2\phi(P_i) + 2\Lambda_i g_{2\lambda}(P_i)\Bigg] f(\tilde{m})(\lambda \mid z_\alpha)\\
&\qquad + \sum_\alpha\Bigg[-2G_{2\lambda}(P_i,z_\alpha)\Bigg(\sum_a \omega_a(z_\alpha)\partial_{\lambda_a} + 2\sum_{\beta\neq\alpha} G(z_\alpha,z_\beta)\Bigg)\\
&\qquad\qquad - 4G_{2\lambda}(P_i,z_\alpha)G(z_\alpha,P_i) + 2kd_{z_\alpha}G_{2\lambda}(z_\alpha,P_i)\Bigg]\\
&\qquad\qquad \times \operatorname{res}_{z=P_i} f(\tilde{m})(\lambda \mid z,(z_\beta)_{\beta\neq\alpha})
\end{aligned} \tag{28}$$

when $m = 0$, this equation simplifies to

$$\begin{aligned}
&2(k+2)\nabla_{\frac{\partial}{\partial P_i}} f(\tilde{m})_\lambda = 2(k+2)\frac{\partial}{\partial P_i} f(\tilde{m})(\lambda)\\
&\quad - \Bigg[-\Lambda_i\sum_a \omega_a(P_i)\partial_{\lambda_a} + \Lambda_i\sum_{j\neq i}\Lambda_j G(P_i,P_j) + \Lambda_i^2\phi(P_i)\\
&\quad + 2\Lambda_i g_{2\lambda}(P_i)\Bigg] f(\tilde{m})(\lambda).
\end{aligned} \tag{29}$$

Remark 7. It would be interesting to express the equations obtained above in terms of dynamical r-matrices, as it was done in [8].

5. Commuting Differential Operators

The operators (23) are differential-evaluation operators acting on functions on $J^0(X)^r \times \prod_{i=1}^r S^{n_i}X$. They make sense for arbitrary complex values of k. When k is critical, one expects these operators to commute with each other. To prove this, we will consider modules $W_{n|m,m'}$ generalizing the twisted Weyl modules.

For generic λ_0 in $J^0(X)$, λ_0-twisted conformal blocks for these modules can be characterized via functions (2) as formal sections of finite-dimensional bundles over $J^0(X)$.

5.1. TWISTED CONFORMAL BLOCKS FOR GENERAL MODULES

Let X be a smooth complex curve of genus $g \geqslant 1$ and let P_0 be a fixed point of X. Denote by $\mathcal{K}$ and $\mathcal{O}$ the local field and ring of X at P_0. Denote also by R the ring $H^0(X - \{P_0\}, \mathcal{O}_X)$ and by $\mathbb{A}$ the adèle ring of X.

Recall that (8) defines a form ψ_λ, depending on formal variables $\lambda_a^{(i)}$, on an arbitrary $\mathfrak{g}$-module V.

For $\mu_1, \ldots, \mu_g$ complex linear combinations of the $\lambda_a^{(i)}$, define $R^{(f)}_{(\mu_i)}$ as the subspace of $\mathcal{K}[[\lambda_a^{(i)}]]$ formed by the functions $f(z, \lambda_a^{(i)})$ depending formally on the $\lambda_a^{(i)}$, such that the coefficients of the monomials in $\lambda_a^{(i)}$ extend to regular functions on $\widetilde{X} - \sigma^{-1}(P_0)$ and we have $f(\gamma_{A_a} z, \lambda_a^{(i)}) = f(z, \lambda_a^{(i)})$ and $f(\gamma_{B_a} z, \lambda_a^{(i)}) = e^{\mu_a} f(z, \lambda_a^{(i)})$.

ψ_λ has the following properties:

LEMMA 5.1 (a) *Set for* $a = 1, \ldots, g$, $\lambda_a = \sum_i \lambda_a^{(i)} h_i$. *Define* $\mathfrak{g}_\lambda^{out(f)}$ *as*

$$\mathfrak{g}_\lambda^{out(f)} = (\bar{\mathfrak{h}} \otimes R)[[\lambda_a^{(i)}]] \oplus \oplus_{\alpha \in \Delta} (\bar{\mathfrak{g}}_\lambda \otimes R^{(f)}_{\langle \alpha, \lambda_1 \rangle, \ldots, \langle \alpha, \lambda_g \rangle}).$$

Then ψ_λ *is* $\mathfrak{g}_\lambda^{out(f)}$*-invariant.*

(b) $\lambda \mapsto \langle \psi_\lambda, v \rangle$ *satisfies the differential equation* $\partial_{\lambda_a^{(i)}} \langle \psi_\lambda, v \rangle = \langle \psi_\lambda, h_i[r_a] v \rangle$ *for any* v *in* V.

Proof. Clearly, $\mathfrak{g}_\lambda^{out(f)}$ is contained in $Ad(e^{-\sum_{i,a} \lambda_a^{(i)} h_i[r_a]})(\mathfrak{g}^{out}[[\lambda_a^{(i)}]][\lambda_a^{(i)-1}])$; this implies (a). We have for any $a, b = 1, \ldots, g$, $\langle dr_a, r_b \rangle = 1/2i\pi \int_{\partial i(X)} dr_a r_b$; the contributions of the paths $\widetilde{B}_c$ and $\widetilde{B}_c^{-1}$ cancel each other, as well as those of the paths $\widetilde{A}_c$ and $\widetilde{A}_c^{-1}$, $c \neq b$; the sum of the contributions of the paths $\widetilde{A}_b$ and $\widetilde{A}_b^{-1}$ is equal to $1/2i\pi \int_{A_b} dr_a$, which is zero as r_a is single-valued along a-cycles. Therefore we have $[h_i[r_a], h_j[r_b]] = 0$ for any i, j, a, b, which proves (b).

5.2. CONFORMAL BLOCKS FOR THE $W_{n|m,m'}$

In this section, we set $\bar{\mathfrak{g}} = \mathfrak{sl}_2$. Let k be an arbitrary complex number.

For m, m' integer numbers with $m + m' \geqslant 0$, define $\mathfrak{g}^{in}_{m,m'}$ by

$$\mathfrak{g}^{in}_{m,m'} = (\bar{\mathfrak{n}}_- \otimes z^m \mathcal{O}) \oplus (\bar{\mathfrak{h}} \otimes \mathcal{O}) \oplus (\bar{\mathfrak{n}}_+ \otimes z^{m'} \mathcal{O}) \oplus \mathbb{C}K.$$

Define $\mathfrak{g}^{in}_{m,\infty}$ and $\mathfrak{g}^{in}_{-\infty,\infty}$ by the convention that $z^\infty \mathcal{O} = 0$ and $z^{-\infty} \mathcal{O} = \mathcal{K}$.

Let n be a positive integer. If $m + m' > 0$, $(m, m') = (-\infty, \infty)$, or $m + m' = 0$ and $n = -km$, define $\chi_{n|m,m'}$ as the character of $\mathfrak{g}^{in}_{m,m'}$ such that $\chi_{n|m,m'}(K) = k$, $\chi_{n|m,m'}(h[z^i]) = -2n\delta_{i,0} k$, $\chi_{n|m,m'}(x[z^i]) = 0$, $x = e, f$.

Define $W_{n|m,m'}$ as the induced module $U\mathfrak{g} \otimes_{U\mathfrak{g}^{in}_{m,m'}} \mathbb{C}_{\chi_{n|m,m'}}$. Denote by v_n the vector $1 \otimes 1$ of this module. (When $m + m' = 0$, $W_{n|m,m'}$ is a twisted Weyl module.) For λ_0 a complex number, define $CB_{\lambda_0}(W_{n|m,m'})$ as the space of $\mathfrak{g}^{out}_{\lambda_0}$-invariant linear forms on $W_{n|m,m'}$ (where $\mathfrak{g}^{out}_{\lambda_0}$ is as in Theorem 2.1).

Let us define $\mathcal{F}^{(n)}_{\lambda_0}$ as the space of forms $f(\lambda \mid z_1, \ldots, z_n)$, depending formally on λ in the neighborhood of λ_0, symmetric in $z_1, \ldots, z_n$, sections of $\Omega_X \mathcal{L}_{-2\lambda}$ in z_i, regular

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outside P_0. Define for any integer p, $\mathcal{F}^{(n)}_{\lambda_0}(p)$ as the subspace of $\mathcal{F}^{(n)}_{\lambda_0}$ consisting of the forms with poles at $z_i = P_0$ of order at most p.

