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IMPROVED BOUNDS FOR THE VARIANCE OF THE BUSY PERIOD OF THE $M/G/\infty$ QUEUE

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Abstract

Bounds obtained by Ramalhoto [1] for the variance of the busy period of an $M/G/\infty$ queue are improved.

LOWER AND UPPER BOUNDS

Ramalhoto [1] obtained bounds for the variance of the busy period (BP) of an $M/G/\infty$ queue in terms of the Poisson parameter λ, the mean (α) and the variance (σ_s^2) of the service-time distribution $G(\cdot)$ where α and σ_s^2 are finite. These bounds are improved in the present note.

1. Notation and definitions

Let $\rho = \lambda\alpha$, $\alpha^2\gamma_s^2 = \sigma_s^2$ and $U(t) = \int_t^\infty [1 - G(x)]\,dx$. Then $U(0) = \alpha$ and

$$2\int_0^\infty U(t)\,dt = 2\int_0^\infty t(1-G(t))\,dt = \alpha^2(\gamma_s^2 + 1).$$

Further, let T be a random variable having p.d.f.

$$f(t) = 2t(1-G(t))/\alpha^2(\gamma_s^2+1) \qquad t \geqq 0$$
$$= 0 \qquad \text{otherwise.}$$

The expression for the variance of BP as given in Ramalhoto [1] can be written as

$$\text{(1)} \qquad \operatorname{Var}(\text{BP}) = \lambda^{-2}\{\exp(\rho)[\rho^2(\gamma_s^2+1)E(\exp\lambda U(T))] - (\exp(\rho)-1)^2\}.$$

We shall first prove the following lemma.

Lemma. Let $a_n = E[U(T)]^n$ and $a_n/b_n = \alpha^n/[(n+1)(n+2)]$. Then for any positive integer n,

$$\text{(i)} \qquad 2\int_0^\infty U^n(t)\,dt = n\alpha(\gamma_s^2+1)a_{n-1}$$

$$\text{(ii)} \qquad 2(\gamma_s^2+1)^{-1} \leqq b_n \leqq 2.$$

Proof. (i) is proved by integration by parts. $U(t) \geqq \max[0, (\alpha - t)]$. Therefore, using

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(i), we get the left-hand side of (ii). Further, $U(t) \leqq \alpha - t(1-G(t))$. Multiplying throughout by $U^n(t)$ and integrating with respect to t over R^+, then using (i) we get $(n+2)a_n \leqq n\alpha a_{n-1}$, i.e. $b_n \leqq b_{n-1}$ for all positive integer n. Therefore, $b_n \leqq b_0 = 2$.

2. Lower and upper bounds for Var (BP)

Proposition. $L_1(\lambda^{-2}, \rho, \gamma_s^2) \leqq \mathrm{Var}\,[\mathrm{BP}] \leqq U_1(\lambda^{-2}, \rho, \gamma_s^2)$ where

$$L_1(\lambda^{-2}, \rho, \gamma_s^2) = \lambda^{-2}\{\max\,[(\exp\,(2\rho) + \exp\,(\rho)\rho^2\gamma_s^2 - 2\rho\exp\,(\rho) - 1), 0]\},$$

$$U_1(\lambda^{-2}, \rho, \gamma_s^2) = \lambda^{-2}\{2\exp\,(\rho)(\gamma_s^2 + 1)(\exp\,(\rho) - 1 - \rho) - (\exp\,(\rho) - 1)^2\}.$$

Proof. Since $\exp\,[\lambda U(T)]$ is a bounded random variable, therefore,

$$E[\exp \lambda U(T)] = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n a_n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\rho^n b_n}{n+2!}.$$

Hence using (ii)

$$1 + 2\rho^{-2}(1+\gamma_s^2)^{-1}\left(\exp\,(\rho) - 1 - \rho - \frac{\rho^2}{2}\right) \leqq E[\exp \lambda U(T)] \leqq 2\rho^{-2}(\exp\,(\rho) - 1 - \rho).$$

Substituting this in (1), the proposition is proved.

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Reference

Ramalhoto, M. F. (1984) Bounds for the variance of the busy period of the $M/G/\infty$ queue. *Adv. Appl. Prob.* **16,** 929–932.