For any ρ in $R_{2\lambda}$, define first order differential operators $\widetilde{f}[\rho]$ by

$$\begin{aligned}(\widetilde{f}[\rho]f)(\lambda \mid z_1, \ldots, z_{n+1}) \\ = \sum_{i=1}^{n+1}\left[-\rho(z_i)\left(\sum_a \omega_a(z_i)\partial_{\lambda_a} + 2\sum_{j\neq i} G(z_i, z_j)\right) + kd\rho(z_i)\right] \\ \times f(\lambda \mid z_1, \ldots \check{i} \ldots z_{n+1}).\end{aligned} \tag{30}$$

$\widetilde{f}[\rho]$ maps $\mathcal{F}^{(n)}_{\lambda_0}$ to $\mathcal{F}^{(n+1)}_{\lambda_0}$.

PROPOSITION 5.1. *Define a map $\imath$ from $CB_{\lambda_0}(W_{n|m,m'}) \to \mathcal{F}^{(n)}_{\lambda_0}$ by*

$$\imath(\psi_{\lambda_0})(\lambda|z_1, \ldots, z_n) = \langle\psi_{\lambda_0}, e^{\sum_a(\lambda-\lambda_0)_a h[r_a]})e(z_1)\cdots e(z_n)v_n\rangle,$$

for ψ_{λ_0} in $CB_{\lambda_0}(W_{n|m,m'})$.

Assume that $H^1(X, \mathcal{L}_{2\lambda_0}(-mP_0))$ is zero. Then $\imath$ is an isomorphism from $CB_{\lambda_0}(W_{n|m,m'})$ to the intersection of the kernels of the $\widetilde{f}[\rho]$ in $\mathcal{F}^{(n)}_{\lambda_0}(m')$, with ρ in $R_{2\lambda} \cap z^m\mathcal{O}$ (which is the same as $H^0(X, \mathcal{L}_{2\lambda}(-mP_0))$).

Proof. The fact that the image of $\imath$ is contained in the kernel of the $\widetilde{f}[\rho]$ follows from the identity

$$\langle\psi_{\lambda_0}, e^{\sum_a(\lambda-\lambda_0)_a h[r_a]}[f[\rho], e(z_1)\cdots e(z_n)]v_n\rangle = 0,$$

which follows from $f[\rho]v_n = 0$ and $\langle\psi_{\lambda_0}, f[e^{-2\sum_a(\lambda-\lambda_0)_a r_a}\rho]v\rangle = 0$ for any vector v.

Let us now consider $f(\lambda \mid z_1, \ldots, z_n)$ in $\mathcal{F}^{(n)}_{\lambda_0}(m')$, in the kernel of the $\widetilde{f}[\rho]$ and let us construct its preimage by $\imath$.

Clearly, $CB_{\lambda_0}(W_{n|m,m'})$ is isomorphic to the space of linear forms ϕ on $U\mathfrak{g}$, such that $\phi(xx^{in}) = \phi(x^{out}x) = 0$, for x^{in} in $\mathfrak{g}^{in}_{m,m'}$ and x^{out} in $\mathfrak{g}^{out}_{\lambda_0}$.

Define $\mathbb{C}\langle h[r_a], e[\varepsilon]\rangle$ as the subalgebra of $U\mathfrak{g}$ generated by the $h[r_a]$ and the $e[\varepsilon]$, ε in $\mathcal{K}$. Since we have $\mathcal{K} = R_{2\lambda_0} + z^m\mathcal{O}$, the map

$$\pi : U\mathfrak{g}^{out}_{\lambda_0} \otimes \mathbb{C}\langle h[r_a], e[\varepsilon]\rangle \otimes U\mathfrak{g}^{in}_{m,m'} \to U\mathfrak{g}$$

given by the product is surjective. It kernel is spanned by the $ae[\varepsilon]\otimes b\otimes c - a\otimes e[\varepsilon]b\otimes b$, ε in $R_{-2\lambda_0}$, the $a\otimes be[\varepsilon]\otimes c - a\otimes b\otimes e[\varepsilon]c$, ε in $z^{m'}\mathcal{O}$, the $ah[1]\otimes b\otimes c - a\otimes b\otimes h[1]c - a\otimes[h[1], b]\otimes c$ and the $af[\varepsilon]\otimes b\otimes c - a\otimes b\otimes f[\varepsilon]c - \sum a[f[\varepsilon], b]'\otimes[f[\varepsilon], b]''\otimes[f[\varepsilon], b]'''\otimes c$, ε in $R_{2\lambda_0}\cap z^m\mathcal{O}$ with a, b, c in $U\mathfrak{g}^{out}_{\lambda_0}$, $\mathbb{C}\langle h[r_a], e[\varepsilon]\rangle$ and $U\mathfrak{g}^{in}_{m,m'}$, and $\sum[f[\varepsilon], b]'\otimes[f[\varepsilon], b]''\otimes[f[\varepsilon], b]'''$ any preimage of $[f[\varepsilon], b]$ by π.

Define a linear form $\bar{\phi}$ on $\mathbb{C}\langle h[r_a], e[\varepsilon]\rangle$ by the formula

$$\bar{\phi}\left(\prod_a h[r_a]^{\alpha_a} e[\varepsilon_1]\cdots e[\varepsilon_{n'}]\right) = \delta_{nn'} res_{z_1=P_0}\cdots res_{z_n=P_0} f_{(\alpha_a)}(z_1, \ldots, z_n)\varepsilon_1(z_1)\cdots\varepsilon_n(z_n),$$

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where we set $f(\lambda \mid z_1, \ldots, z_n) = \sum_{(\alpha_i)} \prod_a (\lambda - \lambda_0)_a^{\alpha_a} f_{(\alpha_a)}(z_1, \ldots, z_n)$. Extend $\bar{\phi}$ to $U\mathfrak{g}_{\lambda_0}^{out} \otimes \mathbb{C}\langle h[r_a], e[\varepsilon]\rangle \otimes U\mathfrak{g}_{m,m'}^{in}$ by the rule that $\bar{\phi}(a \otimes b \otimes c) = \varepsilon(a)\varepsilon(c)\bar{\phi}(b)$, ε denoting the counit.

The functional properties of $f(\lambda \mid z_1, \ldots, z_n)$ imply that the image of the kernel of π is mapped to 0 by $\bar{\phi}$, so that $\bar{\phi}$ defines a linear form of $U\mathfrak{g}$. It is then clear that this form is left $\mathfrak{g}_{\lambda_0}^{out}$-invariant and right $\mathfrak{g}_{m,m'}^{in}$-invariant, and that its image by $\imath$ is $f(\lambda \mid z_1, \ldots, z_n)$. □

LEMMA 5.2. *The operator $T(z)$ acts naturally on $CB_{\lambda_0}(W_{n|m,m'})$. When m is $\leqslant -(g-1)$, this action is expressed on the $f(\lambda \mid z_1, \ldots, z_n)$ by formula (23).*

Remark 8. Since $H^1(X, \mathcal{L}_{2\lambda_0}(-mP_0))$ is zero, $H^1(X, \mathcal{L}_{2\lambda}(-mP_0))$ also vanishes for λ in a neighborhood to λ_0. By the Riemann–Roch theorem, it follows that $H^0(X, \mathcal{L}_{2\lambda}(-mP_0))$ has constant dimension at the neighborhood of λ_0. It follows that the ρ understood in the statement of Proposition 5.1 form a free $\mathbb{C}[[(\lambda - \lambda_0)_a]]$-module with rank equal to this dimension. □

Remark 9. The condition that $H^1(X, \mathcal{L}_{2\lambda_0}(-mP_0))$ vanishes is fulfilled if $m < -(g-1)$ and any λ_0, or if $m = -(g-1)$ and $2\lambda_0$ not in some translate of the theta characteristic containing zero. In the latter case, $CB_{\lambda_0}(W_{n|m,m'})$ is isomorphic to $\mathcal{F}_{\lambda_0}^{(n)}(m')$, because $H^0(X, \mathcal{L}_{2\lambda_0}(-mP_0))$ also vanishes. □

Remark 10. If $m = -(g-1)$ and $2\lambda_0$ is in the translate of the theta-characteristic (for example, if λ_0 is zero), the image of $\imath$ is characterized by some vanishing conditions near λ_0.

5.3. COMMUTING DIFFERENTIAL OPERATORS

THEOREM 5.1. *Suppose that k equals -2.*

(1) *Set for p integer $\geqslant g$ and λ in $J^0(X)$, $\mathcal{F}_\lambda^{(n)}(p) = S^n H^0(X, \Omega_X \mathcal{L}_{-2\lambda}(pP_0))$. $(\mathcal{F}_\lambda^{(n)}(p))_{\lambda \in J^0(X)}$ forms a finite-dimensional vector bundle over $J^0(X)$, denoted $\mathcal{F}^{(n)}(p)$. The operators T_z defined by (23) form a family of commuting differential operators acting on sections of this bundle. This family has rank $\leqslant 3g - 3 + p$. It normalizes the first order operators $\widetilde{f}[\rho]$ defined by (30), ρ in $H^0(X, \mathcal{L}_{2\lambda}(-mP_0))$ for any m (that is, it preserves the intersection of their kernels).*

(2) *Formula (23) also defines a family of commuting differential-evaluation operators, acting on functions of λ in $J^0(X)$ and of $z_1, \ldots, z_n$ in a subset U of X (e.g. the pointed formal disc at P_0), symmetric in the z_i; these operators are indexed by points of U. They normalize the operators $\widetilde{f}[\rho]$, ρ some function on U, defined by formula (30).*

Proof. Let us prove (1). If $p \geqslant g$, the action of T_z on the jets at λ_0 of sections of $\mathcal{F}^{(n)}(p)$ coincides with the action of $T(z)$ on $CB_{\lambda_0}(W_{n|-(g-1),p})$, by Remark 9 and Lemma 5.2. Since the $T(z)$ commute together, this shows that the operators T_z form

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a commutative family. The result on normalization of the $\widetilde{f}[\rho]$ follows from the fact that the action of T_z on the intersection of their kernels coincides with the action of $T(z)$ on $CB_{\lambda_0}(W_{n|m,\infty})$.

LEMMA 5.3 *For any f_λ, $(T_z f_\lambda)(z_1, \dots, z_n)$ is a quadratic form on z, regular on X except for a pole of order $\leqslant p$ at P_0.*

Proof of Lemma. It is clear that the right-hand side of (23) is a quadratic form in z with possible poles at P_0 and the z_α. Since $k = -2$, one checks that this expression has no pole at z_α.

Let us evaluate the pole at P_0. Let z be a local coordinate at P_0. $G_\lambda(z, w)$ has the expansion

$$G_{2\lambda}(z, w) = \frac{z^{g-1} w^{1-g} dz}{z - w} + z^{g-1} w^{1-g} dz \sum_{i,j \geqslant 0} a_{ij}(\lambda) z^i w^j.$$

Therefore, if ω belongs to $H^0(X, \Omega_X)$, then $D^{(2\lambda)}\omega$ is in $H^0(X, \Omega_X^2(P_0))$, because if ω_a is $(z^a + o(z^a))dz$, we have $D_z^{(2\lambda)}\omega_a = [(2g - 2 - a)z^{a-1} + O(z^a)]dz$.

On the other hand, $\omega_{2\lambda}$ has the expansion at P_0

$$\omega_{2\lambda} = -g(g-1)z^{-2}(dz)^2 - 2(g-1)z^{-1}(dz)^2 a_{00}(\lambda) + O(1)(dz)^2.$$

So the two first lines of the right-hand side of (23) have a poles of order $\leqslant 2$ at P_0. Since $\omega_a(z)$, $G(z, z_\alpha)$, $G(z_\alpha, z)$ and $G_{2\lambda}(z, z_\alpha)$ are regular at $z = P_0$, the pole at P_0 of the two last lines of (23) is of order at most p. □

The result on the rank of the family (T_z) now follows from the fact that $h^0(\Omega_X^2(pP_0)) = 3g - 3 + p$.

Let us prove (2). If we set $p = \infty$ in the result of (1), we see that the operators T_z, z in U, commute on all functions of λ and the z_i, which are symmetric in these variables and behave as sections of $\Omega_X \mathcal{L}_{-2\lambda}$, regular outside P_0. The commutator $[T_z, T_{z'}]$ is again a differential-evaluation operator. But no such operator can vanish on these functions without being zero. □

Remark 11. Arguments similar to the proof of Theorem 5.1 imply that the T_z defined by (26) commute when k is critical.

Remark 12. In the case $n = 0$, we find a commuting family of operators

$$\begin{aligned}(T_z f)(\lambda_1, \dots, \lambda_g) \\ = \left[\frac{1}{2}\left(\sum_a \omega_a(z)\partial_{\lambda_a}\right)^2 + \sum_a D_z^{(2\lambda)}\omega_a(z)\partial_{\lambda_a} - 2\omega_{2\lambda}(z)\right] f(\lambda_1, \dots, \lambda_g).\end{aligned} \tag{31}$$

If $g = 1$, we have

$$
\begin{aligned}
&\omega_a = 2i\pi dz,\ G_\lambda(z, z') = \frac{\theta\Big(-\frac{\lambda}{2i\pi} + z - z'\Big)\theta'(0)}{\theta\Big(-\frac{\lambda}{2i\pi}\Big)\theta(z - z')}dz,\\
&D_z^{(2\lambda)}\omega_a = 2\frac{\theta'}{\theta}\left(\frac{\lambda}{i\pi}\right)2i\pi(dz)^2,\\
&\omega_{2\lambda} = -\frac{\theta''}{\theta}\left(\frac{\lambda}{2i\pi}\right)(dz)^2,
\end{aligned}
$$

where θ is the Jacobi theta-function, so that

$$
\begin{aligned}
T_z &= \left[\frac{1}{2}(2i\pi\partial_\lambda)^2 + 2\frac{\theta'}{\theta}\left(\frac{\lambda}{i\pi}\right)2i\pi\partial_\lambda + 2\frac{\theta''}{\theta}\left(\frac{\lambda}{i\pi}\right)\right](dz)^2\\
&= \frac{1}{2}\left(2i\pi\partial_\lambda + 2\frac{\theta'}{\theta}\left(\frac{\lambda}{i\pi}\right)\right)^2(dz)^2,
\end{aligned}
$$

which is conjugate to $\frac{1}{2}(2i\pi\partial_\lambda)^2$.

When $g > 1$, (21) is a generating series for one first order and $3g - 3$ second order operators. The linear operator is $\sum_a 2(1-g)\omega_a(P_0)\partial_{\lambda_a} + (1-g)a_{00}(\lambda)$. From the formula for the variation of the periods matrix $\delta\tau_{ab} = res_{P_0}(\omega_a\omega_b\xi)$ follows that the operator corresponding to a variation $\delta\tau_{ij}$ has leading term $\sum_{a,b}\delta\tau_{ab}\partial_{\lambda_a}\partial_{\lambda_b}$.

Remark 13. In the case of the rational curve, we get the commuting family of operators defined on symmetric functions $f(z_1, \cdots, z_n)$ by

$$
(T(z)f)(z_1, \cdots, z_n) = \sum_{i=1}^{n}\left(\sum_{j\neq i}\frac{1}{z_j - z_i}\right)\frac{f(z_1, \cdots, z, \cdots, z_n) - f(z_1, \cdots, z_n)}{z - z_i}, \quad (32)
$$

where z is at the ith position in the right-hand side.

Remark 14. Relation with the Beilinson–Drinfeld operators. It is not possible to interpret directly the operators T_z directly as Beilinson–Drinfeld (BD) operators ([2]). Indeed, for $g = n_{\mathcal{K}}t[f_\lambda]w$, with $n_{\mathcal{K}}$ in $N(\mathcal{K})$, f_λ in C_λ (see Section 6.2) and $w = \left(\begin{smallmatrix} z^n & 0\\ 0 & z^{-n}\end{smallmatrix}\right)$, the local ring $\hat{\mathcal{O}}_{Bun_{\bar{G}}}([g])$ is $H_0({}^{g^{-1}}\mathfrak{g}^{out}, Ind_{\mathfrak{g}_{\mathcal{O}}}^{\mathfrak{g}}\mathbb{C}_\chi)^*$, where $\mathbb{C}_\chi$ is the $\mathfrak{g}_{\mathcal{O}}$-module associated with the character χ of $\mathfrak{g}_{\mathcal{O}}$, defined by $\chi(K) = -2$ and $\chi(\bar{\mathfrak{g}}\otimes\mathcal{O}) = 0$, and ${}^g x$ denotes the conjugation of x by g for x in $\mathfrak{g}$ and g in $\bar{G}(\mathcal{K})$. This space is isomorphic to $CB_\lambda(W_{-2n|-2n,2n})$, which has no interpretation in terms of the $\mathcal{F}_\lambda^{(n)}(p)$.

However, the vector $f[z^{-2n-1}]^p v_{-n}$ is cyclic in $W_{-2n|-2n,2n}$, which implies that $W_{-2n|-2n,2n}$ is a quotient of $W_{p-2n|-2n,2n+2}$. $CB_\lambda(W_{-2n|-2n,2n})$ may then be viewed as a subspace of $CB_\lambda(W_{p-2n|-2n,2n+2})$, which has a functional interpretation when $p \geqslant 2n$. The BD operators may then be expressed as the commuting family of operators (T_z), acting on some subspace (defined as the intersection of a family $\widetilde{f}[\rho]$ and some vanishing conditions) of some $\mathcal{F}_\lambda^{(n)}(p)$.

Another connection with the BD operators is the following. The BD operators admit lifts to bundles over the moduli space of $\bar{G}$-bundles with parabolic structure at P_0. Such bundles are attached to a weight Λ. The space of local sections of this bundle is then $H_0({}^{g^{-1}}\mathfrak{g}^{out}, Ind_{\mathfrak{g}_{0|0,1}}^{\mathfrak{g}} \mathbb{C}_{\chi_\Lambda})^*$ where $\mathbb{C}_{\chi_\Lambda}$ is the $\mathfrak{g}_{0|0,1}$-module defined by $\chi_\Lambda(K) = -2$, $\chi_\Lambda(h[1]) = \Lambda$ and $\chi_\Lambda(x[t^i]) = 0$ for $x = f$ and $i \geqslant 0$, and $x = h, e$ and $i > 0$. This space is isomorphic to $CB_\lambda(W_{\lambda-2n|-2n,2n+1})$ which is isomorphic to the intersection of kernels of some $\widetilde{f}[\rho]$ in some $\mathcal{F}_\lambda^{(n)}(p)$ if $-2n \leqslant 1 - g$ and $\lambda > 2n$.

The commuting family of operators (T_z), acting on the intersection of kernels of the $\widetilde{f}[\rho]$, gets then identified with the BD operators. The commuting family (T_z) acting on $\mathcal{F}_\lambda^{(n)}(p)$ itself gets then identified with the lift of the BD operators to some moduli space of B-bundles with additional structure.

Appendix A. Proof of Theorem 2.1

A.1. ADELIZATION

For any point s of X, denote by $\mathcal{K}_s$ and $\mathcal{O}_s$ the local field and ring at this point. For a finite subset S of X, set $\mathcal{K}_S = \oplus_{s\in S}\mathcal{K}_s$ and $\mathcal{O}_S = \oplus_{s\in S}\mathcal{O}_s$. Set also $R_S = H^0(X - S, \mathcal{O}_X)$; we view R_S as a subring of $\mathcal{K}_S$. Define $\mathfrak{g}_S$ as the Lie algebra $(\bar{\mathfrak{g}} \otimes \mathcal{K}_S) \oplus \mathbb{C}K$, endowed with the Lie bracket

$$[x[\varepsilon], y[\varepsilon']] = [x, y][\varepsilon\varepsilon'] + K\langle d\varepsilon, \varepsilon'\rangle, \tag{33}$$

with $\langle \omega, \varepsilon\rangle = \sum_{s\in S} res_s(\omega\varepsilon)$ and $x[\varepsilon] = (x \otimes \varepsilon, 0)$. Set $\mathfrak{g}_S^{out} = \bar{\mathfrak{g}} \otimes R_S$; we view $\mathfrak{g}_S^{out}$ as a Lie subalgebra of $\mathfrak{g}_S$, by the embedding $x \otimes r \mapsto x[r]$. For any s in X, let $\mathfrak{g}_s$ be the space $(\bar{\mathfrak{g}} \otimes \mathcal{K}_s) \oplus \mathbb{C}K$, endowed with the bracket analogous to (23), is a Lie subalgebra of $\mathfrak{g}^{\mathbb{A}}$; the associated embedding is denoted by i_s.

Let k be a positive integer, (Λ, k) be an integrable weight of $\mathfrak{g}$ and $(\rho_{\Lambda,k}, L_{\Lambda,k})$ be the associated integrable module over $\mathfrak{g}$.

Define $(\rho_{0,k}, L_{0,k})$ as the integrable module over $\mathfrak{g}$ with highest weight $(0, k)$ (the vacuum module of level k). Denote by v_{top} its highest weight vector. Define V^S as the vector space $L_{\Lambda,k} \otimes \otimes_{s\in S, s\neq P_0} L_{0,k}$; there is a map $\rho_S : \mathfrak{g}_S \to End(V^S)$ defined by the condition that the action of $\mathfrak{g}_s$ by $\rho_S \circ i_s$ on V^S is identical to $\rho_{\Lambda,k}^{(P_0)}$ if $s = P_0$ and to $\rho_{0,k}^{(s)}$ else.

Define $\mathfrak{g}^{\mathbb{A}}$ as the space $(\bar{\mathfrak{g}} \otimes \mathbb{A}) \oplus \mathbb{C}K$, endowed with the Lie bracket analogous to (23); the map $x \mapsto (x, 0)$ makes $\bar{\mathfrak{g}} \otimes \mathbb{C}(X)$ a Lie subalgebra of $\mathfrak{g}^{\mathbb{A}}$. For x in $\bar{\mathfrak{g}}$, $\varepsilon = (\varepsilon_s)_{s\in X}$ in $\mathbb{A}$, we sometimes denote by $x^{(s)}[\varepsilon]$ the element of $\mathfrak{g}_s$ equal to $(x \otimes \varepsilon_s, 0)$.

Define $V^{\mathbb{A}}$ as the $\mathfrak{g}^{\mathbb{A}}$-module $\otimes'_{x\in X} V_x$, with $V_x = L_{0,k}$ for $x \neq P_0$ and $V_{P_0} = L_{\Lambda,k}$. (Here $\otimes'$ means that the module is spanned by the products $\otimes_{x\in X} v_x$ with v_x in V_x equal to the vacuum vector $v_{top}^{(x)}$ for all but finitely many x.) The proof of the following Lemma is a variant of that of [19], Prop. 2.2.3:

LEMMA A.1. *Let ψ be a $\mathfrak{g}^{out}$-invariant linear form on $L_{\Lambda,k}$. For any finite subset S of X containing P_0, there is a unique linear form ψ_S on V^S, which is $\mathfrak{g}_S^{out}$-invariant and such that $\psi_S(\otimes_{x\in S, x\neq P_0} v_{top}^{(x)} \otimes v) = \psi(v)$ for any v in $L_{\Lambda,k}$.*

There is also a unique linear form $\psi^{\mathbb{A}}$ on $V^{\mathbb{A}}$, which is $\bar{\mathfrak{g}} \otimes \mathbb{C}(X)$-invariant and such that $\psi^{\mathbb{A}}(\otimes_{x\in X, x\neq P_0} v_{top}^{(x)} \otimes v) = \psi(v)$ for any v in $L_{\Lambda,k}$.

Proof. Let us set $\mathfrak{g}_{P_0,x}^{out} = H^0(X - \{P_0, x\}, \bar{\mathfrak{g}})$. Let us denote by $W_{0,k}$ the Weyl module $U\mathfrak{g} \otimes_{U\mathfrak{g}^{in}} \mathbb{C}$, where $\mathbb{C}$ is the $\mathfrak{g}^{in}$-module on which $\bar{\mathfrak{g}} \otimes \mathcal{O}$ acts by zero and K acts by k. Let us prove that there is a bijective correspondence between

(i) the forms ψ_{P_0} on $L_{\Lambda,k}$, which are $\mathfrak{g}^{out}$-invariant,
(ii) the forms $\psi_{P_0,x}$ on $W_{0,k} \otimes L_{\Lambda,k}$, which are $\mathfrak{g}_{P_0,x}^{out}$-invariant
and
(iii) the forms $\bar{\psi}_{P_0,x}$ on $L_{0,k} \otimes L_{\Lambda,k}$, which are $\mathfrak{g}_{P_0,x}^{out}$-invariant, the correspondence being such that

$$\psi_{P_0}(v) = \psi_{P_0,x}(v_{top} \otimes v) = \bar{\psi}_{P_0,x}(v_{top} \otimes v).$$

The proof of the general statement of the Lemma is similar.

Let us construct a form as in (ii) from a form as in (i). Fix a family of functions $(\rho_i)_{i>0}$ in $H^0(X - \{P_0, x\}, \mathcal{O}_X)$, such that ρ_i has the expansion $z_x^{-i} + O(1)$ near x, and a basis $(x_\alpha)_{\alpha\in A}$ of $\bar{\mathfrak{g}}$. Choose an order of the index set A. By the PBW theorem, a basis of $W_{0,k}$ is given by the $\prod_\alpha x_\alpha^{(x)}[\rho_{i_1(\alpha)}] \dots x_\alpha^{(x)}[\rho_{i_{n(\alpha)}(\alpha)}] v_{top}$, for sequences of integers $n(\alpha)$ and of indices $i_1(\alpha) \leqslant i_2(\alpha) \cdots \leqslant i_{n(\alpha)}(\alpha)$, where the product is performed according to the order of A. Set then

$$\begin{aligned}\psi_{P_0,x}&\left(\prod_\alpha x_\alpha^{(x)}[\rho_{i_1(\alpha)}] \dots x_\alpha^{(x)}[\rho_{i_{n(\alpha)}(\alpha)}] v_{top} \otimes v\right) \\ &= \psi_{P_0}\left({\prod_\alpha}' x_\alpha^{(P_0)}[-\rho_{i_{n(\alpha)}(\alpha)}] \dots x_\alpha^{(P_0)}[-\rho_{i_1(\alpha)}] v\right).\end{aligned}$$

Here $\prod'$ means that the product over all α's is taken in the order inverse to the order of A. We have then

$$\psi_{P_0,x}\left(\prod_{\alpha\in A} x_\alpha^{(P_0,x)}[\rho_{i_1(\alpha)}] \dots x_\alpha^{(P_0,x)}[\rho_{i_{n(\alpha)}(\alpha)}](v_{top} \otimes v)\right) = 0,$$

for all v in $V_{\Lambda,k}$, if the product is nonempty. Since the elements of $W_{0,k} \otimes V_{\Lambda,k}$ are combinations of the $\prod_{\alpha\in A} x_\alpha^{(P_0,x)}[\rho_{i_1(\alpha)}] \dots x_\alpha^{(P_0,x)}[\rho_{i_{n(\alpha)}(\alpha)}](v_{top} \otimes v)$, it follows that $\psi_{P_0,x}$ is $\mathfrak{g}_{P_0,x}^{out}$-invariant.

Let us now show that any form as in (ii) is of the type (iii). We follow the argument of [6], based on [10].

For any integer $N \geqslant 2g$, we can construct an element $\rho_{(N)}$ in $H^0(X - \{P_0, x\}, \mathcal{O}_X)$ with the expansions $\rho_{(N)} = z_x^{-1} + O(1)$ near x and $\rho_{(N)} = z_{P_0}^{-N}(\alpha + O(z_{P_0}))$ near P_0, with $\alpha \neq 0$. For that, it suffices to add to ρ_1 some function of $H^0(X - \{P_0\}, \mathcal{O}_X)$.

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Fix $\alpha^\vee$ in the coroot lattice, such that $\langle\alpha^\vee, \theta\rangle \neq 0$. Let N be an integer $\geqslant 2g$ and of the form $1 + d\langle\alpha^\vee, \theta\rangle$, with d integer.

$L_{0,k}$ is the quotient $W_{0,k}/I$, where I is the submodule of $W_{0,k}$ generated by $e_\theta[z_x^{-1}]^{k+1}v_{top}$, where e_θ is the root vector associated to the maximal root θ. I is isomorphic to some Verma module. From [10] follows that $e_\theta^{(x)}[z_x^{-1}]$ is surjective on I. One may use some element of the form $\exp(h^{(P_0)}[\varepsilon])$, with ε in $z_x\mathbb{C}[[z_x]]$, to conjugate $e_\theta^{(x)}[z_x^{-1}]$ to $e_\theta^{(x)}[\rho_{(N)}]$. Therefore, $e_\theta^{(x)}[\rho_{(N)}]$ is also surjective on I.

Let us now show that $e_\theta^{(P_0)}[\rho_{(N)}]$ is locally nilpotent on $L_{\Lambda,k}$. $e_\theta^{(P_0)}[\rho_{(N)}]$ is conjugated by some element of the form $\exp(h^{(P_0)}[\varepsilon])$, with ε in $z_{P_0}\mathbb{C}[[z_{P_0}]]$, to $\alpha e_\theta[z_{P_0}^{-N}]$. Recall that the affine Weyl group contains a translation element w_ω associated to any ω in the coroot lattice; the action of w_ω on the nilpotent loop generators is $w_\omega \cdot e_\alpha^{(P_0)}[f] = e_\alpha[(z_{P_0})^{\langle\omega,\alpha\rangle}f]$, for e_α the root vector associated to any root α. Moreover, the module $L_{\Lambda,k}$ endowed with the composition of the action of $\mathfrak{g}$ with an affine Weyl group automorphism is again integrable. It follows that the action of $w \cdot e_\theta[z_{P_0}^{-1}]$, for w any affine Weyl group element, is locally nilpotent. In particular, for $w = w_{-d\alpha^\vee}$, we find that $e_\theta[z_{P_0}^{-N}]$ is locally nilpotent on $L_{\Lambda,k}$, as well as $e_\theta^{(P_0)}[\rho_{(N)}]$.

These two results imply that $\bar\psi_{P_0,x}$ vanishes on $I \otimes L_{\Lambda,k}$: indeed, any v, v' in I and $L_{\Lambda,k}$, fix m such that $(e_\theta^{(P_0)}[\rho_{(N)}])^m v'$ vanishes; we may write $v = (-e_\theta^{(x)}[\rho_{(N)}])^m v''$, with v'' in I. $\bar\psi_{P_0,x}(v \otimes v')$ is then equal to $\bar\psi(v'' \otimes (-e_\theta^{(P_0)}[\rho_{(N)}])^m v')$, which is zero. $\square$

A.2. FORMULA FOR THE TAME SYMBOL

Denote by σ the *tame symbol* defined in $(\mathbb{A}^\times)^2$ by

$$\sigma((f_x)_{x\in X}, (g_x)_{x\in X}) = (-1)^{\Sigma_{x\in X} v_x(f)v_x(g)} \prod_{x\in X} g'(x)^{v_x(f)} f'(x)^{-v_x(g)};$$

we fix a coordinate z_x at each point x of X and set $f_x = z_x^{v_x(f)}(f'(x) + O(z_x))$.

Fix a lift i of the universal covering $\widetilde{X} \to X$ of X, such that the boundary of $i(X)$ is a union of paths $\widetilde{A}_a$, $\widetilde{B}_a$ projecting to a standard system (A_a), (B_a) of a- and b-cycles. We will identify the local field and ring at any point x of X with the local field and ring at $i(x)$. For $\lambda = (\lambda_a)$ in $\mathbb{C}^g$, define C_λ as the set of the adeles of the meromorphic functions $f : \widetilde{X} \to \mathbb{C}^\times$, such that $f(\gamma_{A_a}z) = f(z)$ and $f(\gamma_{B_a}z) = e^{-\lambda_a}f(z)$.

We then have

LEMMA A.2. (a) *For any λ in $\mathbb{C}^g$, C_λ is not empty; moreover, we can find elements of C_λ without any zero or pole on the $\widetilde{A}_a$.*

(b) *For f in $\mathbb{C}(X)^\times$, without any zero or pole on the cycles A_a, and f_λ in C_λ, we have $\sigma(f, f_\lambda) = e^{\Sigma_a n_a(f)\lambda_a}$, with $n_a(f) = 1/2i\pi\int_{A_a} df/f$.*

Proof. Let us prove (a). Denote by Θ the Riemann theta-function on the Jacobian on X, and by A the Abel map. Let a be any vector of the Jacobian of X, then the

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function

$$z \mapsto \frac{\Theta(A(z) + a - \lambda/2i\pi)}{\Theta(A(z) + a)}$$

belongs to C_λ. That the zero-poles requirement can be satisfied follows from a transversality argument.

Let us prove (b). Suppose that f, g are nonzero meromorphic functions on $i(X)$, such that

$$\sum_{x \in X} res_x \frac{df}{f} = \sum_{x \in X} res_x \frac{dg}{g} = 0.$$

Then we may introduce cuts on $\widetilde{X}$, connecting the zeroes and the poles of f, and choose a determination of $\ln(f)$ which is single-valued along $\partial i(X)$. The same can be done for g. We have then

$$\sigma(f, g) = \exp\left(\frac{1}{4i\pi} \int_{\partial i(X)} \frac{df}{f} \ln g - \frac{dg}{g} \ln f\right).$$

This formula may be proved by deforming $\partial i(X)$ to a set of contours encircling the cuts of $\ln f$ and $\ln g$.

Then in the case where f and g belong to $\mathbb{C}(X)^\times$ and C_λ, we evaluate the integral comparing the contributions of the paths above A_a and A_a^{-1}, and above B_a and B_a^{-1}. For example, in case the zeroes and poles of f and g form disjoint sets, integration by parts gives

$$\begin{aligned}
\frac{1}{4i\pi} \int_{\partial i(X)} \frac{df}{f} \ln g - \frac{dg}{g} \ln f &= \frac{1}{2i\pi} \int_{\partial i(X)} \frac{df}{f} \ln g \\
&= \frac{1}{2i\pi} \sum_a \int_{A_a} \frac{df}{f} (\ln g(z) - \ln g(\gamma_B a z)) \\
&= \frac{1}{2i\pi} \sum_a \int_a \frac{df}{f} \lambda_a,
\end{aligned}$$

which implies (b). □

Remark 15. Lemma A.2, (b) implies that $\sigma(f, g) = 1$ for any f, g in $\mathbb{C}(X)^\times$, which is a well-known fact. One could also prove that for any f in C_λ and f' in $C_{\lambda'}$, without any zero or pole on the $\widetilde{A}_a$, we have

$$\sigma(f, f') = e^{\sum_a n_a(f)\lambda'_a - n_a(f')\lambda_a}. \tag{34}$$

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6.3. CONSTRUCTION OF $\widetilde{\psi}_\lambda$

We now follow the classical procedure to construct operators in $End(V^{\mathbb{A}})$ integrating the Lie algebra action on $V^{\mathbb{A}}$. For f in $\mathbb{A}$, $e_i[f]$ and $f_i[f]$ are locally nilpotent on $V^{\mathbb{A}}$. We set

$$n_i^+[f] = \exp(e_i[f]), \quad n_i^-[f] = \exp(f_i[f])$$

for f in $\mathbb{A}$. Set also, for ρ in $\mathbb{A}^\times$, $w_i[\rho] = n_i^+[\rho]n_i^-[-\rho^{-1}]n_i^+[\rho]$, and

$$t_i[\rho] = w_i[\rho]w_i[1]^{-1}.$$

We have then

$$t_i[\rho\rho'] = \sigma(\rho, \rho')^{-k(h_i|h_i)/2} t_i[\rho]t_i[\rho'] \tag{35}$$

for i simple, and

$$t_i[\rho]t_j[\rho']t_i[\rho]^{-1}t_j[\rho']^{-1} = \sigma(\rho, \rho')^{k(h_i|h_j)}, \tag{36}$$

for any indices i, j (observe that $(h_i|h_j)$ is always integer and $(h_i|h_i)$ always even).

The first identity is a consequence of [12], Thm. 12.24, and the second is a consequence of this identity and [17], 7.3) e) (see also [15], Lemma 8.2, formula (3)).

PROPOSITION–DEFINITION A.1. *Let us fix $\lambda^{(1)}, \ldots, \lambda^{(r)}$ in $\mathbb{C}^g$. For $f_{\lambda^{(i)}}$ in C_{λ^i}, such that the $f_{\lambda^{(i)}}$ have no zero or pole on the A_a, and v in $V_{\Lambda,k}$, the quantity*

$$\exp\left[\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{(i)}}) + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^i n_a(f_{\lambda^{(j)}})\right]$$
$$\times \left\langle \psi^{\mathbb{A}}, t_1[f_{\lambda^{(1)}}] \cdots t_r[f_{\lambda^{(r)}}]\left(v \otimes \otimes_{x \neq P_0} v_{top}^{(x)}\right)\right\rangle \tag{37}$$

is independent of the choice of the $f_{\lambda^{(i)}}$. We will set $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ and

$$\langle \tilde{\psi}_\lambda, v\rangle = \exp\left[\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{(i)}} + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{(j)}})\right]$$

$$\left\langle \psi^{\mathbb{A}}, t_1[f_{\lambda^{(1)}}] \cdots t_r[f_{\lambda^{(r)}}]\left(v \otimes \bigotimes_{x \neq P_0} v_{top}^{(x)}\right)\right\rangle$$

for any such $f_{\lambda^{(i)}}$.

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Proof. Let $f'_{\lambda^{(i)}}$ be other elements of $C_{\lambda^{(i)}}$, satisfying the same zero-poles condition as $f_{\lambda^{(i)}}$. Then $f'_{\lambda^{(i)}} = f_i f_{\lambda^{(i)}}$, with f_i in $\mathbb{C}(X)^{\times}$, without zero or pole on the A_a. We have

$$\exp\left[\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{(i)} n_a(f'_{\lambda^{(i)}}) + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^{(i)} n_a(f'_{\lambda^{(j)}})\right]$$
$$\times \left\langle \psi^{\mathbb{A}}, t_1[f'_{\lambda^{(1)}}] \cdots t_r[f'_{\lambda^{(r)}}]\left(v \otimes \bigotimes_{x \neq P_0} v_{top}^{(x)} \right) \right\rangle$$
$$= \exp\left[\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{i} n_a(f_i) + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^{(i)} n_a(f_i)\right] \times$$
$$\times \exp\left[\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{(i)}} + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{(j)}}\right] \times$$
$$\times \left\langle \psi^{\mathbb{A}}, t_1[f_1 f_{\lambda^{(1)}}] \cdots t_r[f_r f_{\lambda}^{(r)}]\left(v \otimes \bigotimes_{x \neq P_0} v_{\top}^{(x)} \right) \right\rangle$$

The identities (35) and (36) imply that this is equal to

$$\exp\left[\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{(i)} n_a(f_i) + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^{i} n_a(f_j)\right]$$
$$\times \exp\left[\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{i}(f_{\lambda^{(i)}}) + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^{i} n_a(f_{\lambda^{j)}})\right] \times$$
$$\times \prod_i \sigma(f_i, f_{\lambda^{(i)}})^{-k(h_i|h_i)/2} \prod_{i<j} \sigma(f_j, f_{\lambda^{(i)}})^{-k(h_i|h_i)}$$
$$\times \left\langle \psi^{\mathbb{A}}, t_1[f_1] \cdots t_r[f_r] t_1[f_{\lambda^{(1)}}] \cdots t_r[f_{\lambda^{(r)}}\left(v \otimes \bigotimes_{x \neq P_0} v_{top}^{(x)} \right) \right\rangle$$

Now, as the $t_i[f_i]$ are products of exponentials of elements of the $\bar{\mathfrak{g}} \otimes \mathbb{C}(X)$ and $\psi^{\mathbb{A}}$ is $\bar{\mathfrak{g}} \otimes \mathbb{C}(X)$-invariant, we have $\langle \widetilde{\psi}_{\lambda}, \prod_{i=1}^{r} t_i[f_i] v' \rangle = \langle \widetilde{\psi}_{\lambda}, v' \rangle$ for any v' in $V^{\mathbb{A}}$. Applying Lemma A.2., (b), we find that (38) is equal to

$$\exp\left[\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{(i)}} + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{j}})\right] \times$$
$$\times \left\langle \psi^{\mathbb{A}}, t_1[f_{\lambda^{(1)}}] \cdots t_r[f_{\lambda^{(r)}}]\left(v \otimes \bigotimes_{x \neq P_0} v_{top}^{(x)} \right) \right\rangle.$$

$\square$

Remark 16. In view of (36) and (34), it is clear that (37) is independent of the chosen ordering of simple coroots.

Let us now give an expression of $\widetilde{\psi}_{\lambda}$ in terms of extremal vectors.

LEMMA A.3. *Define the vectors* $v_{i;[n]}$ *in* $L_{0,k}$ *by the formulas*

$$v_{i;[0]} = v_{top} v_{i;[n+1]} = \frac{(-1)^k}{k!} f_i[z^{-2n-1}]^k v_{i;[n]}$$

and

$$v_{i;[-n-1]} = \frac{1}{k!} e_i[z^{-2n-1}]^k v_{i;[-n]} \quad \text{for} \quad n \geqslant 0.$$

Then we have $v_{i;[n]} = t_i[z^n] v_{top}$.

Proof. It is enough to prove this statement for the case $\bar{\mathfrak{g}} = \mathfrak{sl}_2$. The formulas for $v_{i;[1]}$ and $v_{i;[-1]}$ are derived by direct expansions. The other formulas are obtained by applying the affine Weyl group translation associated with the coroot h_i (which preserves $t_i[z]$). □

We have then

PROPOSITION A.1. *Assume that the sets* S_i *of zeroes and poles of the* $f_{\lambda^{(i)}}$ *are distinct. Then we have for* v *in* $V_{\Lambda,k}$,

$$\begin{aligned}\langle \tilde{\psi}_\lambda, v\rangle = \exp\Bigg[&\sum_i \frac{k(h_i \mid h_i)}{2} \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{(i)}}) + \sum_{i<j} k(h_i \mid h_j) \sum_a \lambda_a^{(i)} n_a(f_{\lambda^{(j)}}) \times \\ &\times \prod_{i=1}^r \prod_{s\in S_i} (f'_{\lambda^{(i)}(s)^{-k}}) \Bigg\langle \psi_{\{P_0\}\cup(\cup_i S_i)}, \bigotimes_{i=1}^r \Bigg(\bigotimes_{s\in S_i} v^{(s)}_{i;[v_s(f_{\lambda^{(i)}})]} \Bigg) \otimes \\ &\otimes \prod_{i=1}^r t_i^{(P_0)}[f_{\lambda^{(i)}}] v \Bigg\rangle,\end{aligned}$$

where we set $f_{\lambda^{(i)}}(z) = f'_{\lambda^{(i)}}(s) z_s + o(z_s)$ *for* s *in* S_i. *Recall that* $\psi_{\{P_0\}\cup(\cup_i S_i)}$ *denotes the prolongation of* ψ *to the product of* $L_{\Lambda,k}$ *and vacuum modules at the points of* S_i.

A.4. PROOF OF THEOREM 2.1

To prove Thm. 2.1, 1), we first prove

LEMMA A.4. *For any* v *in* $L_{\Lambda,k}$, *the function* $\lambda \mapsto \langle \widetilde{\psi}_\lambda, v\rangle$ *depends analytically on* λ *and satisfies* $\partial_{\lambda_a^{(i)}} \langle \widetilde{\psi}_\lambda, v\rangle = \langle \widetilde{\psi}_\lambda, h_i[r_a] v\rangle$, $a = 1, \ldots, g$, $i = 1, \ldots, r$.

Proof of Lemma. Let us prove this first in the case $\bar{\mathfrak{g}} = \mathfrak{sl}_2$. In that case, we work in a neighborhood of some point λ_0 of $J^0(X)$. Let $P_i(\lambda)$ be points on X ($i = 1, \ldots, g$) such that f_λ has simple zeroes at the $P_i(\lambda)$ and a pole of order g at P_0. Let $z_{P_i(\lambda_0)}$ a coordinate at $P_i(\lambda_0)$; we will again denote by $P_i(\lambda)$ the coordinate of the point $P_i(\lambda)$ in the coordinate system. We will assume that the local coordinate at $P_i(\lambda)$ is $z_{P_i(\lambda)} = z_{P_i(\lambda_0)} - P_i(\lambda)$.

Let for P in X, ρ_P be a meromorphic function on X, with only poles at P_0 and at P, with the expansion $\rho = z_P^{-1} + \mathrm{O}(1)$. We assume that the expansions at P_0 the functions ρ_P depend smoothly on P, for P near any of the $P_i(\lambda_0)$. We set also $f_\lambda(z) = f_\lambda'(P_i(\lambda))z_{P_i(\lambda)} + \frac{1}{2}f_\lambda''(P_i(\lambda))z_{P_i(\lambda)}^2 + \cdots$.

Then Proposition A.1 implies that

$$\langle\tilde\psi_\lambda, v\rangle = \exp\left[k\sum_a \lambda_a n_a(f_\lambda)\prod_i f_\lambda'(P_i(\lambda))^{-k}\right]$$
$$\left\langle\psi_{\{P_0,P_i(\lambda)\}}, \bigotimes_{i=1}^g \frac{f[-z_{P_i(\lambda)}^{-1}]^k}{k!}v_\top^{(P_i(\lambda))} \otimes t^{(P_0)}[f_\lambda]v\right\rangle.$$

As we have seen, $f[-z_{P_i(\lambda)}^{-1}]^k v_{top}^{(P_i(\lambda))}$ is equal to $f^{(P_i(\lambda))}[-\rho_{P_i(\lambda)}]^k v_{top}^{(P_i(\lambda))}$. By the coinvariance of ψ, and the fact that $v_{[1]}$ is annihilated by the $f[\phi]$, ϕ in $\mathcal{O}$, the right-hand side of this equation is equal to

$$\frac{1}{(k!)^g}\exp\left[k\sum_a \lambda_a n_a(f_\lambda)\prod_i f_\lambda'(P_o(\lambda))^{-k}\right]\left\langle\psi, \prod_{i=1}^g\left(f[\rho_{P_i(\lambda)}]^k\right)t[f_\lambda^{(P_0)}]v\right\rangle.$$

This formula shows that $\langle\widetilde\psi_\lambda, v\rangle$ depends smoothly on λ. Let us compute its differential. Let $\delta\lambda$ be a variation of λ. A computation of adjoint actions shows that

$$\delta t\left[f_\lambda^{(P_0)}\right] = \left(h\left[\frac{\delta f_\lambda^{(P_0)}}{f_\lambda^{(P_0)}}\right] + k\left\langle df_\lambda^{(P_0)}, \frac{\delta f_\lambda^{(P_0)}}{(f_\lambda^{(P_0)})^2}\right\rangle\right)t\left[f_\lambda^{(P_0)}\right],$$

so that

$$\begin{aligned}
\delta\langle\widetilde\psi_\lambda, v\rangle = {} & k\left(\sum_a n_a(f_\lambda)\delta\lambda_a\right)\langle\widetilde\psi_\lambda, v\rangle + \\
& + \frac{1}{(k!)^g}\exp^k\left[\sum_a \lambda_a(f_\lambda)\right]\times \\
& \times\sum_{i=1}^g\left\langle\psi, \prod_{j\neq i}\left(f[\rho_{P_j(\lambda)}]^k\right)kf[\rho_{P_i(\lambda)}]^{k-1}f[\delta\rho_{P_i(\lambda)}]t[f_\lambda^{(P_0)}]v\right\rangle\times \\
& \times\prod_i f_\lambda'(P_i(\lambda))^{-k} \\
& + \frac{1}{(k!)^g}\exp^k\left[\sum_a \lambda_a n_a(f_\lambda)\right]\times \\
& \times\left\langle\psi, \prod_{i=1}^g\left(f[\rho_{P_i(\lambda)}]^k\right)\left(h\left[\frac{\delta f_\lambda^{(P_0)}}{f_\lambda^{(P_0)}}\right] + k\left\langle df_\lambda^{(P_0)}, \frac{\delta f_\lambda^{(P_0)}}{(f_\lambda^{(P_0)})^2}\right\rangle\right)t[f_\lambda^{(P_0)}]v\right\rangle\times
\end{aligned}$$

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$$\times \prod_i f'_\lambda(P_i(\lambda))^{-k} + \\ + \left(-k \sum_{i=1}^{g} \frac{\delta f'_\lambda(P_i(\lambda))}{f'_\lambda(P_i(\lambda))}\right) \langle \widetilde{\psi}_\lambda, v\rangle,$$

which can be rewritten (using coinvariance) as

$$\begin{aligned} \delta\langle\widetilde{\psi}_\lambda, v\rangle = & \left(k \sum_a \delta\lambda_a n_a(f_\lambda) - k \sum_i \frac{\delta f'_\lambda(P_i(\lambda))}{f'_\lambda(P_i(\lambda))} + k \left\langle \frac{df_\lambda}{f_\lambda}, \frac{\delta f_\lambda}{f_\lambda}\right\rangle_{P_0}\right) \langle\widetilde{\psi}_\lambda, v\rangle + \\ & + \prod_i f'_\lambda(P_i(\lambda))^{-k} \times \\ & \times \left\langle \psi_{\{P_0, P_i(\lambda)\}}, \sum_i \bigotimes_{j\neq i} v_{[1]}^{(j)} \otimes \frac{(-1)^k}{k!} f[\delta P_i(\lambda) z_{P_i(\lambda)}^{-2}] f[z_{P_i)(\lambda)}^{-1}]^{k-1} v_{top}^{(i)} \otimes \right. \\ & \left. \otimes\, t^{(P_0)}[f_\lambda] v \right\rangle + \prod_i f'_\lambda(P_i(\lambda))^{-k} \times \\ & \times \left\langle \psi_{\{P_0, P_i(\lambda)\}}, \sum_i h^{(P_0)}\left[\frac{\delta f_\lambda}{f_\lambda}\right] (\otimes_{i=1}^g v_{[1]}^{(i)} \otimes t^{(P_0)}[f_\lambda] v) \right\rangle. \end{aligned}$$

The penultimate term is rewritten as

$$\prod_i f'_\lambda(P_i(\lambda))^{-k} \left\langle \psi_{\{P_0, P_i(\lambda)\}}, -\sum_i \delta P_i(\lambda) h^{(i)}[z_{P_i(\lambda)}^{-1}] \left(\bigotimes_{i=1}^g v_{[1]}^{(i)}\right) \otimes t^{(P_0)}[f_\lambda] v\right\rangle,$$

using the identity in $L_{0,k}$

$$h[z^{-1}] v_{[1]} = \frac{(-1)^{k-1}}{(k-1)!} f[z^{-2}] f[z^{-1}]^{k-1} v_{top},$$

which follows from

$$\begin{aligned} h[z^{-1}] f[z^{-1}]^k v_{top} &= \big(e[z] f[z^{-1}] - f[z^{-2}] e[z]\big) f[z^{-1}]^k v_{top} \\ &= -k f[z^{-2}] f[z^{-1}]^{k-1} v_{top}, \end{aligned} \tag{39}$$

because $f[z^{-2}] f[z^{-1}]^k v_{top} = 0$, which is a consequence of the integrability conditions.

On the other hand, we have

$$t[f_\lambda^{(P_0)}] h[r_a] t[f_\lambda^{(P_0)}]^{-1} = h[r_a] + 2k \left\langle \frac{df_\lambda^{(P_0)}}{f_\lambda^{(P_0)}}, r_a\right\rangle$$

so that $\sum_a \delta\lambda_a \langle\widetilde{\psi}_\lambda, h[r_a]v\rangle$ is equal to

$$\frac{1}{(k!)^g}\exp\left[\sum_a \lambda_a n_a(f_\lambda \prod_i f'_\lambda(P_i(\lambda))^{-k}\right]\times$$
$$\times\sum_a \delta\lambda_a\left\langle\psi, \prod_{i=1}^g\left(f[\rho_{P_i(\lambda)}]^k\right)\left(h[r_a]+2k\left\langle\frac{df_\lambda^{(P_0)}}{f_\lambda^{(P_0)}}, r_a\right\rangle\right)t[f_\lambda^{(P_0)}]v\right\rangle.$$

Therefore, we have

$$\delta\langle\widetilde{\psi}_\lambda, v\rangle - \sum_a \delta\lambda_a\langle\widetilde{\psi}_\lambda, h[r_a]v\rangle$$
$$=\left[k\sum_a \delta\lambda_a n_a(f_\lambda) + k\left\langle\frac{df_\lambda}{f_\lambda}, \frac{\delta f_\lambda}{f_\lambda}\right\rangle_{P_0} - 2k\sum_a \delta\lambda_a\left\langle\frac{df_\lambda}{f_\lambda}, r_a\right\rangle_{P_0} - k\sum_i \frac{\delta f'_\lambda(P_i(\lambda))}{f'_\lambda(P_i(\lambda))}\right]+$$
$$\langle\widetilde{\psi}_\lambda, v\rangle + \prod_i f'_\lambda(P_i(\lambda))^{-k}\times$$
$$\times\left\langle\psi_{\{P_0,P_i(\lambda)\}}, \left(h^{(P_0)}\left[\frac{\delta f_\lambda}{f_\lambda}\right] - \sum \delta P_i(\lambda)h^{(i)}[z_{P_i(\lambda)}^{-1}] - \sum_a \delta\lambda_a h^{(P_0)}[r_a]\right)\times\right.$$
$$\left.\times\left(\bigotimes_i v_{[1]}\right)\otimes t[f_\lambda^{(P_0)}]v\right\rangle.$$

On the other hand,

$$\varrho = \frac{\delta f_\lambda^{(P_0)}}{f_\lambda^{(P_0)}} - \sum_a \delta\lambda_a r_a$$

is single-valued on X and has simple poles at the $P_i(\lambda)$. Therefore,

$$\left\langle\psi_{\{P_0,P_i(\lambda)\}}, \left(h^{(P_0)}[\varrho] + \sum_i h^{(i)}[\varrho])((\otimes_i v_{[1]}^{(i)})\otimes t^{(P_0)}v\right)\right\rangle$$

is zero, so that $\delta\langle\widetilde{\psi}_\lambda, v\rangle - \sum_a \delta\lambda_a\langle\widetilde{\psi}_\lambda, h[r_a]v\rangle$ is proportional to

$$\begin{aligned}&-\sum_{i=1}^g \frac{\delta f'_\lambda(P_i(\lambda))}{f'_\lambda(P_i(\lambda))} + \sum_a \delta\lambda_a n_a(f_\lambda) + \left\langle\frac{df_\lambda}{f_\lambda}, \frac{\delta f_\lambda}{f_\lambda}\right\rangle_{P_0} - \\ &-2\sum_a \delta\lambda_a\left\langle\frac{df_\lambda}{f_\lambda}, r_a\right\rangle_{P_0} + 2\sum_i\left[\left(\frac{\delta f_\lambda}{f_\lambda}\right)^{reg}(P_i(\lambda)) - \sum_a \delta\lambda_a r_a(P_i(\lambda))\right],\end{aligned} \tag{40}$$

where we set

$$\left(\frac{\delta f_\lambda}{f_\lambda}\right)(z) = \alpha_{\lambda,i}z_{P_i(\lambda)}^{-1} + \left(\frac{\delta f_\lambda}{f_\lambda}\right)^{reg}(P_i(\lambda)) + O(z_{P_i(\lambda)}).$$

The vanishing of (40) then follows from the identities

$$\left\langle \frac{df_\lambda}{f_\lambda}, \frac{\delta f_\lambda}{f_\lambda} \right\rangle_{P_0} = -\sum_i \left\langle \frac{df_\lambda}{f_\lambda}, \frac{\delta f_\lambda}{f_\lambda} \right\rangle_{P_i} + \sum_a n_a(f_\lambda)\delta\lambda_a,$$

$$\left\langle \frac{df_\lambda}{f_\lambda}, r_a \right\rangle_{P_0} = -\sum_i r_a(P_i(\lambda)) + n_a(f_\lambda)$$

and

$$-\frac{\delta f'_\lambda(P_i(\lambda))}{f'_\lambda(P_i(\lambda))} - \left\langle \frac{df_\lambda}{f_\lambda} \right\rangle_{P_i} + 2\left(\frac{\delta f_\lambda}{f_\lambda}\right)^{reg}(P_i) = 0;$$

the latter identity follows from the expansions

$$\frac{df_\lambda}{f_\lambda} = \frac{dz}{z - P_i(\lambda)} + \frac{1}{2}\frac{f''_\lambda}{f'_\lambda}(P_i(\lambda))dz + \mathrm{O}(z - P_i(\lambda))dz,$$

$$\frac{\delta f_\lambda}{f_\lambda} = -\frac{\delta P_i(\lambda)}{z - P_i(\lambda)} + \left[\frac{\delta f'_\lambda(P_i(\lambda))}{f'_\lambda(P_i(\lambda))} - \frac{1}{2}\sum_a \frac{f''_\lambda}{f'_\lambda}(P_i(\lambda))\delta P_i(\lambda)\right] + \mathrm{O}(z - P_i(\lambda)),$$

$$\left(\frac{\delta f_\lambda}{f_\lambda}\right)^{reg}(P_i(\lambda)) = \frac{\delta f'_\lambda(P_i(\lambda))}{f'_\lambda(P_i(\lambda))} - \frac{1}{2}\sum_i \frac{f''_\lambda}{f'_\lambda}(P_i(\lambda))\delta P_i(\lambda).$$

This ends the proof of Lemma 6.4 in the case $\bar{\mathfrak{g}} = \mathfrak{sl}_2$. In the case of general $\bar{\mathfrak{g}}$, this result allows to compute $\partial_{\lambda_a^{(1)}} \langle \widetilde{\psi}_\lambda, v \rangle$; the additional prefactors of the expression of $\langle \widetilde{\psi}_\lambda, v \rangle$ allow to transfer the $h_1[r_a]$ in front of v. Using Remark 16, we can treat the case of any simple coroot in the same way. $\square$

Let us now show why Lemma 6.4 implies Theorem 2.1(1). The differential equation of Lemma 6.4 and the equality $\widetilde{\psi}_0 = \psi$ imply that the formal expansion of $\langle \widetilde{\psi}_\lambda, v \rangle$ for λ near 0 is equal to $\langle \psi_\lambda, v \rangle$. This implies Theorem 2.1(1).

Theorem 2.1(2) follows from the equality $\psi_\lambda = \widetilde{\psi}_\lambda$ and the fact that for any $f_{\lambda^{(i)}}$ in $C_{\lambda^{(i)}}$, we have

$$Ad(t_1[f_{\lambda^{(1)}}] \cdots t_r[f_{\lambda^{(r)}}])(\mathfrak{g}_\lambda^{out}) = \mathfrak{g}^{out}.$$

Finally, Theorem 2.1(3) follows from the equality $\widetilde{\psi}_\lambda = \psi_\lambda$ and the fact that if f_λ belongs to C_λ, $f_\lambda e^{\zeta_a}$ belongs to $C_{\lambda+\Omega_a}$. This ends the proof of Theorem 2.1.

Remark 17. Equation (39) is translated through the states-fields correspondence into the identity

$$\frac{d}{dz}(f(z)^k) = -{:}h(z)f(z)^k{:},$$

which is valid in level k modules (see [14]), and means that $f(z)^k$ is a vertex operator.

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The connection between this vertex algebra and the Abel–Jacobi map was noticed in [7].

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